# The Tractability Border of Reachability in Simple Vector Addition Systems with States


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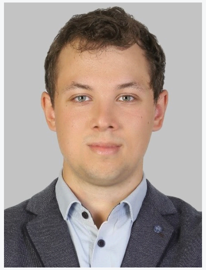

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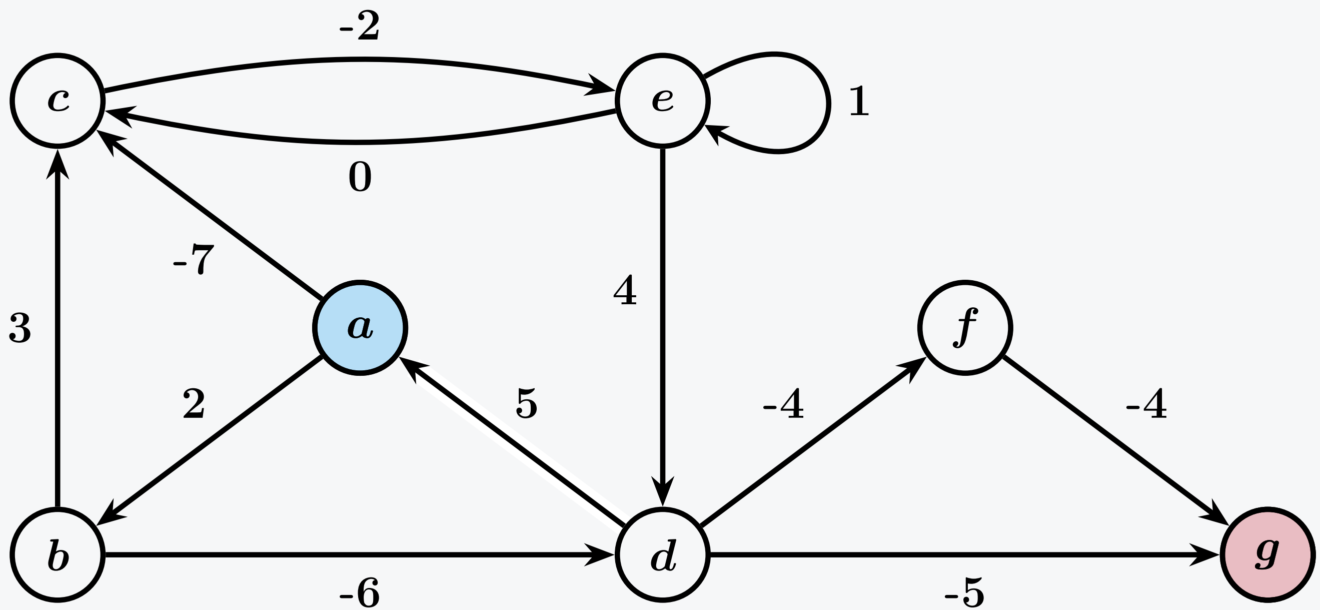
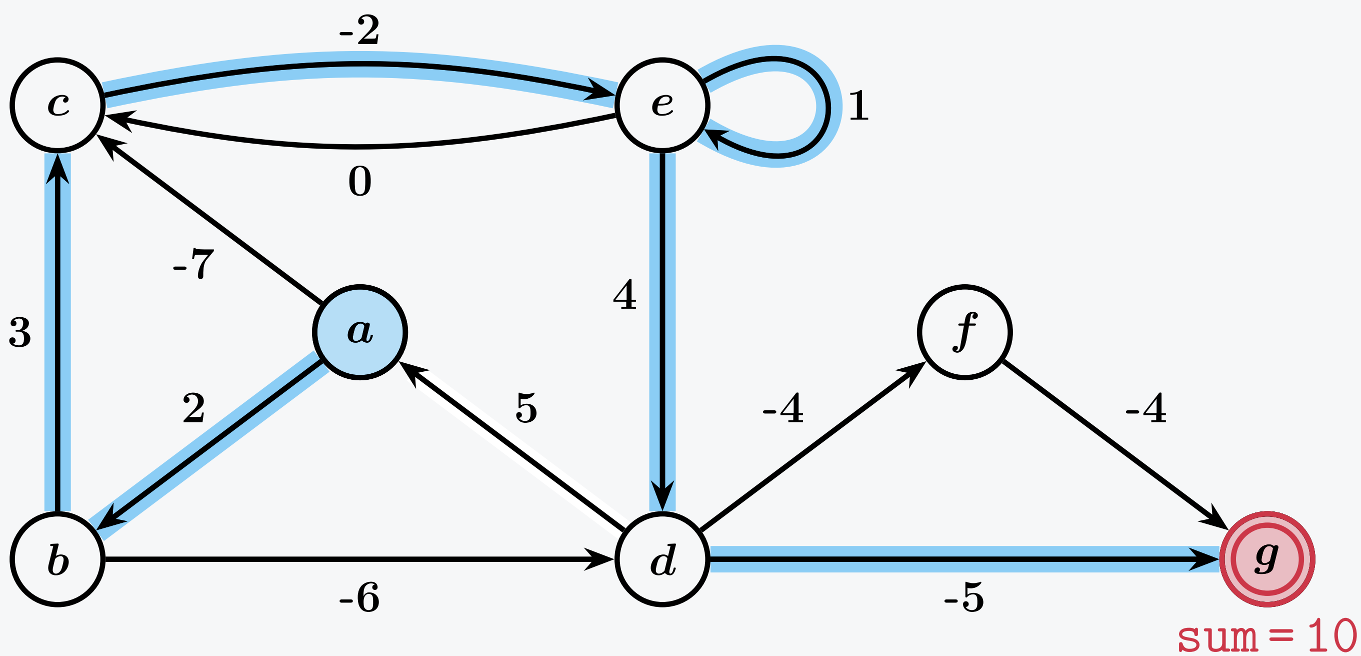
# Never Negative and Exact Weight Reachability



