Asymptotic Behaviour of Max-Plus and Min-Plus Automata

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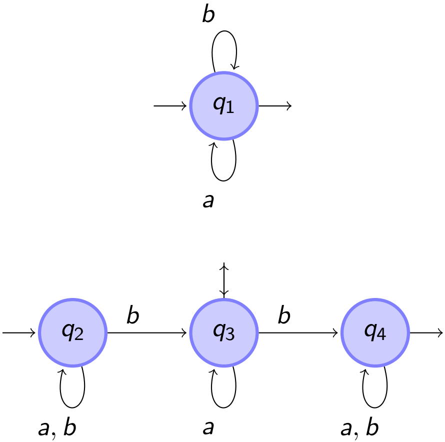
based on joint works with Thomas Colcombet (LIAFA)
and Florian Zuleger (TU Vienna)

FREC 2014

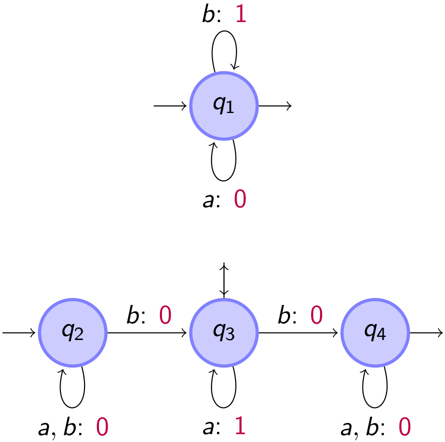
Max-Plus and Min-plus Automata

From NDFA to Min-Plus and Max-Plus Automata



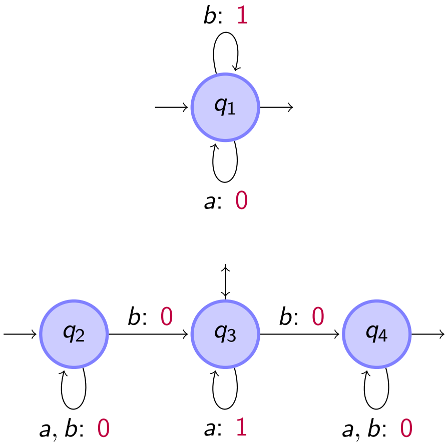
Non deterministic finite automata : $A^* \to \{0, \infty\}$

From NDFA to Min-Plus and Max-Plus Automata



Min-plus automata : $A^* \to \mathbb{N} \cup \{+\infty\}$
Max-plus automata : $A^* \to \mathbb{N} \cup \{-\infty\}$

Min-Plus and Max-Plus Automata



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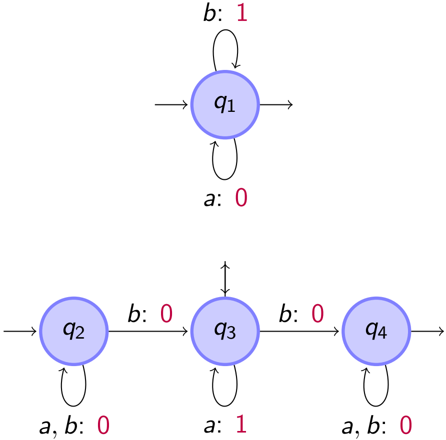

Weight of a run:
sum of the weights of the transitions

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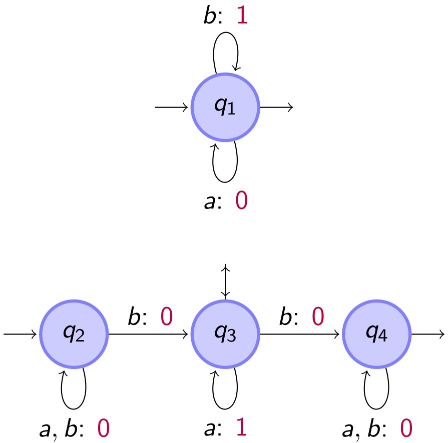
Min-Plus and Max-Plus Automata



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$w \mapsto$ minimum of the weights of the accepting runs labelled by w

