First order and regular list functions

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University of Warwick

Joint work with Mikołaj Bojańczyk and Krishna S.

LCS seminar, Lyon, 26-04-2018

Transduction: strings to strings function

$$A^* \rightarrow B^*$$

Transduction: strings to strings function

$$A^* \to B^*$$

Two classes:

- Regular transductions
  MSOT = Det two-way transducers = copyless-SST

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$\to$ Another characterisation of these classes.

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$\to$ Another characterisation of these classes.

Why ?... see later

Our running example

$$\{a, b\}^* \quad \to \quad \{a, b\}^*$$

$$w \quad \mapsto \quad w\overline{w}a^{|w|_a}$$

number of a in w

$\overline{w}$: mirror of w

$$abaabbbab \mapsto abaabbbabbabbbaabaaaaa$$

Deterministic Two-way Transducers [Engelfriet-Hoogeboom]

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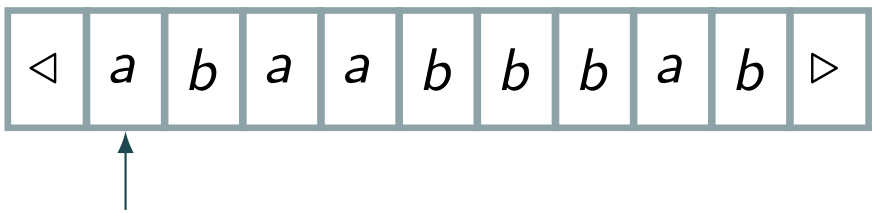

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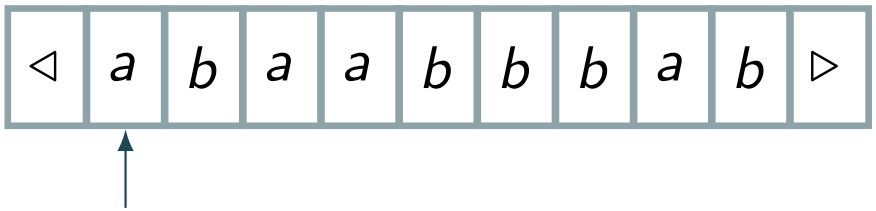

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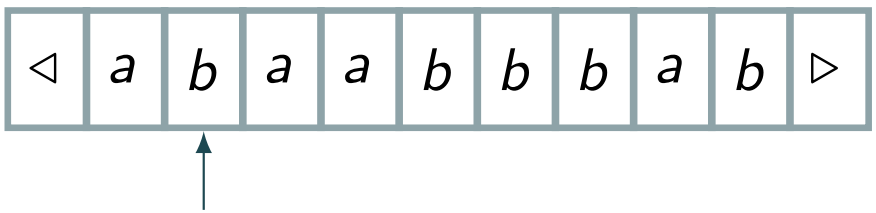

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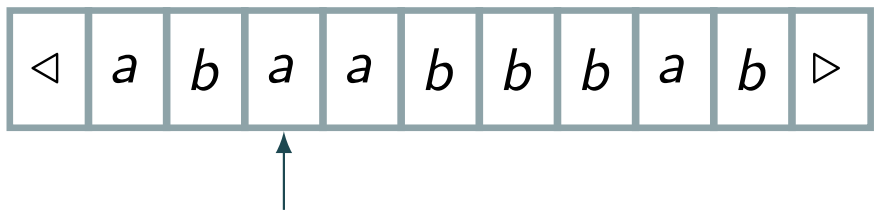

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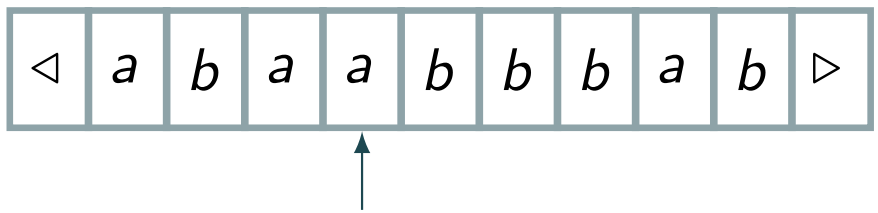