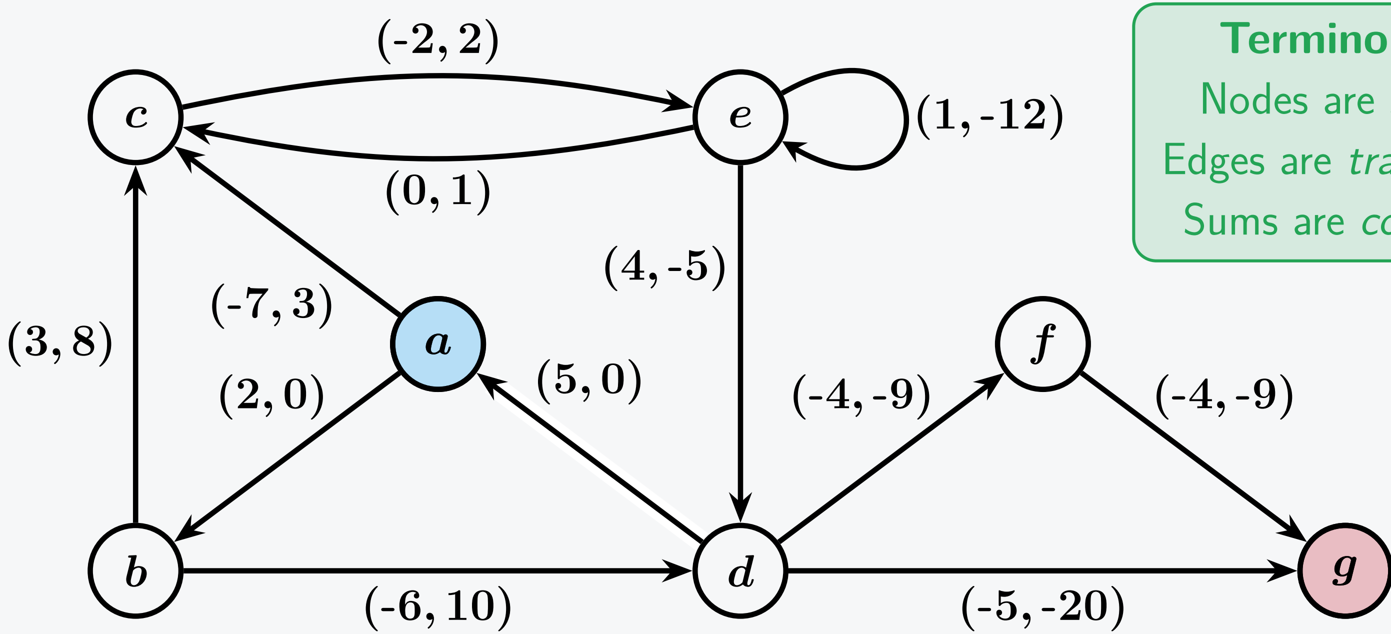

Does there exist a never negative path from $a$ to $g$ of weight 10 ?

# Never Negative and Exact Weight Reachability




Does there exist a never negative path from $a$ to $g$ of weight 10 ? **YES!**

