Weighted automata and Identities

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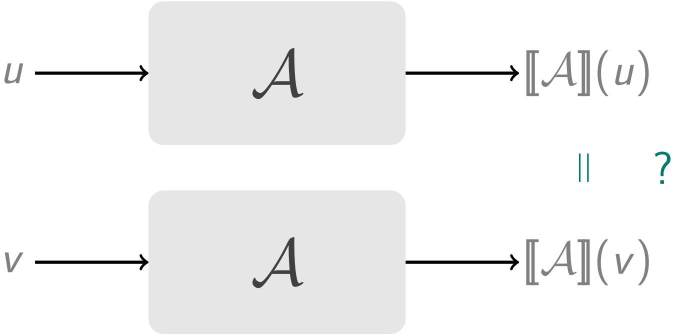
A natural and fundamental question:



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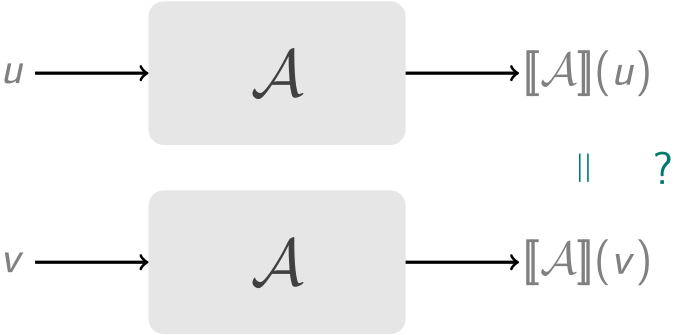

Which pairs of inputs can be distinguished
by a given computational model?

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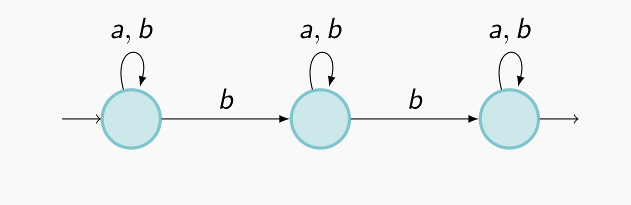
3. Minimal size to distinguish two given input words?

Boolean Automata

$[\![\mathcal{A}]\!] : A^* \to \{Acc, Rej\}$



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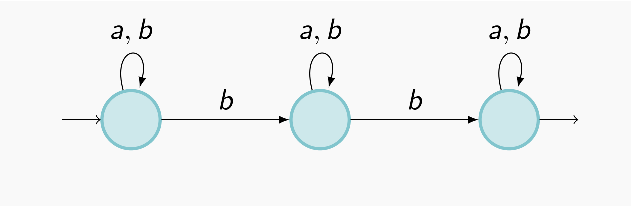

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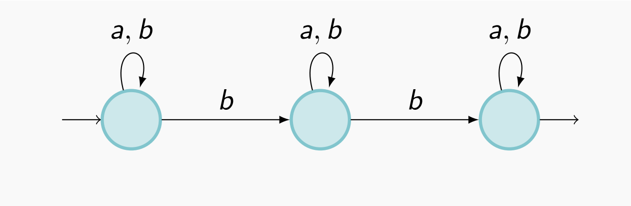

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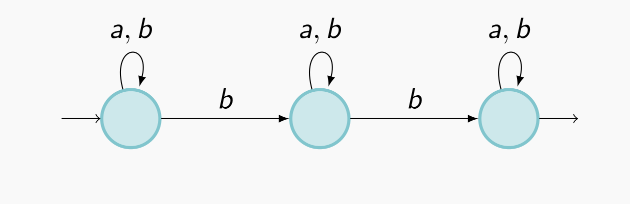

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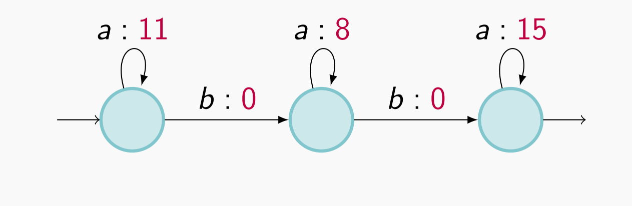
3. Minimal size to distinguish two given input words?
   $\to$ Profinite theory...

