The Tractability Border of Reachability in Simple Vector Addition Systems with States

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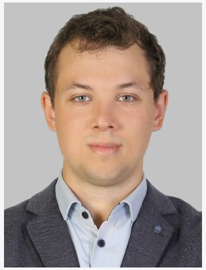

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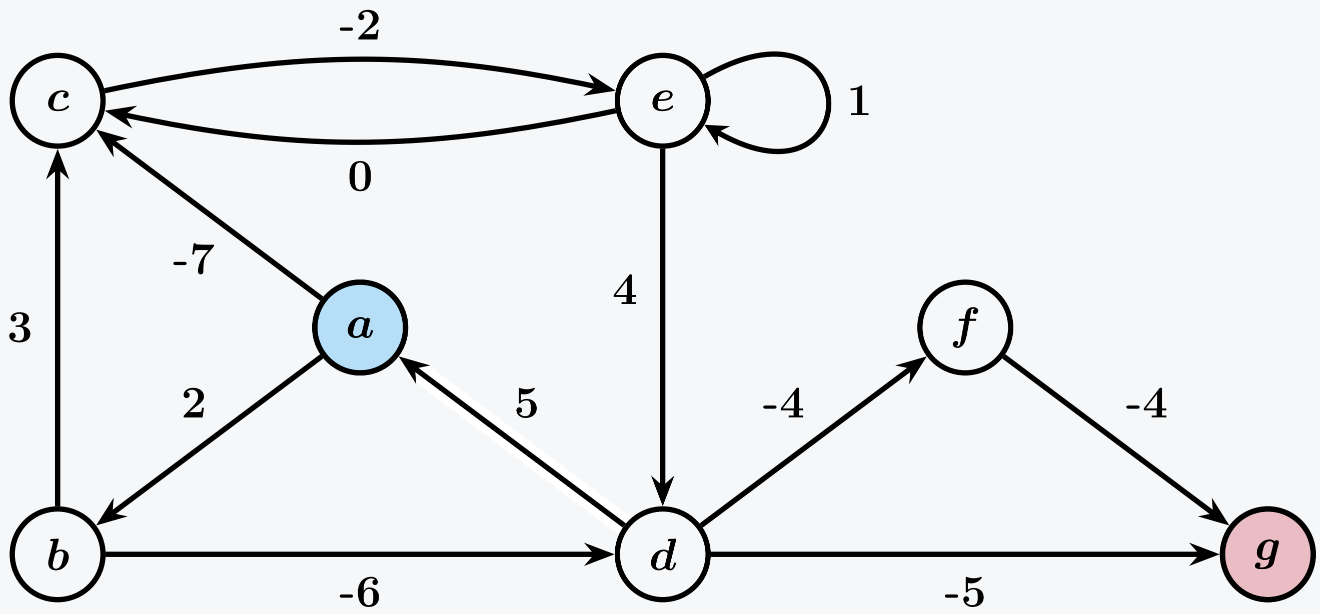
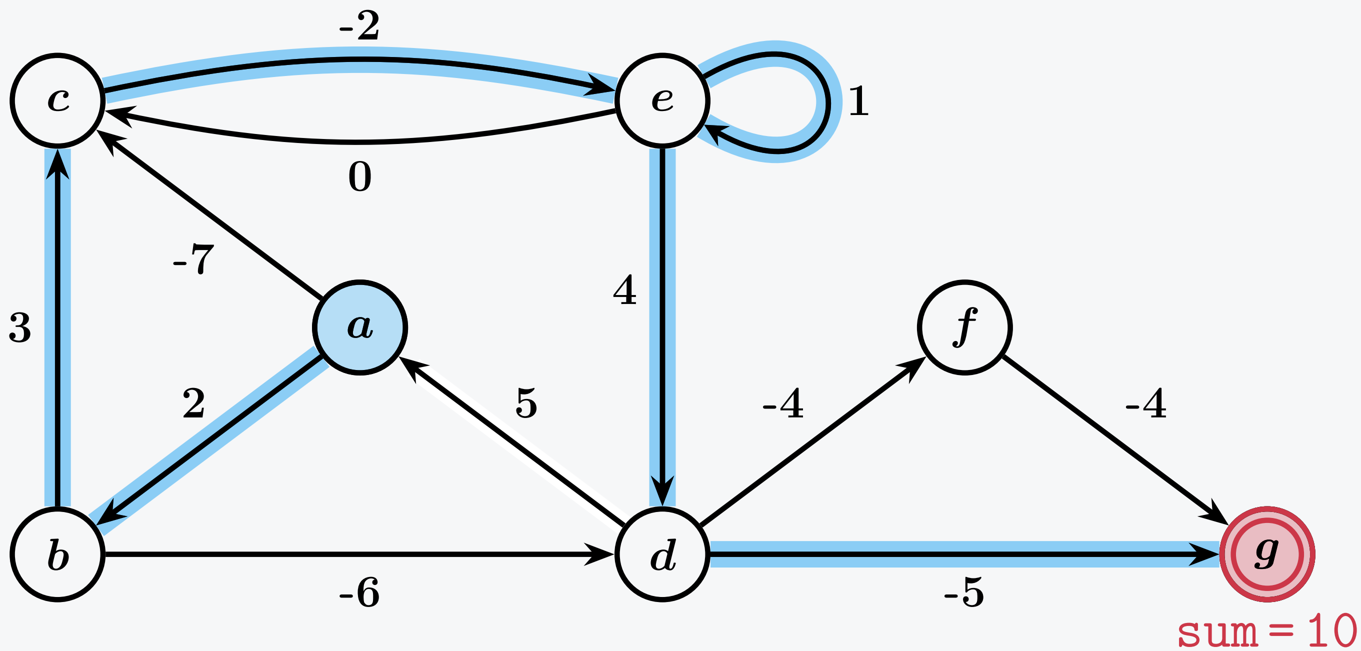
Never Negative and Exact Weight Reachability



