Max-plus automata and Size-change abstraction

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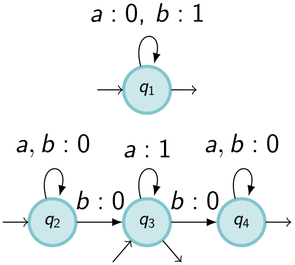
Joint work with Thomas Colcombet (LIAFA)
and Florian Zuleger (Vienna University)

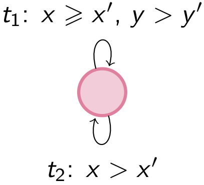
Journées ALGA, 30-31 mars 2015

Max-plus automata



- automata theory
- tropical algebra

Size-change abstraction



- verification
- program analysis

Max-plus automata

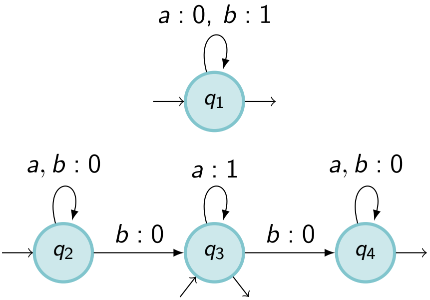
Syntax :

Non deterministic finite automaton for which each transition is labelled by a non negative integer (weight).

Semantic :

Weight of a run = sum of the weights of the transitions.

$$\begin{array}{lcl} \mathbb{A}^* & \rightarrow & \mathbb{N} \cup \{-\infty\} \\ w & \mapsto & \text{Maximum of the weights} \\ & & \text{of accepting runs} \\ & & \text{labelled by } w \\ & & (-\infty \text{ if no such run}) \end{array}$$



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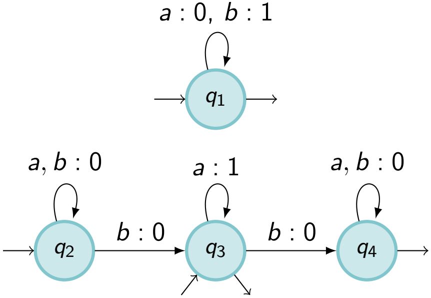

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$$a^{n_0} b a^{n_1} b \cdots b a^{n_k} \mapsto \max(n_0, n_1, \ldots, n_k, k)$$

Max-plus automata: Decidability

$$f, g : \mathbb{A}^* \to \mathbb{N} \cup \{-\infty\}$$

Decidable

Undecidable

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$\exists w,\ f(w) = k$?

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$$\forall w \in \mathbb{A}^*,\ f(w) \leqslant g(w)\ ?$$

[Krob '92]

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[Krob '92]

- $\forall w,\ |w| \leqslant f(w)$
- $\exists b,\ \forall w,\ |w| \leqslant f(w) + b$

Max-plus automata: main theorem

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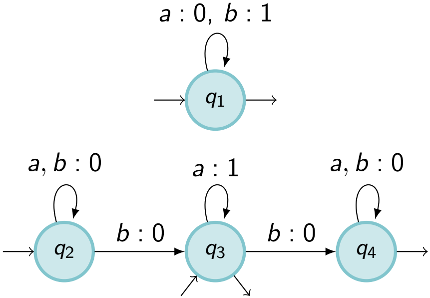
Theorem [Colcombet, D., Zuleger]

There is an algorithm such that, given a max-plus automaton, computes a rational $\alpha \geqslant 1$ such that:

$$g(n) = \Theta(n^{\alpha})$$

Max-plus automata: example



$$f(a^{n_0}ba^{n_1}b\cdots a^{n_{k-1}}ba^{n_k}) \\ = \max(n_0, n_1, \ldots, n_k, k)$$

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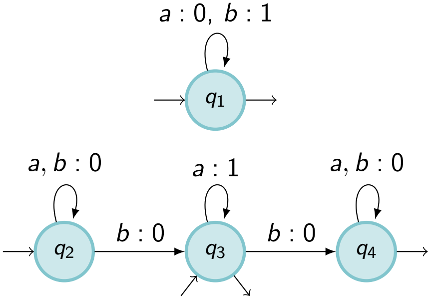
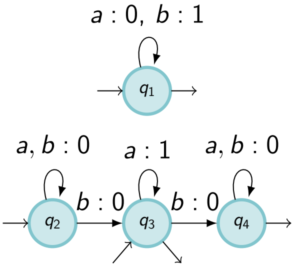


$$f(a^{n_0}ba^{n_1}b\cdots a^{n_{k-1}}ba^{n_k}) = \max(n_0, n_1, \ldots, n_k, k)$$