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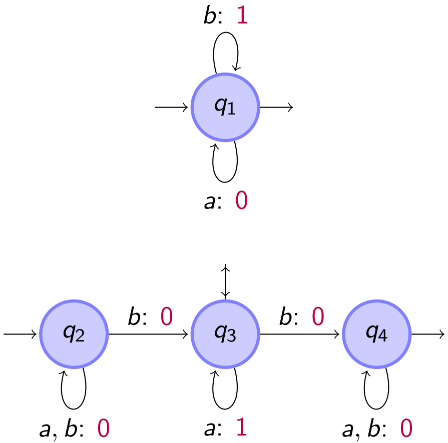

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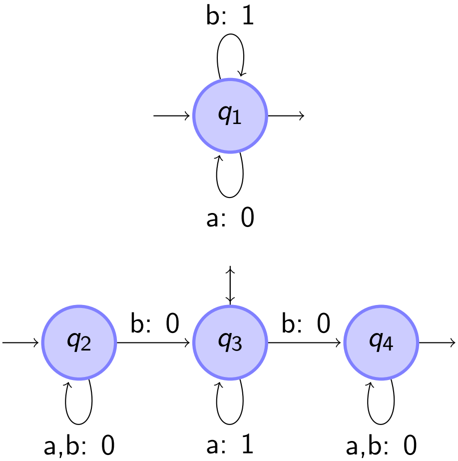
An example



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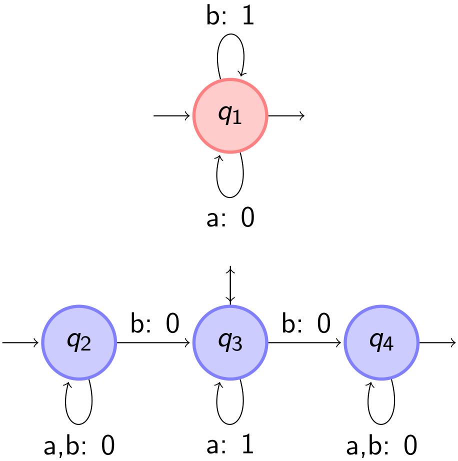

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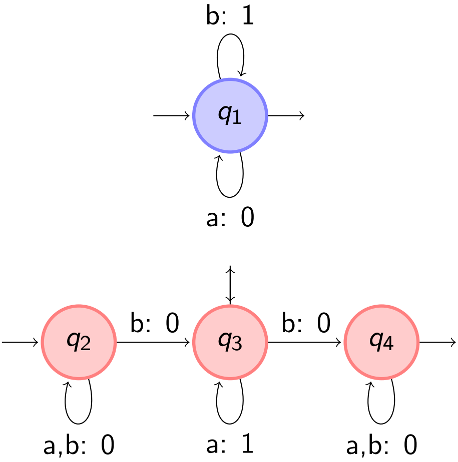

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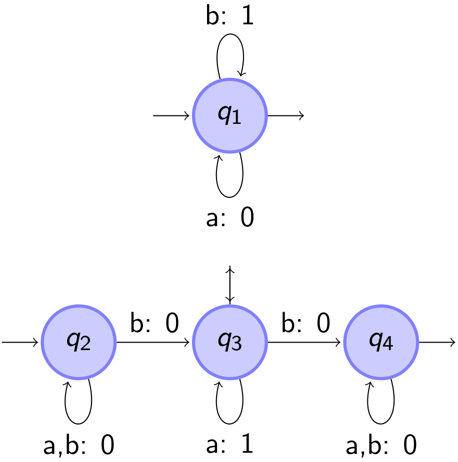
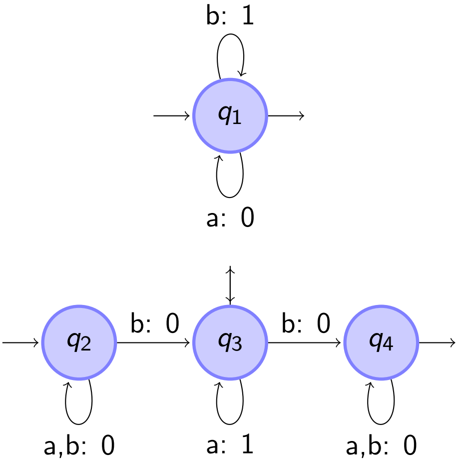


Min-plus: $a^{n_0} b a^{n_1} b \cdots b a^{n_k} \mapsto \min(n_0, n_1, \ldots, n_k, k)$

An example



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Max-plus: $a^{n_0}ba^{n_1}b\cdots ba^{n_k} \mapsto \max(n_0, n_1, \ldots, n_k, k)$

Description of the behaviour of computed functions

Asymptotic behaviour

Undecidable: Given f, g computed by min-plus (resp. max-plus) automata, is $f(w) \leqslant g(w)$ for all words w ? [Krob, 92]

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$$h(n) = \sup_{|w| \leqslant n} f(w)$$

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Max-Plus

Undecidable:
for all w, $|w| \leqslant f(w)$?

$$h(n) = \inf_{|w| \geqslant n} f(w)$$

for all n, $n \leqslant h(n)$?