Weighted automata *[Schützenberger]*

$[\![\mathcal{A}]\!] : A^* \to S$



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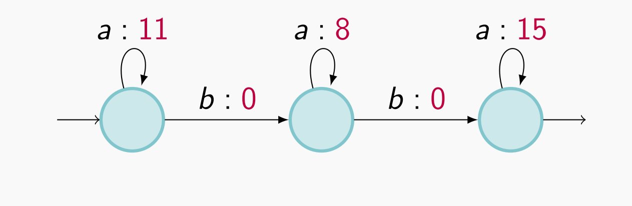

Semiring $(S, \oplus, \otimes)$: transitions are weighted by elements of S

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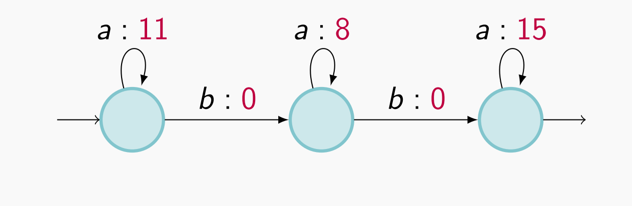

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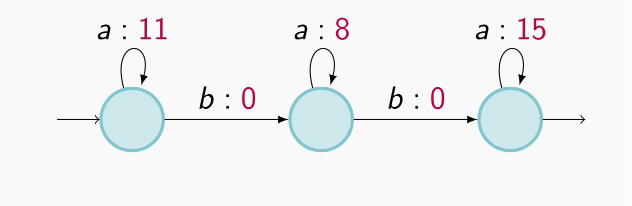

Semiring $(S, \oplus, \otimes)$: transitions are weighted by elements of S

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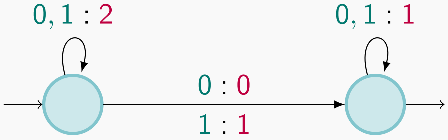
$$[\![\mathcal{A}]\!] : w \mapsto \bigoplus_{\substack{\rho \text{ accepting path} \\ \text{labelled by } w}} \left(\rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_{|w|}\right)$$

Automata weighted over $(\mathbb{R}, +, \times)$

$[\![\mathcal{A}]\!] : A^* \to \mathbb{R}$

An example with $A = \{0, 1\}$



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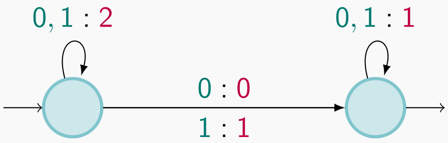

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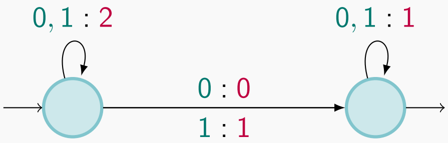

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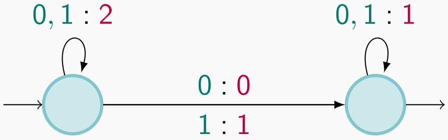

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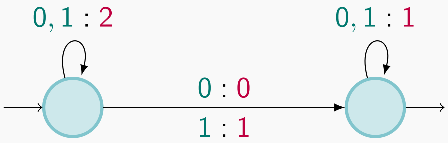

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3 Minimal size to distinguish two given input words?
$\to$ 1 or 2 states

Max-plus automata

Semiring $(\mathbb{N} \cup \{-\infty\}, \max, +)$

$[\![\mathcal{A}]\!] : A^* \to \mathbb{N} \cup \{-\infty\}$

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   $\to$ No

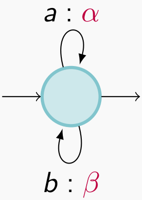
3. Minimal size to distinguish two given input words?
   $\to$??????