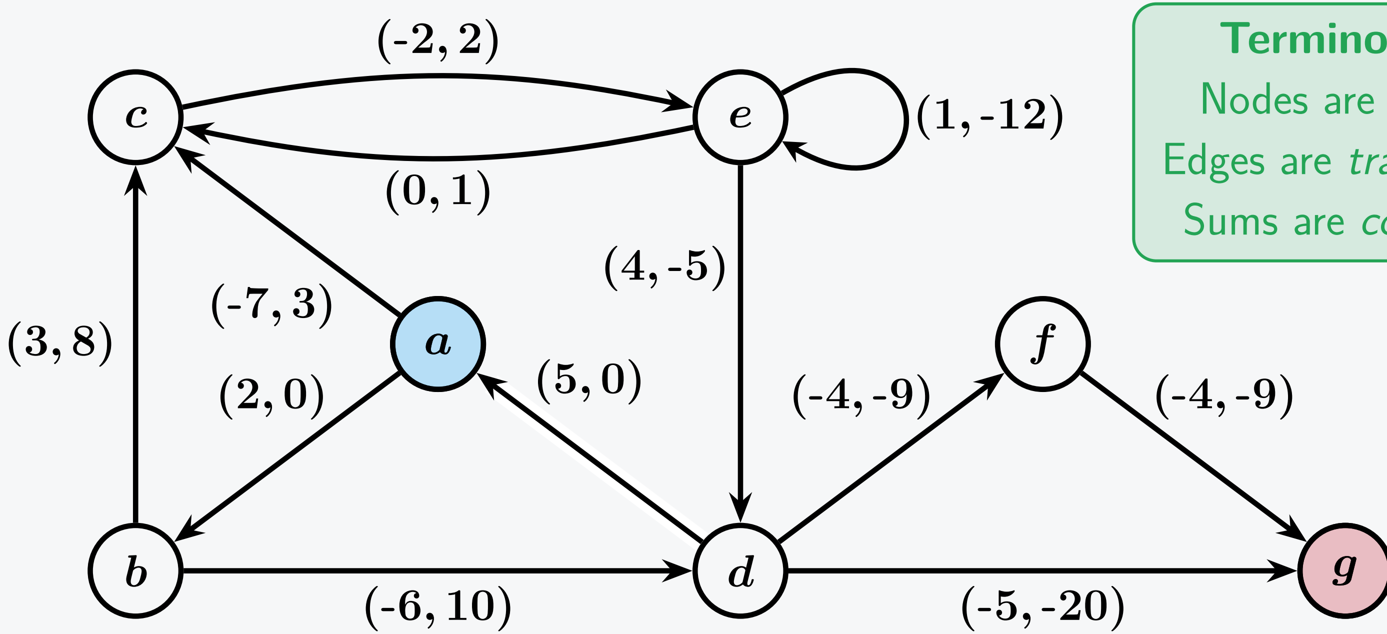
Does there exist a never negative path from a to g of weight 10 ?