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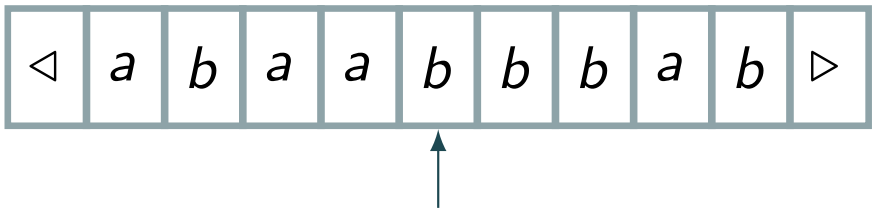

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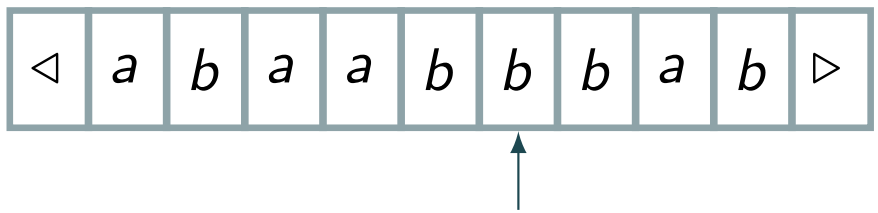

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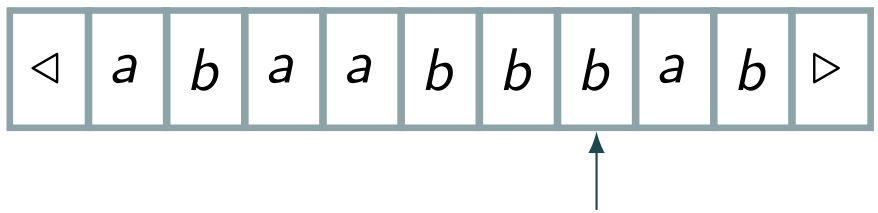

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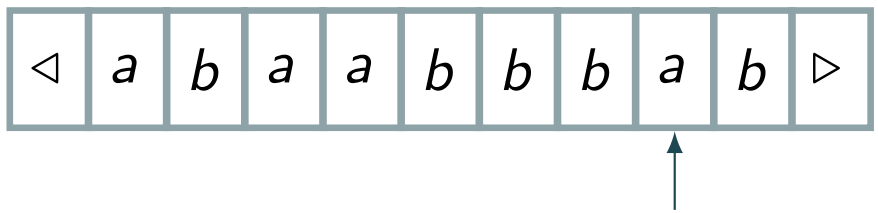

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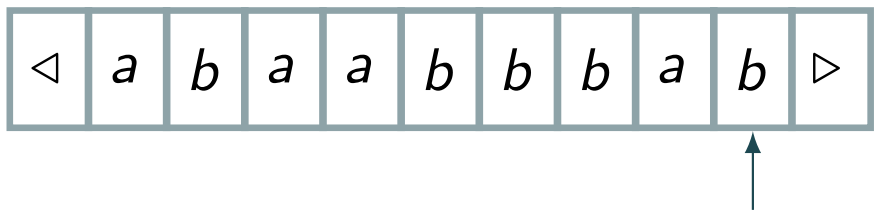

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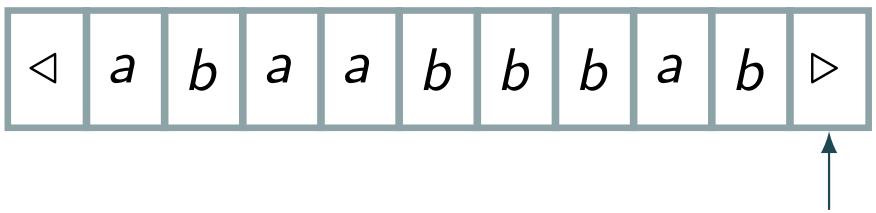