# Reachability in 2-VASS




**Terminology**

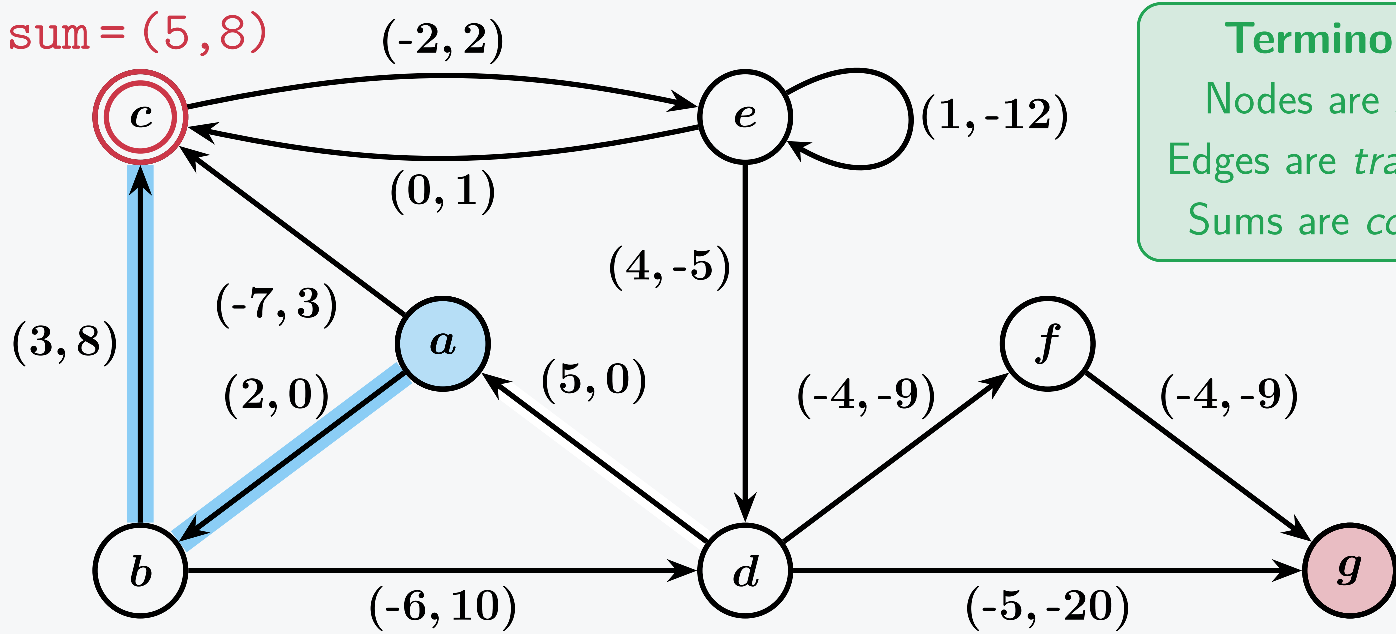
Nodes are *states*

Edges are *transitions*

Sums are *counters*

Does there exist a never negative path from $a$ to $g$ of weight (4,5) ?

