# Asymptotic Behaviour of Max-Plus Automata and Size-Change Abstraction

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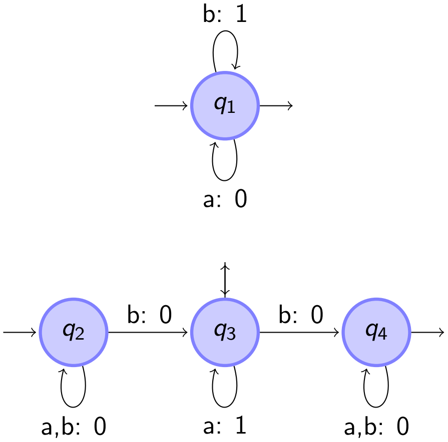
joint work with Thomas Colcombet and Florian Zuleger

Highlights of Logic, Games and Automata 2014

LIAFA

# Max-Plus Automata

Non deterministic finite automaton for which each transition is also labelled by a non-negative integer (= weight of the transition).



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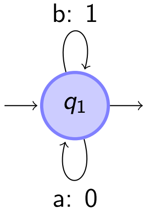
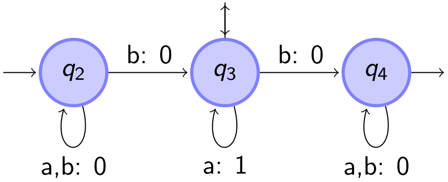



**Weight of a run**

Sum of the weights of the transitions.



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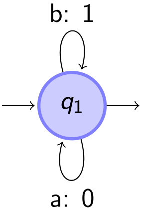


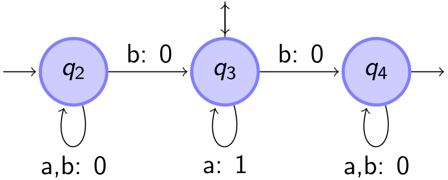


**Weight of a run**

Sum of the weights of the transitions.

**Computed function**

$$\begin{array}{ccl} \mathbb{A}^* & \to & \mathbb{N} \cup \{-\infty\} \\ w & \mapsto & \text{maximum of the weights of} \\ & & \text{the runs labelled by } w \text{ going} \\ & & \text{from an initial state} \\ & & \text{to a final state} \\ & & (-\infty \text{ if no such run}) \end{array}$$

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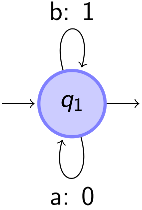


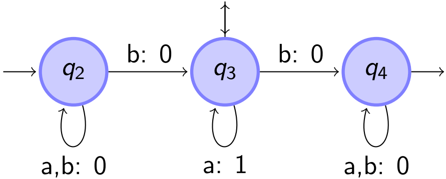


**Weight of a run**

Sum of the weights of the transitions.

**Computed function**

$\mathbb{A}^* \to \mathbb{N} \cup \{-\infty\}$

$w \mapsto$ maximum of the weights of the runs labelled by $w$ going from an initial state to a final state ($-\infty$ if no such run)

$$a^{n_0} b a^{n_1} b \cdots b a^{n_k} \mapsto \max(n_0, n_1, \ldots, n_k, k)$$

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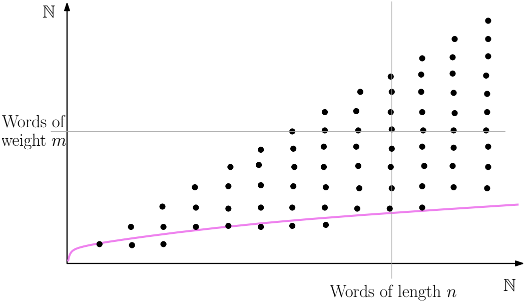

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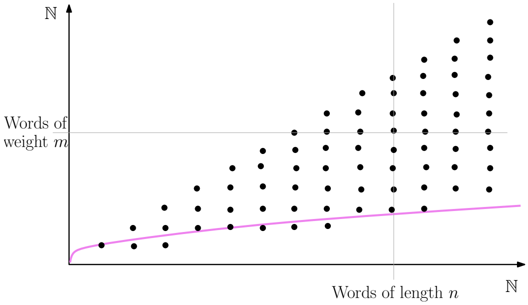
**Theorem [Krob]**

Undecidable:
for all $n$, $f_{\min}(n) \leqslant n$.