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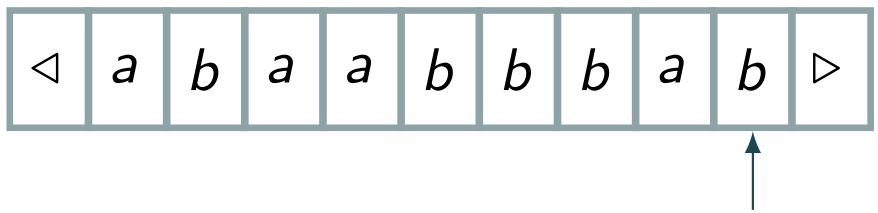

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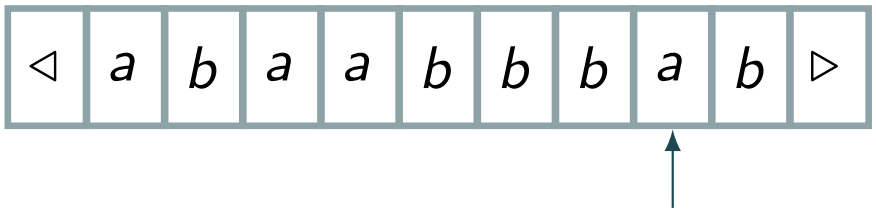

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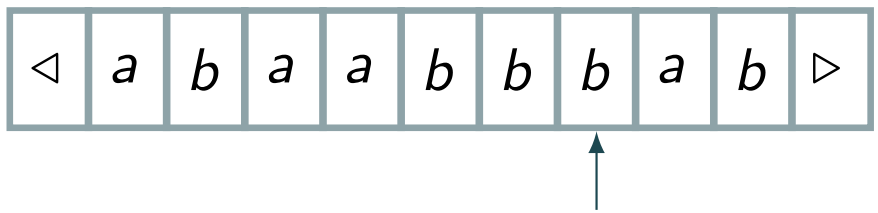

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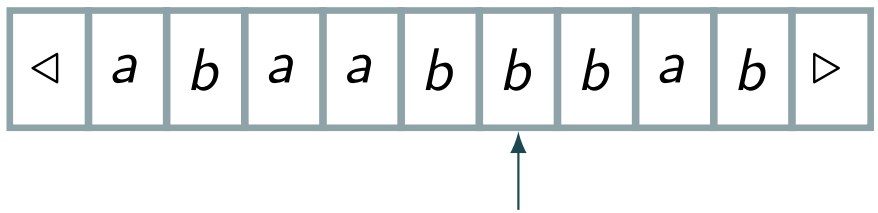

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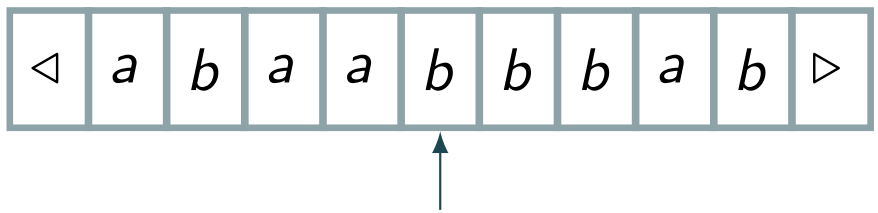

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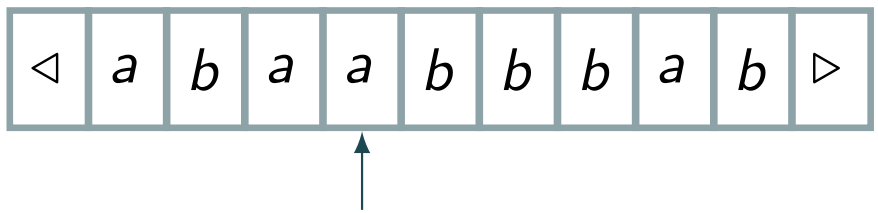