Longest word with weight at most n

$$f((a^n b)^n a^n) = n$$

Max-plus automata



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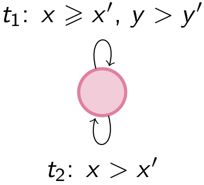

- verification
- program analysis

Size-change abstraction

```
Input x, y :
    while x>=0 {
        y--;
        if y=0 {
            x--;
            y=random();
        }
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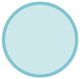

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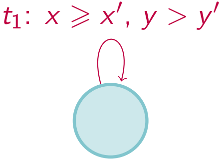

t_1: $x \geqslant x'$, $y > y'$

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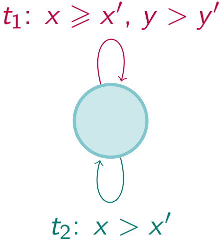

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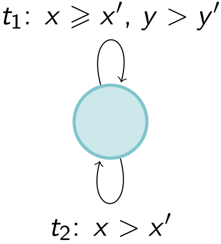
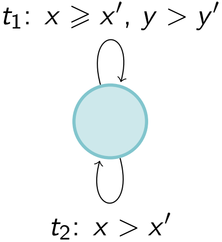
Size-change abstraction



- Finite number of variables (values in $\mathbb{N}$)
- Transition: conjunction of a finite number of predicates of the form $x_i > x'_j$ or $x_i \geqslant x'_j$
- Trace: sequence of transitions and valuations compatible

Size-change abstraction



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$$(5,5) \xrightarrow{t_1} (5,4) \xrightarrow{t_2} (4,8) \xrightarrow{t_1} (3,6) \xrightarrow{t_1} (3,2) \xrightarrow{t_2} (2,10)...$$

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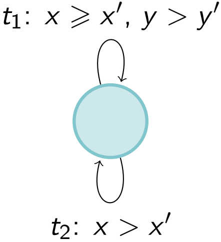

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Terminating sca: no infinite trace

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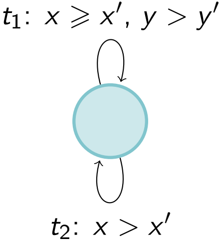

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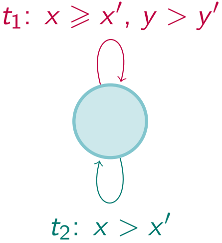
Terminating sca: no infinite trace

Theorem [Lee, Jones, Ben-Amram]

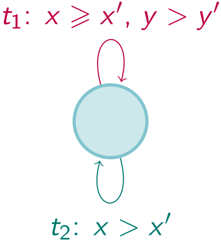
It is decidable whether a given sca instance is terminating.

Size-change abstraction: longest traces



- Terminating
- Traces of unbounded lengths

Size-change abstraction: longest traces



- Terminating
- Traces of unbounded lengths

Restriction to $[0, n]$, what is the length of the longest trace ?

$$\begin{array}{cccccccccc}