Never Negative and Exact Weight Reachability



Does there exist a never negative path from a to g of weight 10 ? **YES!**

Reachability in 2-VASS



Terminology

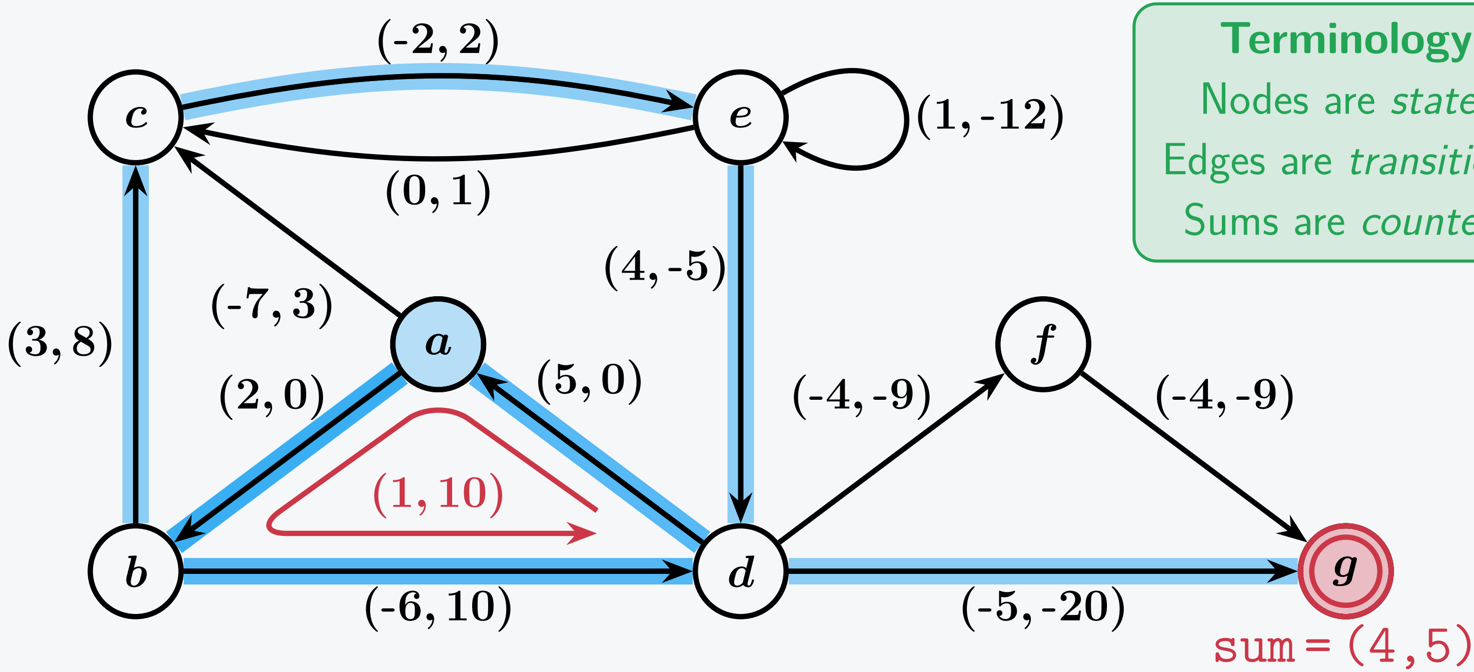
Nodes are *states*

Edges are *transitions*

Sums are *counters*

Does there exist a never negative path from a to g of weight (4,5) ?