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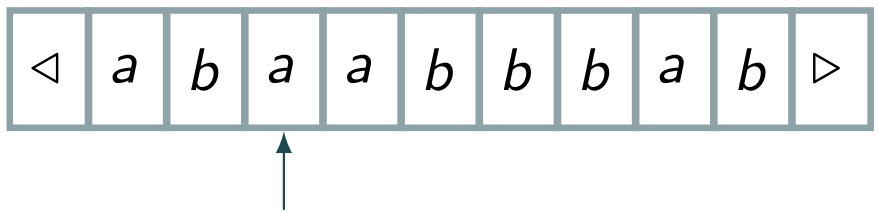

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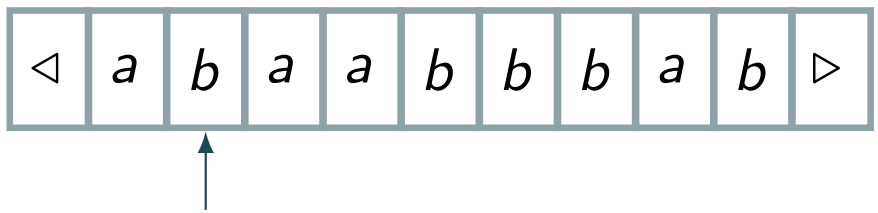

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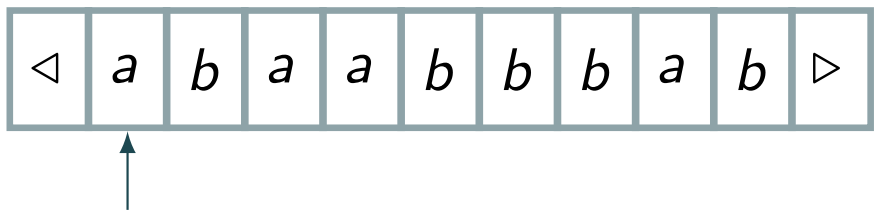

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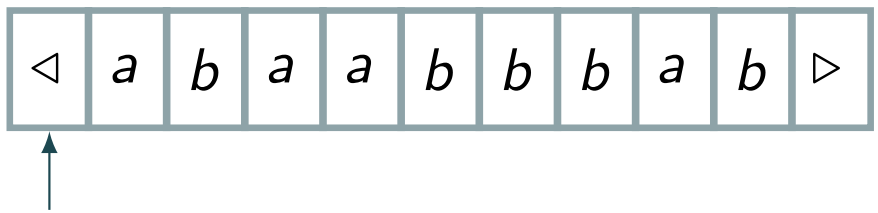

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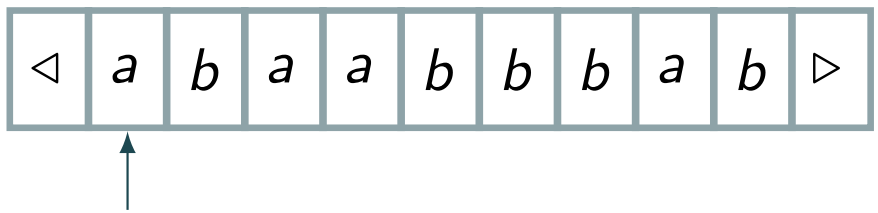

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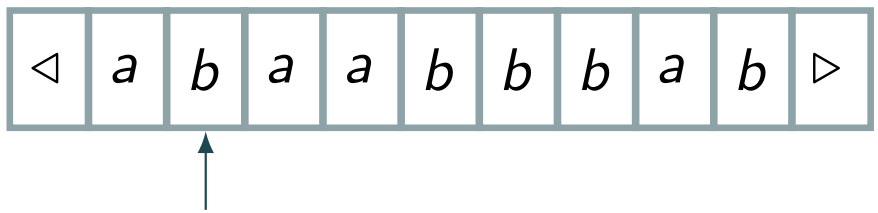