# Reachability in 2-VASS




**Terminology**

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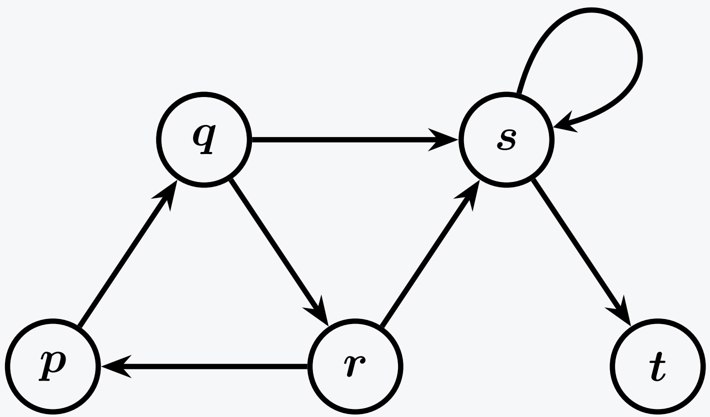
Does there exist a never negative path from $a$ to $g$ of weight `(4,5)` ?

**Homework ;)**

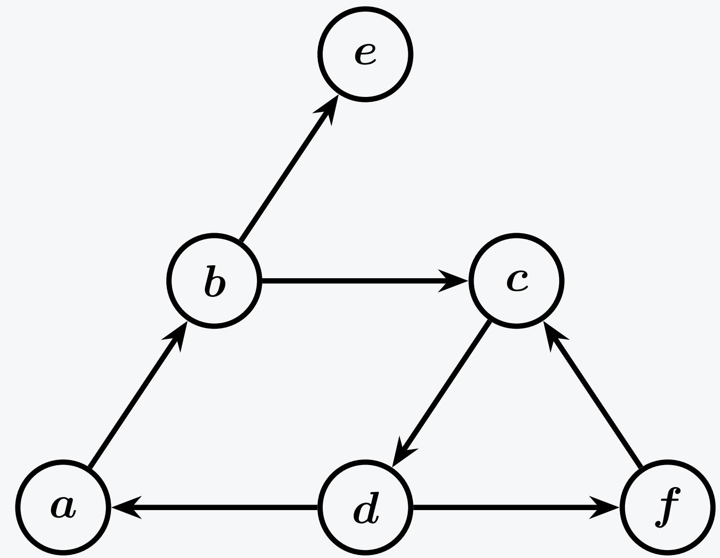
(or watch this talk's video)

# “Simple” Vector Addition Systems with States

**Definition** (Flat)**.** For every state $q$, there is at most one simple cycle that contains $q$.




Flat :)



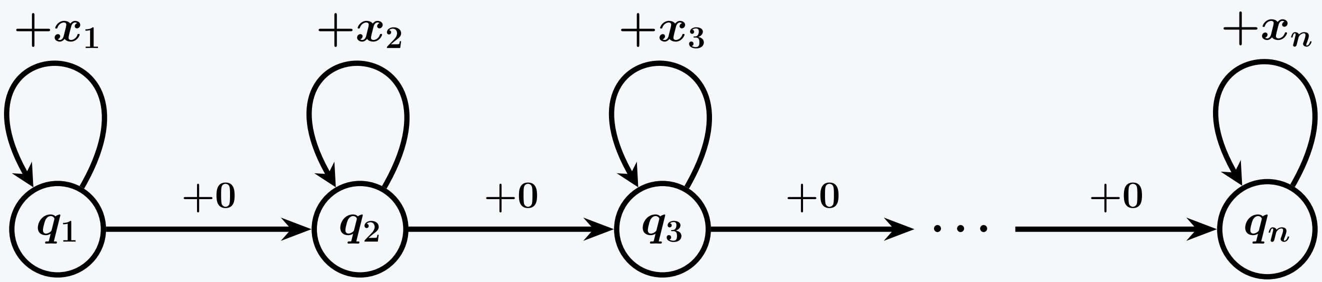

Not flat :(

# Reachability in Flat VASS

**Theorem.** Reachability in flat VASS is in NP (even with binary encoding). [Fribourg and Olsén '97]
[Czerwiński, Lasota, Lazić, Leroux, and Mazowiecki '20]

**Theorem.** Reachability in binary flat 1-VASS is NP-hard. [Rosier and Yen '85]

*Proof sketch.* Let $(\{x_1, \ldots, x_n\}, t)$ be an instance of (unbounded) subset sum.