Reachability in 2-VASS



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Nodes are *states*

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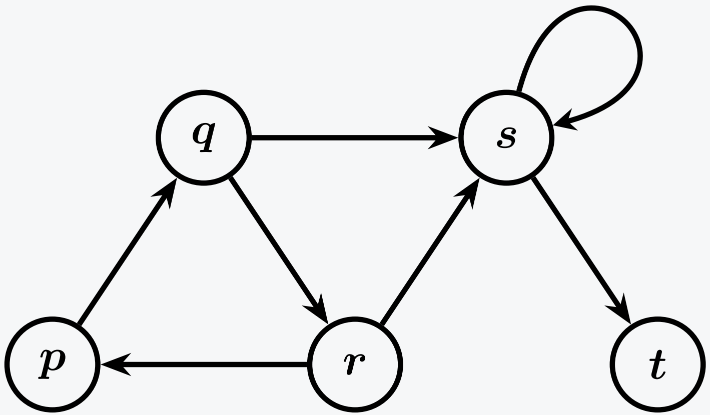
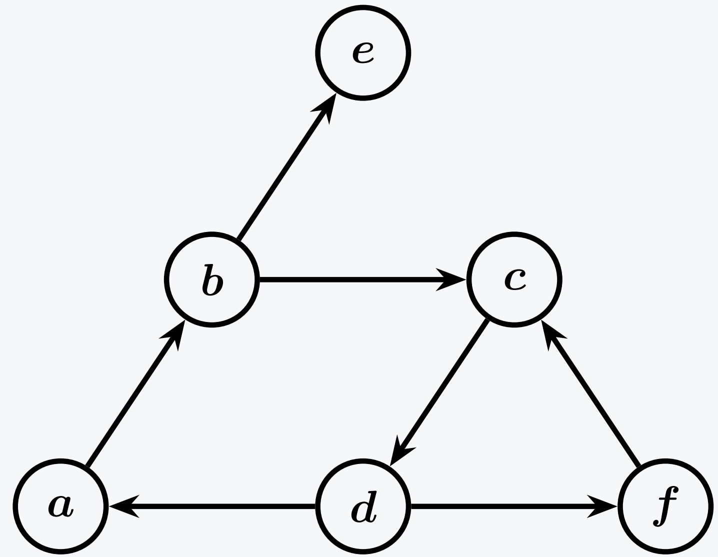
Does there exist a never negative path from a to g of weight `(4,5)` ? **Yes**

“Simple” Vector Addition Systems with States

Definition (Flat)**.** For every state q, there is at most one simple cycle that contains q.



Flat :)



Not flat :(

Reachability in Flat VASS

Theorem. Reachability in flat VASS is in NP (even with binary encoding). [Fribourg and Olsén '97]
[Czerwiński, Lasota, Lazić, Leroux, and Mazowiecki '20]

Theorem. Reachability in binary flat 1-VASS is NP-hard. [Rosier and Yen '85]

Theorem. Reachability in unary (flat) 1-VASS and 2-VASS is in NL. [Valiant and Paterson '73]
[Englert, Lazić, and Totzke '16]

Theorem. Reachability in unary flat d-VASS is NP-hard for $d = 7$.
[Czerwiński, Lasota, Lazić, Leroux, and Mazowiecki '20]

... for $d = 5$. [Dubiak '20]

... for $d = 4$. [Czerwiński and Orlikowski '22]

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What is the complexity of reachability in unary flat $\mathbf{3}$-VASS?