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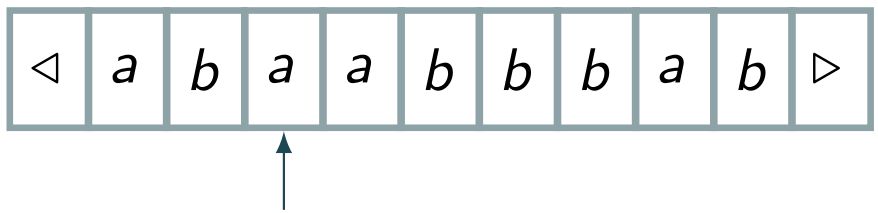

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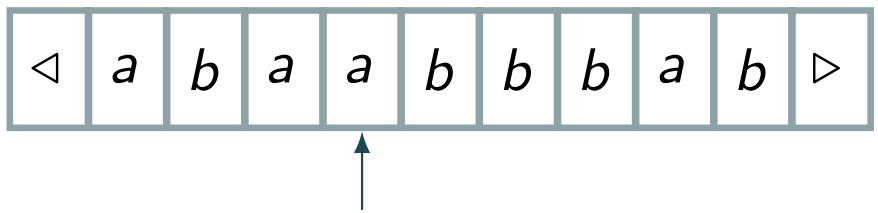

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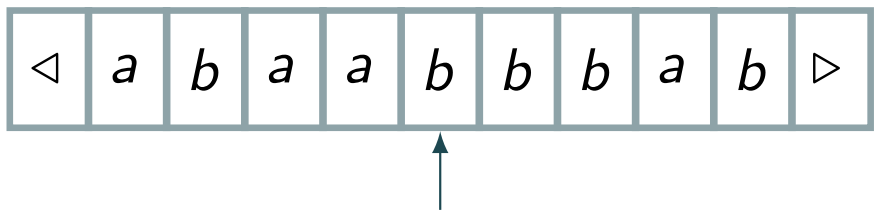

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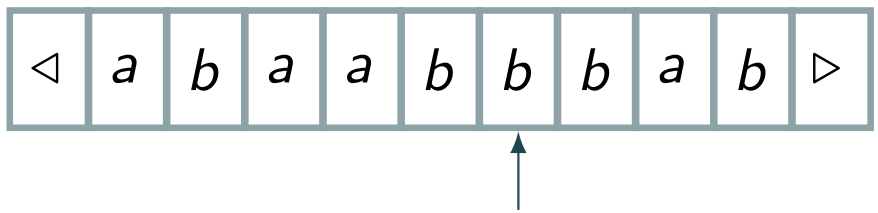

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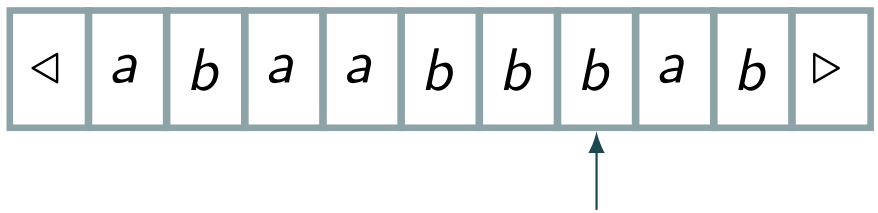

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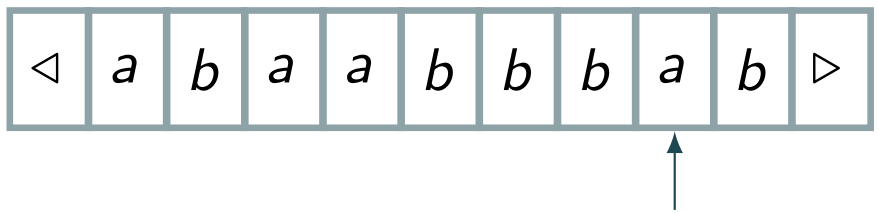