Solution to subset sum $\iff$ there is a path for reachability.

# Reachability in Flat VASS

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**Theorem.** Reachability in binary flat 1-VASS is NP-hard. [Rosier and Yen '85]

**Theorem.** Reachability in unary (flat) 1-VASS and 2-VASS is in NL. [Valiant and Paterson '73]

[Englert, Lazić, and Totzke '16]

**Theorem.** Reachability in unary flat $d$-VASS is NP-hard for $d = 7$.

[Czerwiński, Lasota, Lazić, Leroux, and Mazowiecki '20]

... for $d = 5$. [Dubiak '20]

... for $d = 4$. [Czerwiński and Orlikowski '22]

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What is the complexity of reachability in unary flat $\mathbf{3}$-VASS?

[Blondin, Finkel, Göller, Haase, and McKenzie '15]
[Englert, Lazić, and Totzke '16]

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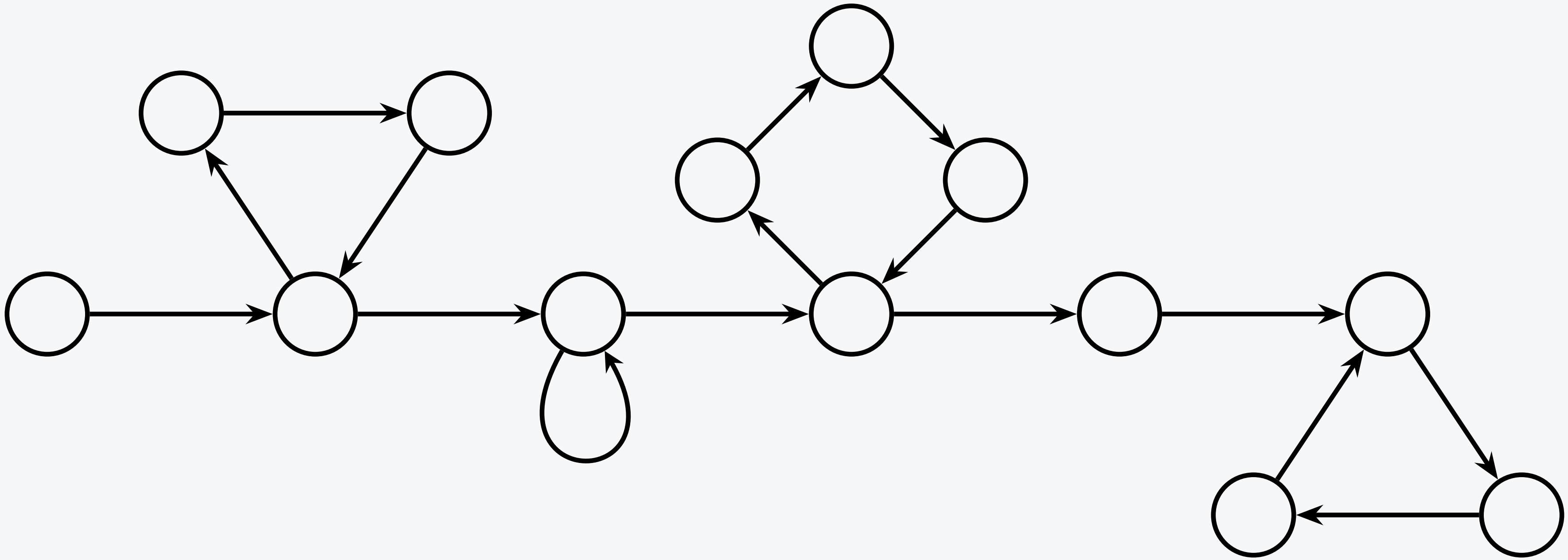
# ~~Flat VASS~~ Linear Path Schemes