[Blondin, Finkel, Göller, Haase, and McKenzie '15]

[Englert, Lazić, and Totzke '16]

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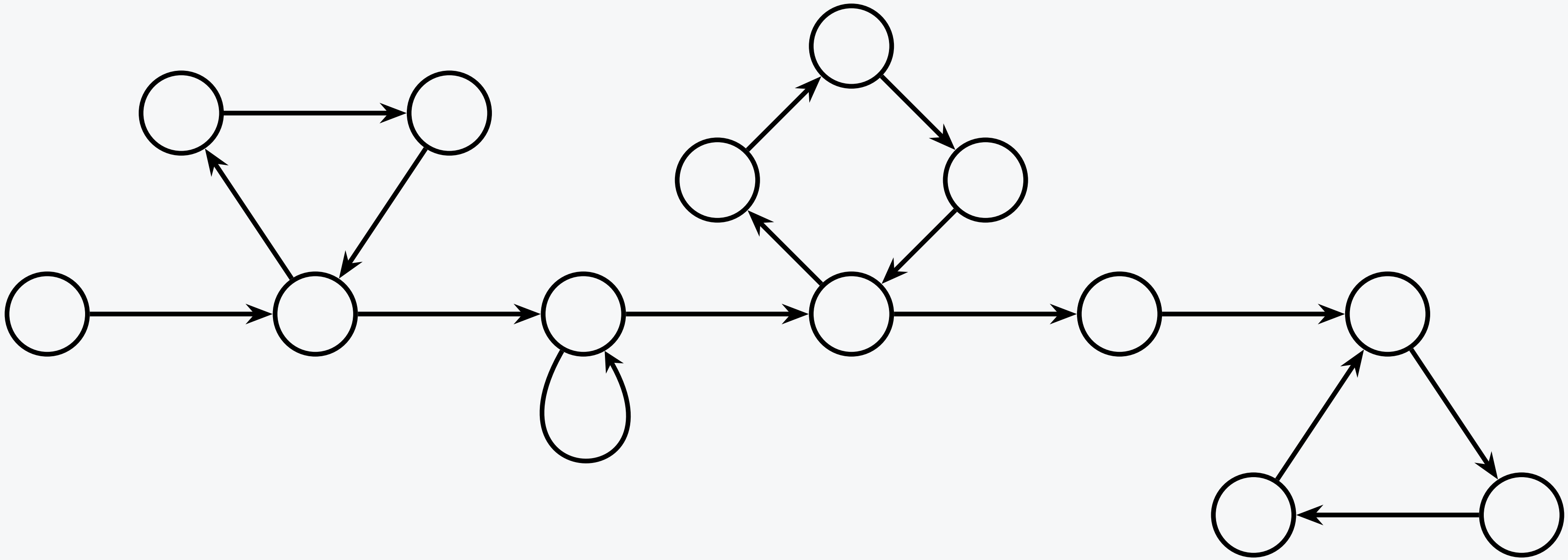
~~Flat VASS~~ Linear Path Schemes

Definition (LPS)**.** A VASS where the states and transitions form a simple path between disjoint cycles.



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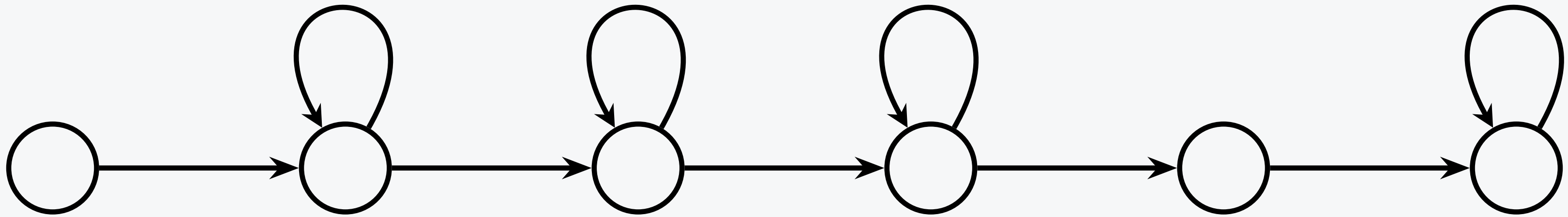
Definition (SLPS)**.** A *Simple* LPS has cycles of length one ("self-loops").