Given a positive integer n,
are there $u \neq v$ such that
for all max-plus automata $\mathcal{A}$ with at most n states:

$$[\![\mathcal{A}]\!](u) = [\![\mathcal{A}]\!](v) \quad ?$$

If $n = 1$

$A = \{a, b\}$



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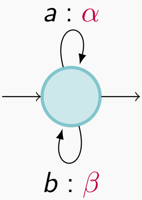

$w \mapsto \alpha|w|_a + \beta|w|_b$

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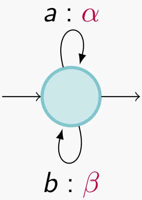

$w \mapsto \alpha|w|_a + \beta|w|_b$

Max-plus automata with one state can distinguish words with different contents (in particular different lengths), and only these ones.

If $n = 2$ or $n = 3$

There exist pairs of distinct words with the same values for all automata with at most 3 states...

But we do not know much more.

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2 states [Izhakian, Margolis] - words of length 20

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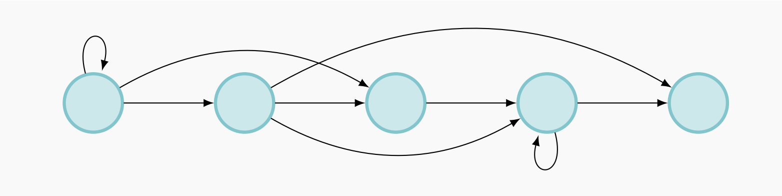
2 states [Izhakian, Margolis] - words of length 20

3 states [Shitov] - words of length 1795308

Triangular automata



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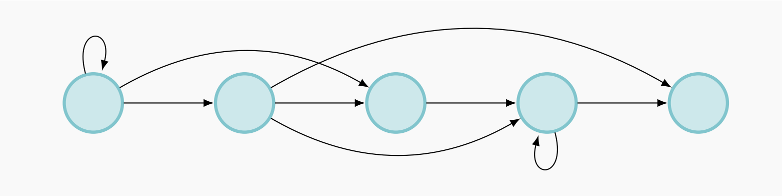


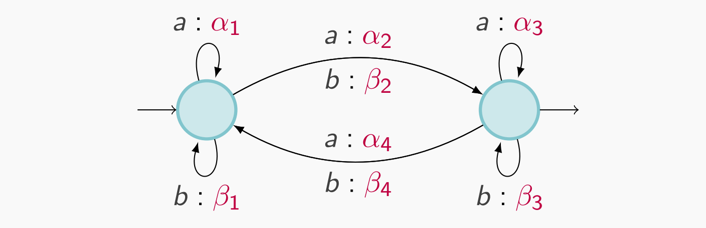
Theorem [Izhakian]

For all n, there exist a pair of distinct words $u \neq v$ such that for all triangular automata $\mathcal{A}$ with at most n states,

$$[\![\mathcal{A}]\!](u) = [\![\mathcal{A}]\!](v)$$

Let's go back to automata with 2 states

$A = \{a, b\}$



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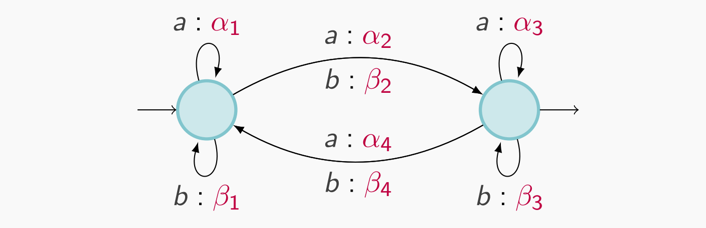

First attempt: Restrict the class of automata we have to consider

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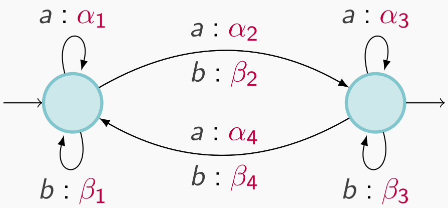

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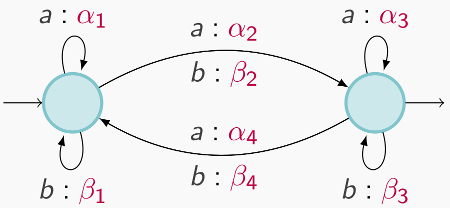