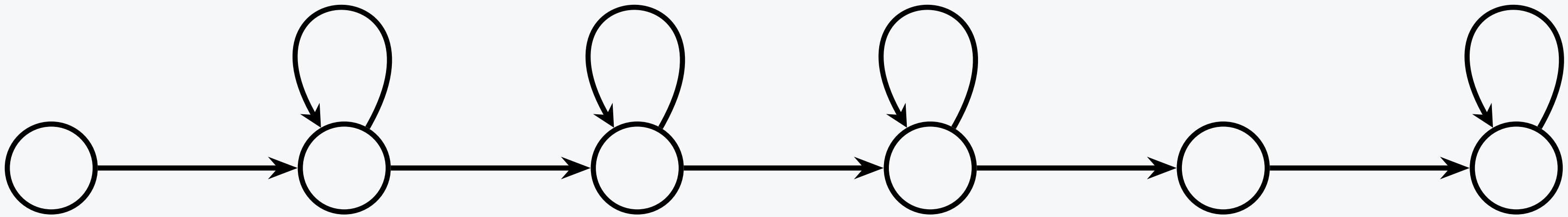
**Definition** (LPS)**.** A VASS where the states and transitions form a simple path between disjoint cycles.



# ~~Flat VASS~~ Linear Path Schemes

**Definition** (LPS)**.** A VASS where the states and transitions form a simple path between disjoint cycles.

**Definition** (SLPS)**.** A *Simple* LPS has cycles of length one ("self-loops").



# ~~Flat VASS~~ Linear Path Schemes

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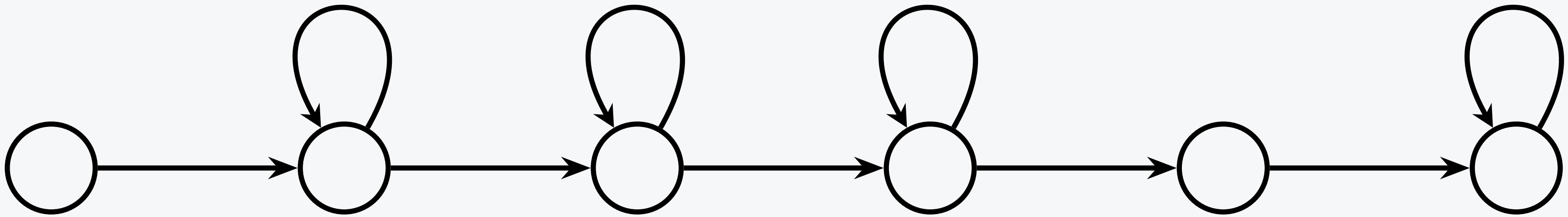

For $d \geq 3$, is reachability in unary $d$-dimensional linear path schemes in P?

[Englert, Lazić, and Totzke '16]

[Leroux '21]

# Main Contribution

**Theorem.** Reachability in unary 3-SLPS is NP-complete.

*Proof approach.* Recall that reachability in (binary encoded) flat VASS is in NP.

For NP-hardness, reduce from 3-SAT.

1) Use “Chinese remainder encoding” for SAT.
2) Encode satisfiability as a conjunction of non-divisibility assertions.
3) Design a 2-SLPS with zero tests for asserting non-divisibility.
4) Concatenate these 2-SLPSs with zero tests so that reachability coincides with the conjunction of non-divisibility assertions.
5) Use an additional third counter to simulate the zero tests.

# Encoding SAT as a Conjunction of Non-Divisibility Assertions

Chinese remainder encoding for SAT with $k$ variables $x_1, \ldots, x_k$ and $m$ clauses.

- Let $p_1, \ldots, p_k$ be the first $k$ primes.
- Let $n \in \mathbb{N}$ such that $n \equiv 0 \bmod p_i \iff x_i$ is false and $n \equiv 1 \bmod p_i \iff x_i$ is true.

There exists $\ell = \mathcal{O}(k^2 \cdot \log(k) + m)$ and $q_1, r_1, \ldots, q_\ell, r_\ell$ such that ...

... SAT holds if and only if $(n \equiv r_1 \bmod q_1) \wedge (n \equiv r_2 \bmod q_2) \wedge \cdots \wedge (n \equiv r_\ell \bmod q_\ell)$

if and only if $\quad q_1 \nmid (n - r_1) \quad \wedge \quad q_2 \nmid (n - r_2) \quad \wedge \cdots \wedge \quad q_\ell \nmid (n - r_\ell)$.