 & (n, n) & \xrightarrow{t_1} & (n, n-1) & \xrightarrow{t_1} & \cdots & \xrightarrow{t_1} & (n, 0) \\
\xrightarrow{t_2} & (n-1, n) & \xrightarrow{t_1} & (n-1, n-1) & \xrightarrow{t_1} & \cdots & \xrightarrow{t_1} & (n-1, 0) \\
 & & & \cdots & & & & \\
\xrightarrow{t_2} & (0, n) & \xrightarrow{t_1} & (0, n-1) & \xrightarrow{t_1} & \cdots & \xrightarrow{t_1} & (0, 0)
\end{array}$$

Size-change abstraction: main theorem

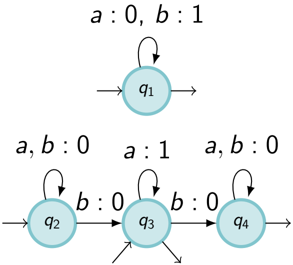
Theorem [Colcombet, D., Zuleger]

Given a terminating sca instance,
there is a computable rational $\alpha \geqslant 1$ such that

$$f = \Theta(n^{\alpha})$$

where f associates a positive integer n to the length of the longest traces if the variables are restricted to be in $[0, n]$.

Max-plus automata



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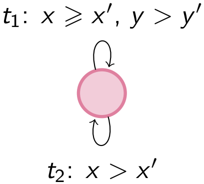

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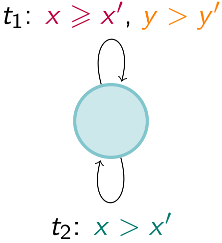
From sca to max-plus automata



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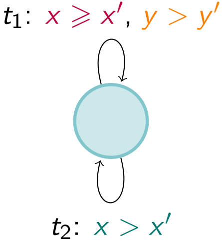

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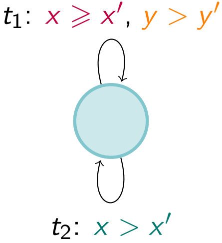

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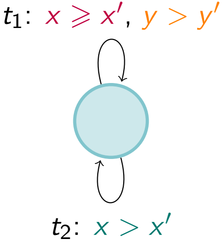

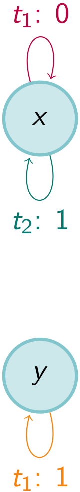

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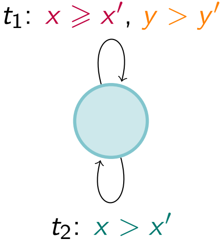

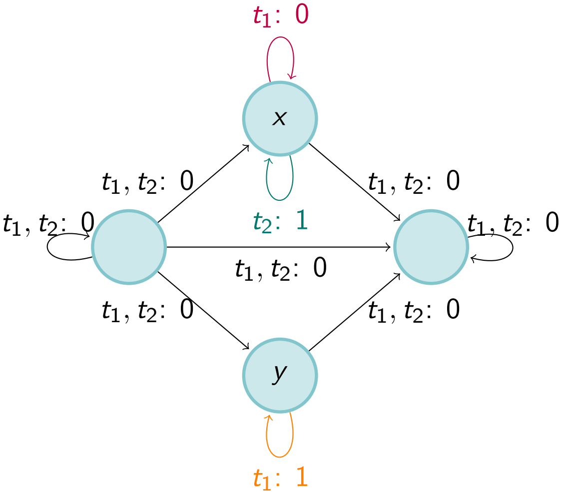

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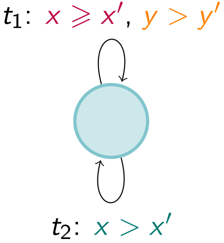

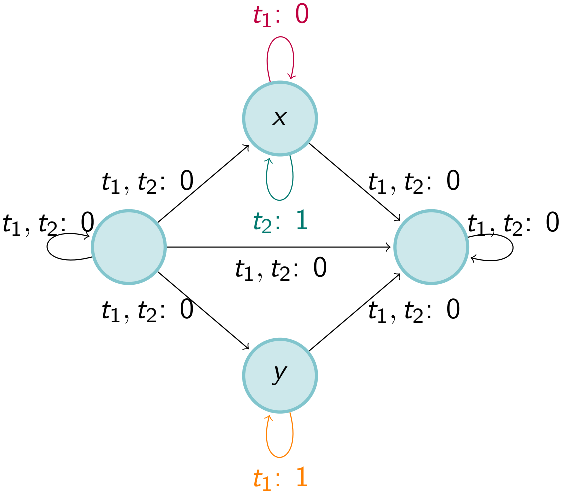

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y > y' & y > y' & & & & & & \\
 & & & y > y' & y > y' & & & \\
 & & & & & & y > y' & y > y'
\end{array}$$

Longest trace when variables do not exceed n
$\longrightarrow$ Sequences of inequalities with at most n strict inequalities
$\longrightarrow$ Runs of weight at most n
Longest word of weight at most n

Conclusion and further questions

- More precise description of functions computed by max-plus automata
- Min-plus automata
- Complexity EXPSPACE, PSPACE ongoing work
- Other applications ?