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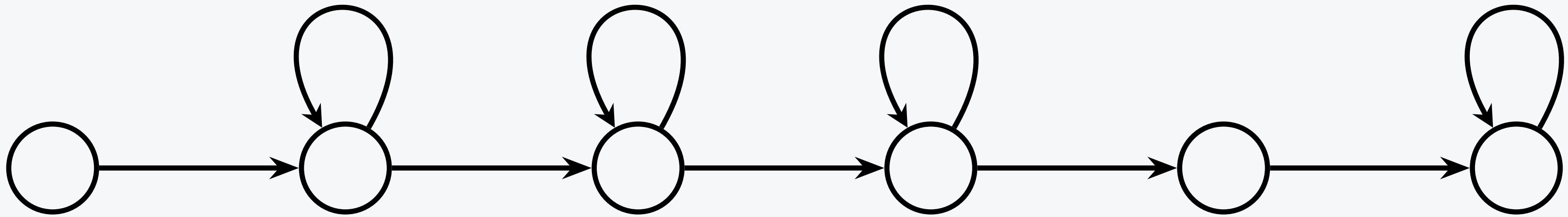

For $d \geq 3$, is reachability in unary d-dimensional linear path schemes in P?

[Englert, Lazić, and Totzke '16]

[Leroux '21]

Main Contribution

Theorem. Reachability in unary 3-SLPS is NP-complete.

Proof approach. Recall that reachability in (binary encoded) flat VASS is in NP.

For NP-hardness, reduce from 3-SAT.

1) Use "Chinese remainder encoding" for SAT.
2) Encode satisfiability as a conjunction of non-divisibility assertions.
3) Design a 2-SLPS with zero tests for asserting non-divisibility.
4) Concatenate these 2-SLPSs with zero tests so that reachability coincides with the conjunction of non-divisibility assertions.
5) Use an additional third counter to simulate the zero tests.

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Encoding SAT as a Conjunction of Non-Divisibility Assertions

Chinese remainder encoding for SAT with k variables $x_1, \dots, x_k$.

- Let $p_1, \dots, p_k$ be the first k primes.
- Let $n \in \mathbb{N}$ such that $n \equiv 0 \bmod p_i \iff x_i$ is false and $n \equiv 1 \bmod p_i \iff x_i$ is true.

First, enforce assignment validity.

- Want to verify that $n \equiv 0 \bmod p_i$ OR $n \equiv 1 \bmod p_i$ (for every i).
- Instead, check $p_i \nmid n - 2$ AND $p_i \nmid n - 3$ AND $\cdots$ AND $p_i \nmid n - (p_i - 1)$.

Second, enforce satisfiability.

- A clause $x_1 \vee \neg x_2 \vee x_3$ is satisfied if $n \equiv 1 \bmod 2$ OR $n \equiv 0 \bmod 3$ OR $n \equiv 1 \bmod 5$.
- This is only falsified when $n \equiv 10 \bmod 2 \cdot 3 \cdot 5$.
- Therefore, check $2 \cdot 3 \cdot 5 \nmid n - 10$.

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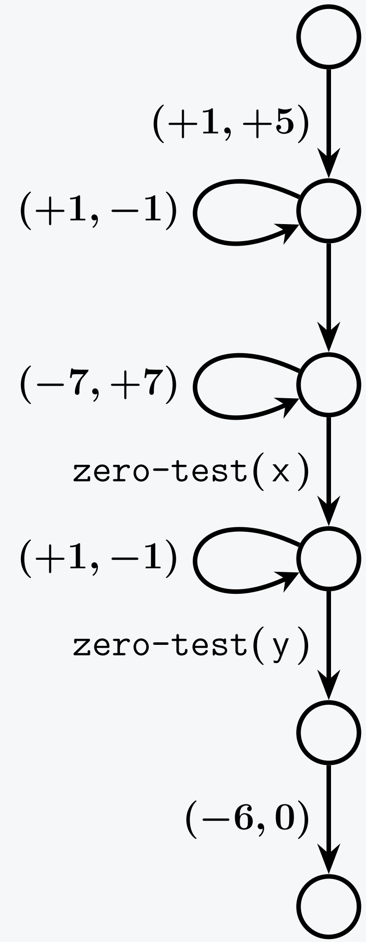
Simple Linear Path Schemes Asserting Non-Divisibility

Suppose we want to assert $\mathbf{7} \nmid \boldsymbol{v}$.