We were inspired by [Schöning '97]

Idea first appears in [Stockmeyer and Meyer '73]

**Takeaway:** SAT can be encoded into a (polynomial size) conjunction of non-divisibility assertions.

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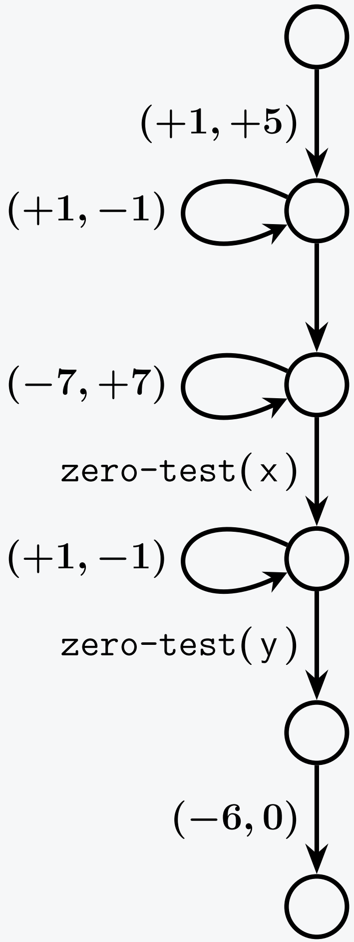
# Simple Linear Path Schemes Asserting Non-Divisibility

Suppose we want to assert $\mathbf{7} \nmid \boldsymbol{v}$.

Let's construct a 2-SLPS with zero tests that:

- starts with x $= \boldsymbol{v}$, y $= \mathbf{0}$,
- can only be passed if $\mathbf{7} \nmid \boldsymbol{v}$, and
- ends with x $= \boldsymbol{v}$, y $= \mathbf{0}$.



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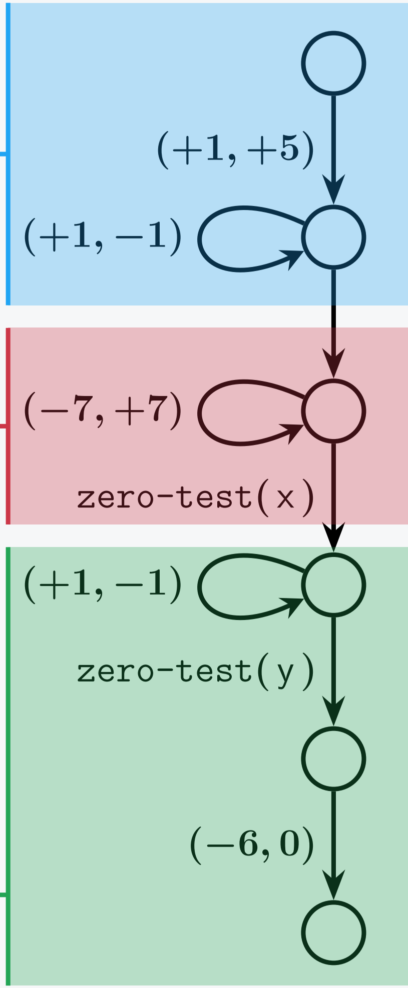
- can only be passed if $\mathbf{7} \nmid \boldsymbol{v}$, and
- ends with $\mathsf{x} = \boldsymbol{v}$, $\mathsf{y} = \mathbf{0}$.

**(i)** Choose $\boldsymbol{r} \in \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}\}$ ...

**(ii)** ... such that $\mathbf{7} \mid \boldsymbol{v} + \boldsymbol{r}$.

**(iii)** Restore $\mathsf{x} = \boldsymbol{v}$, $\mathsf{y} = \mathbf{0}$.



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**Ask me for details!**

# The Tractability Border of Reachability in Simple Vector Addition Systems with States

**Theorem.** Reachability in unary 3-SLPS is NP-complete.

**Theorem.** Reachability in unary *ultraflat* 4-VASS is NP-complete.

**Theorem.** Reachability in *unitary* inverse-Ackermann-dimensional SLPS is NP-complete.

**Theorem.** Reachability in unary 2-SLPS with *binary encoded initial and target configurations* is in P.

## Thank You!

FOCS 2024

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