Example

Min-Plus

$$a^{n_0} b a^{n_1} b \cdots b a^{n_k} \mapsto \min(n_0, n_1, \ldots, n_k, k)$$

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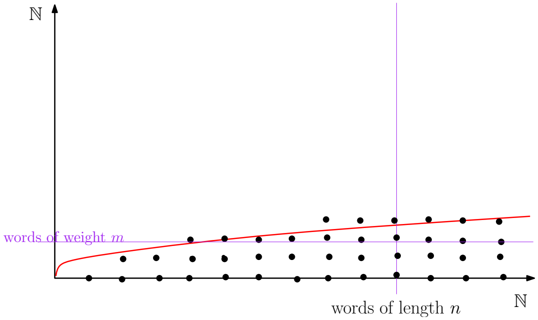


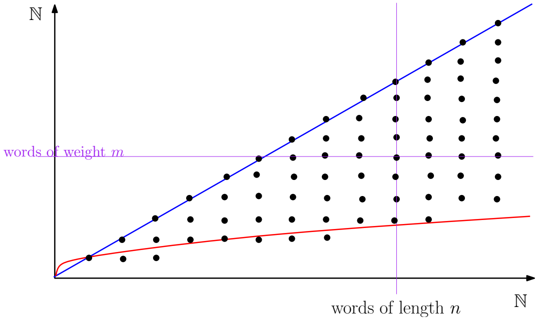
Example

Max-Plus

$$a^{n_0} b a^{n_1} b \cdots b a^{n_k} \mapsto \max(n_0, n_1, \ldots, n_k, k)$$

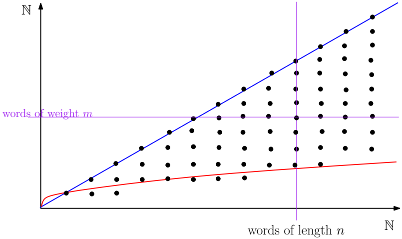
$$h(n) = \inf_{|w| \geqslant n} f(w)$$

What is the minimal weight for words of length at least n ?



Asymptotic equivalent

Max-Plus: $h(n) = \inf_{|w| \geqslant n} f(w)$



Theorem: There exists effectively $\alpha \in (\mathbb{Q} \cap [0,1]) \cup \{-\infty\}$ such that:

$$h(n) = \Theta(n^{\alpha})$$

Asymptotic equivalent

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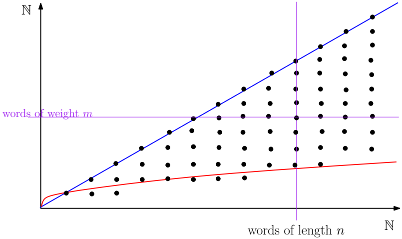

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Min-Plus: $h(n) = \sup_{|w| \leqslant n} f(w)$

$h_1 \leqslant h \leqslant h_2$ with $h_1(n) = O(n^{\frac{1}{p+1}})$, $h_2 = O(n^{\frac{1}{p}})$ for some integer p [Simon].

Ratio function-length

Min-Plus

$$h(n) = \sup_{|w| \leqslant n} f(w)$$

$$r = \sup_{n} \frac{h(n)}{n}$$

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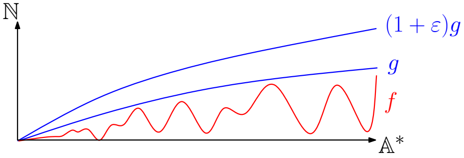
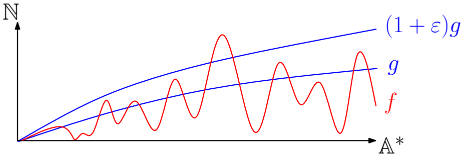
Theorem: There is an algorithm that, given a min-plus automaton (resp. a max-plus automaton), and $\varepsilon > 0$, computes r up to ε.