Let's construct a 2-SLPS with zero tests that:

- starts with x $= \boldsymbol{v}$, y $= \mathbf{0}$,
- can only be passed if $\mathbf{7} \nmid \boldsymbol{v}$, and
- ends with x $= \boldsymbol{v}$, y $= \mathbf{0}$.



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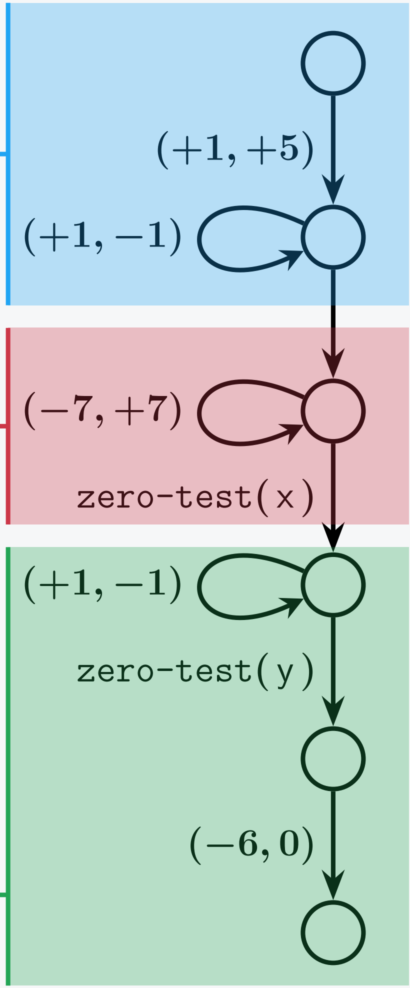
(i) Choose $\boldsymbol{r} \in \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}\}$ ...

(ii) ... such that $\mathbf{7} \mid \boldsymbol{v} + \boldsymbol{r}$.

(iii) Restore x $= \boldsymbol{v}$, y $= \mathbf{0}$.



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4) Concatenate these 2-SLPSs with zero tests so that reachability coincides with the conjunction of non-divisibility assertions. And add an $\mathsf{x} + 1$ loop for guessing an assignment $\mathsf{x} = v$.

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Simulating Zero Tests

Lemma 2.2 (Controlling Counter Technique). *Let $\mathcal{Z}$ be a d-VASS with zero tests and let $s(\mathbf{x}), t(\mathbf{y})$ be two configurations. Suppose $\mathcal{Z}$ has the property that on any accepting run from $s(\mathbf{x})$ to $t(\mathbf{y})$, at most m zero tests are performed on each counter. Then there exists a $(d+1)$-VASS $\mathcal{V}$ and two configurations $s'(\mathbf{0}), t'(\mathbf{y}')$ such that:*

(1) $s(\mathbf{x}) \xrightarrow{*}_{\mathcal{Z}} t(\mathbf{y})$ *if and only if* $s'(\mathbf{0}) \xrightarrow{*}_{\mathcal{V}} t'(\mathbf{y}')$,

(2) $\mathcal{V}$ *can be constructed in* $\mathcal{O}((\mathit{size}(\mathcal{Z}) + \|x\|) \cdot (m+1)^d)$ *time, and*

(3) $\|\mathbf{y}'\| \leq \|\mathbf{y}\|$.

Moreover, if $\mathcal{Z}$ is a flat VASS or a (simple) linear path scheme in which no zero-testing transition lies on a cycle, then $\mathcal{V}$ can be assumed to be a flat VASS or a (simple) linear path scheme, respectively.

Idea first appears in [Czerwiński, Lasota, Lazić, Leroux, and Mazowiecki '19]

First formalised in [Czerwiński and Orlikowski '21]

Reformulated and extended in [this paper]

Takeway message: A "small" number of zero tests can be simulated by an additional counter.

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Theorem. Reachability in unary 3-SLPS is NP-complete.

Theorem. Reachability in unary *ultraflat* 4-VASS is NP-complete.

Theorem. Reachability in *unitary* inverse-Ackermann-dimensional SLPS is NP-complete.

Theorem. Reachability in unary 2-SLPS with *binary encoded initial and target configurations* is in P.

Thank You!

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