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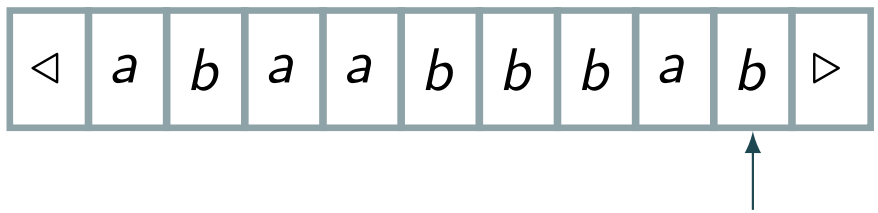

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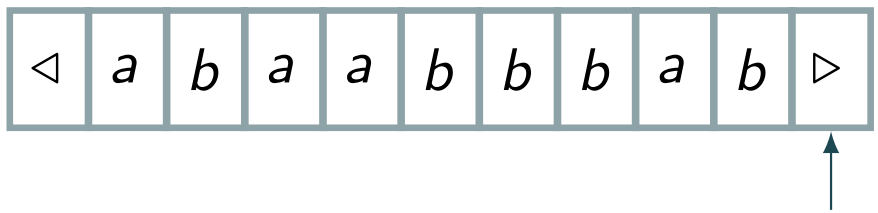

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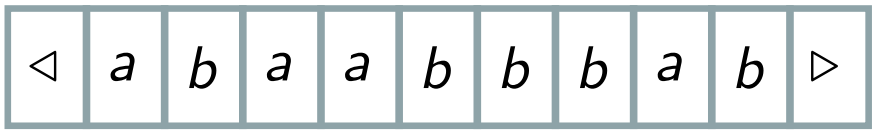

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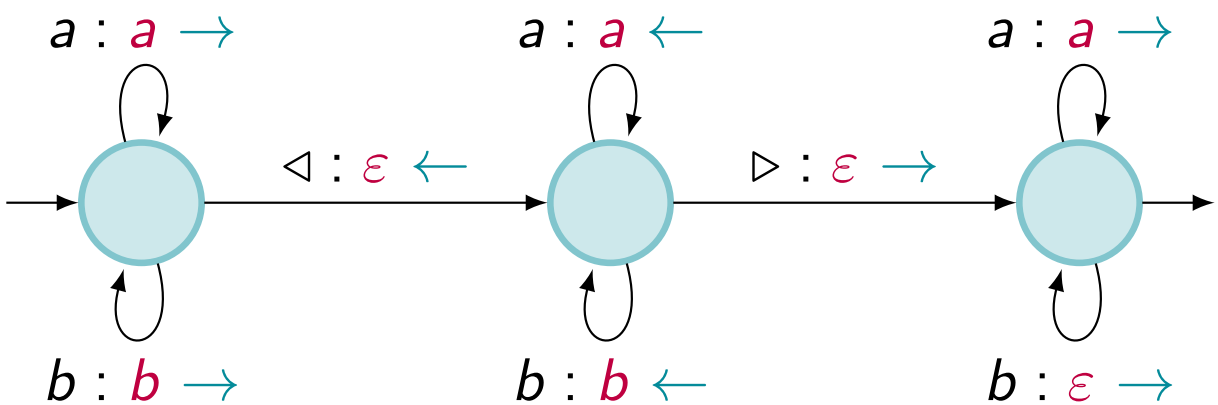

Copyless Streaming String Transducers [Alur-Cerny]

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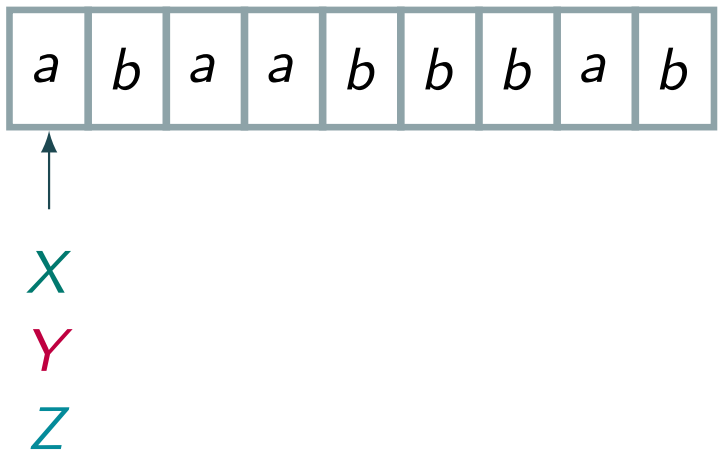

