Walks of Given Length in One-Counter Systems

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One-Counter Systems

Directed Graph

- Integer weighted edges
  Represented in binary

- Controls an integer counter
  Counter starts with zero

- Counter must remain non-negative
  "Counter condition"

Counter = 0
Walks of Given Length

Walks - a sequence of edges
(Valid if counter condition not violated)

Length - the number of edges
(Not the sum of the weights)

**INPUT:**
\( \mathcal{A} \) one-counter system, \( u \) start node, \( v \) end node, \( n \) walk length.
(given in binary)

**QUESTION:**
Does there exist a valid walk in \( \mathcal{A} \) from \( u \) to \( v \) of length \( n \)?
“YES” instance:
   a start node and d end node, and \( n = 42 \) walk length.

“NO” instance:
   a start node and f end node, and \( n = 42 \) walk length.

\( \text{NL-hard and in } \text{NP} \)
...but not known to be \( \text{NP-hard or in } \text{P} \)
Motivation

Petri Nets and Vector Addition Systems with States:
- Related to short paths in VASS, and thus short paths in Petri Nets.
- Generalises 1-VASS Coverability (known to be in $P$).

Compressed Algorithmics:
- Length compression applied to stack actions of pushdown automata with a unary stack.

Verification of safety conditions:
- Systems with control flow represented by a single integer variable.
Special Case: Linear Path Schemes

A sequence of non-overlapping cycles connected by a simple path

Conjecture: For linear path schemes, the given length walks decision problem is in $P$. 
Polynomial Time Algorithm Features

**Idea:** compute the set of all reachable points. *(NP-hard!)*

$(n, c)$ is a reachable point if there exists a valid walk of length $n$ ending with counter value $c$.

**New idea:** compute some useful subset of reachable points.

Points of interest: $c$ is the greatest counter value achievable for a walk of length $n$.

**Requirements:** careful representation of subset of reachable points.

Succinct: poly-sized subset output & Efficient: easy to query target walk lengths.

**Dynamic programming approach.**

Sequentially considers each cycle, then edge, then cycle, ...
Algorithm Behaviour by Example

Initialisation:
Only (0,0) is reachable.
Algorithm Behaviour by Example

After first edge:
Walk length increments, so (1,0) is reachable.
Algorithm Behaviour by Example

After first cycle:

\[
\begin{align*}
(1,0) \\
(5,4) \\
(9,8)
\end{align*}
\]

period + (4,4)
...

Counter Value

Walk Length
After second edge:

\[
\begin{align*}
(22,0) \\
(26,4) \\
(30,8)
\end{align*}
\]

\{ period still \(+(4,4)\) \}

...
Algorithm Behaviour by Example

After second cycle:
- First period $(+4, 4)$
- Second period $+(3, 6)$...better!
Extensions

*Case when several first cycles are negative is in progress*

Beyond linear path schemes

Series parallel based, directed acyclic graph based, arbitrary.

Beyond walks of a given length

Other decision problems, e.g.: “is there a walk of every length?"
Questions?

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