Approximate comparison

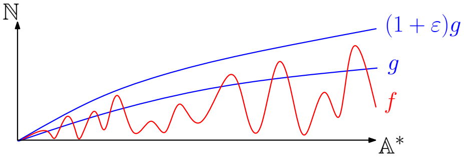
f, g computed by min-plus automata - Case $\varepsilon = 0$: undecidable result of Krob.



$\Longrightarrow$ YES



$\Longrightarrow$ NO

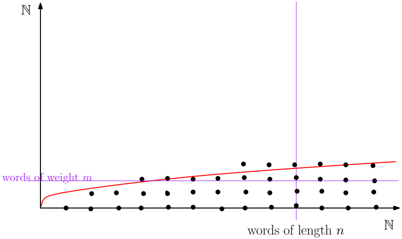


$\Longrightarrow$ YES or NO

Conclusion and further questions

Min-plus

$$h(n) = \sup_{|w| \leqslant n} f(w)$$



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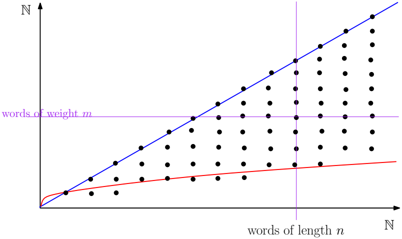

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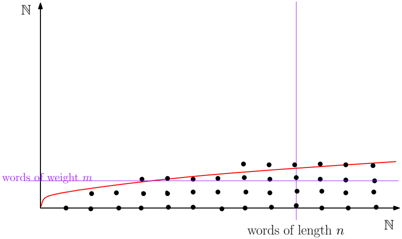

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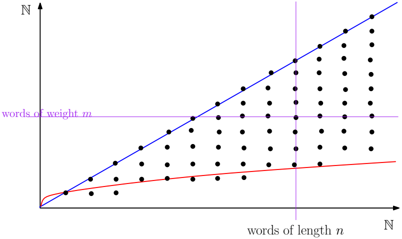

Approximation of $\sup_n \frac{h(n)}{n}$	Approximation of $\inf_n \frac{h(n)}{n}$

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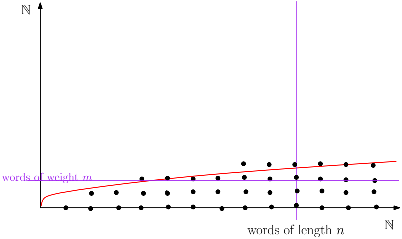

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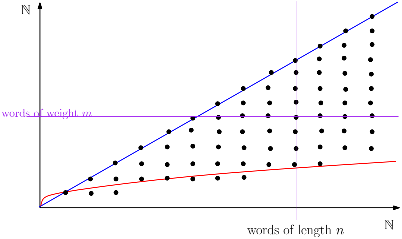

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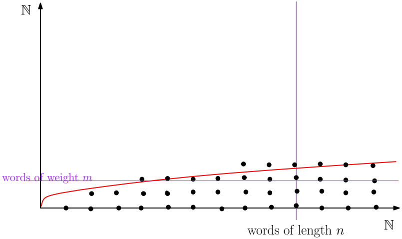

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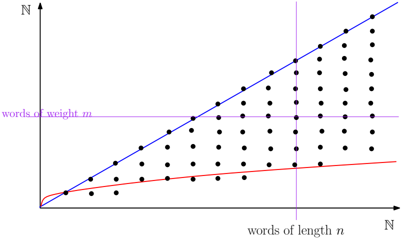

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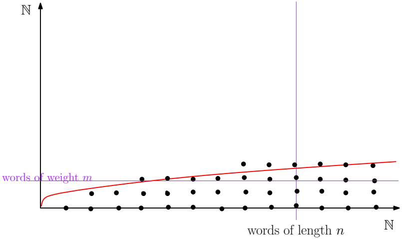

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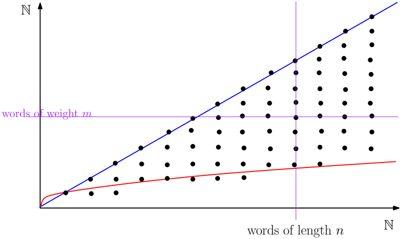

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Describe h as cn^α with α rationnal and c up to ε