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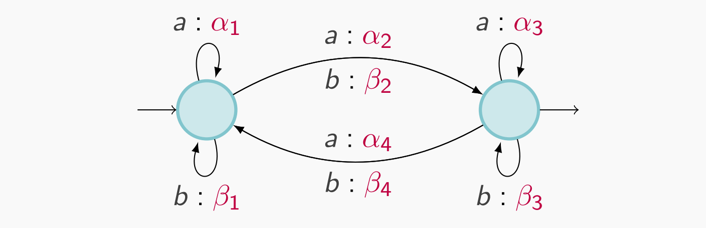

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- Only one initial and one final states

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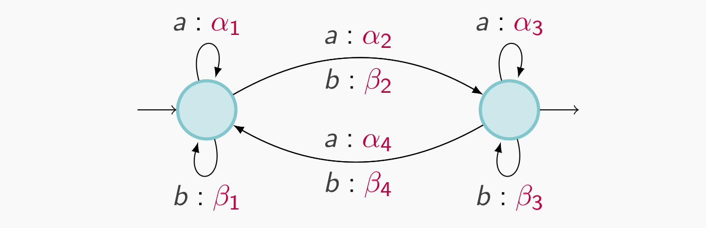

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- $\mathbb{R} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{N}$
- Complete automaton
- Only one initial and one final states
- Reduce the number of parameters

Let's go back to automata with 2 states

Second attempt: Give a list of criteria which can be checked

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Theorem [D., Johnson] - counter-example to a conjecture of Izhakian

There are two pairs of distinct words of minimal length which cannot be distinguished by any max-plus automata with two states:

$$a^2b^3a^3babab^3a^2 = a^2b^3ababa^3b^3a^2 \text{ and } ab^3a^4baba^2b^3a = ab^3a^2baba^4b^3a$$

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$\longrightarrow$ Eliminate the shortest pairs by using the list of criteria
$\longrightarrow$ Checking the pairs directly using the restrictions

A closer look at the list of criteria

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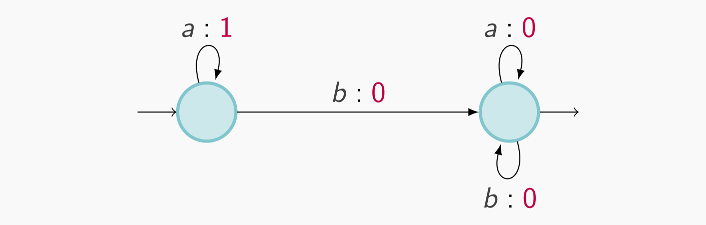
- First and last blocks



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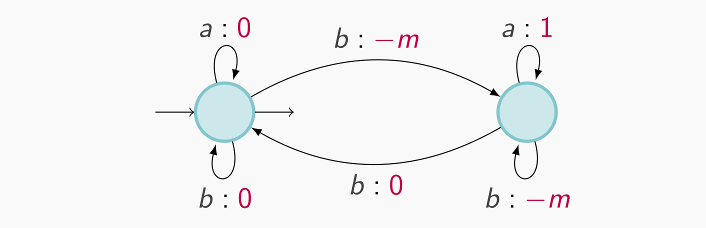

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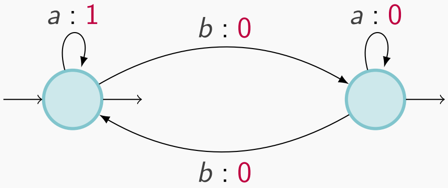
- Bloc-permutation
- "Counting modulo 2" criteria

Number of a's after an even number of b's



A closer look at the list of criteria

- First and last blocks
- Bloc-permutation
- "Counting modulo 2" criteria
- Triangular automata with two states

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Characterize all the identities holding for the class of max-plus automata with at most n states, for all n...

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Characterize all the identities holding for the class of max-plus automata with at most n states, for all n...

- Is there a strict subset of max-plus automata containing all their computational power?
- Link with decidability/undecidability of the equivalence problem?