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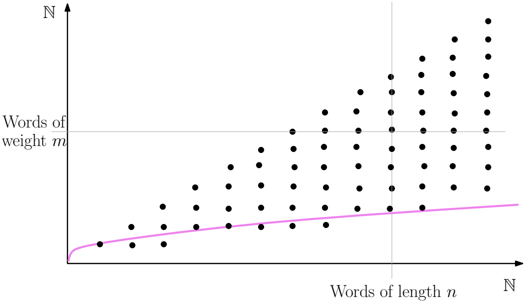


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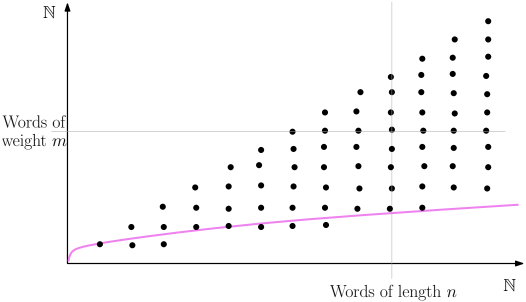


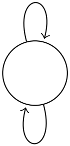

**Theorem**

There exists effectively $\alpha \in (\mathbb{Q} \cap [0,1]) \cup \{-\infty\}$ such that $f_{\min}(n) = \Theta(n^{\alpha})$.

Length of the longest word having value at most $n$: $\Theta(n^{1/\alpha})$.

# Size-Change Abstraction

$t_1$: $x \geqslant x'$, $y > y'$



$t_2$: $x > x'$

Variables: $x$ and $y$

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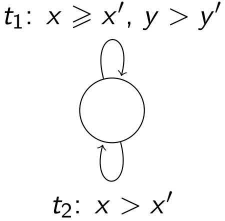


Variables: $x$ and $y$

$(5,5) \xrightarrow{t_1} (5,4) \xrightarrow{t_2} (4,8) \xrightarrow{t_1} (3,6) \xrightarrow{t_1} (3,2) \xrightarrow{t_2} (2,10)\ldots$

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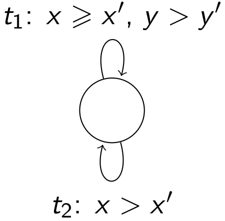


Variables: $x$ and $y$

Terminating SCA instance:
no infinite trace.

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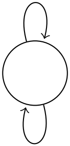

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**Theorem [Lee, Jones, Ben-Amram]**

It is decidable whether a given SCA instance is terminating.

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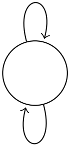

$t_2$: $x > x'$

Variables: $x$ and $y$

Terminating SCA instance: no infinite trace.

Restriction to $[0, n]$:

**Theorem**

Given a terminating SCA instance, there is $\beta \geqslant 1$, rationnal, computable such that the longest trace is of order $\Theta(n^\beta)$.

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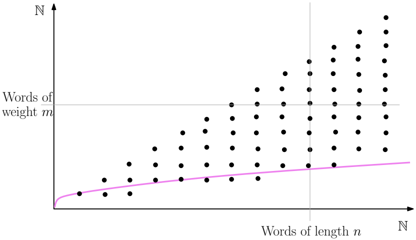
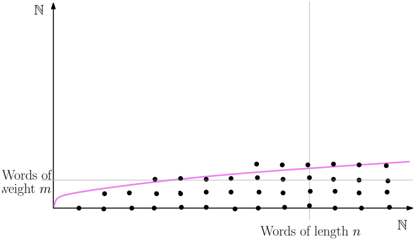
# Conclusion and further questions

- What about min-plus automata?




- Complexity ?