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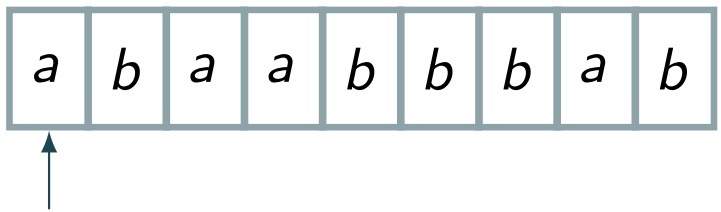

$X := a$

$Y := \qquad\qquad\qquad a$

$Z := a$

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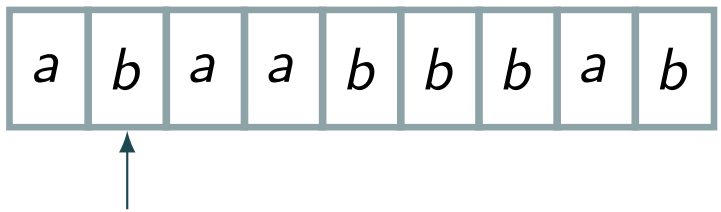

$X := a\ b$

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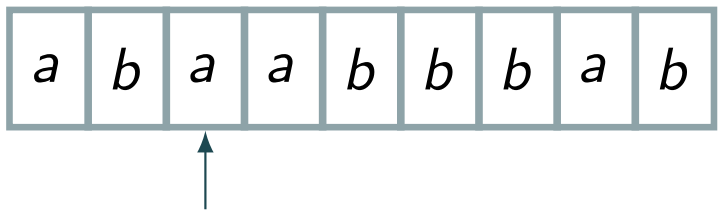

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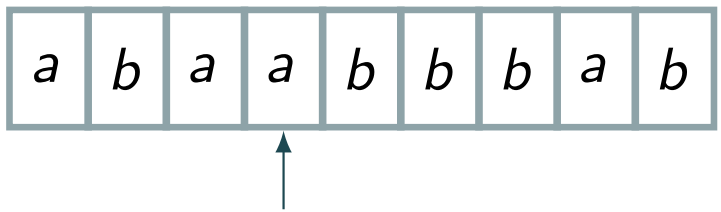

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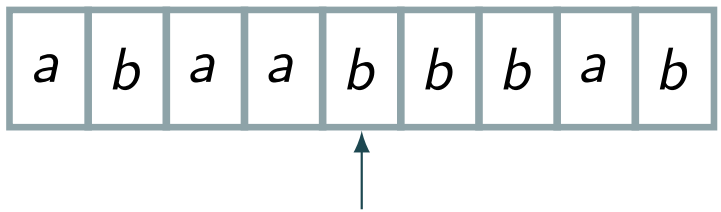

$X := a\ b\ a\ a\ b$

$Y := \qquad\qquad b\ a\ a\ b\ a$

$Z := a\ a\ a$

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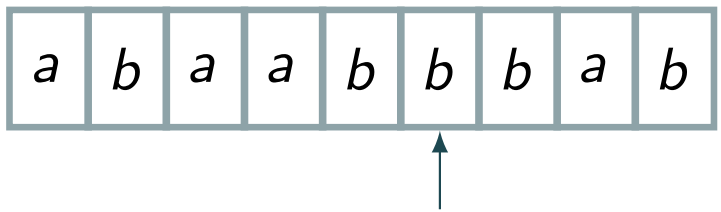

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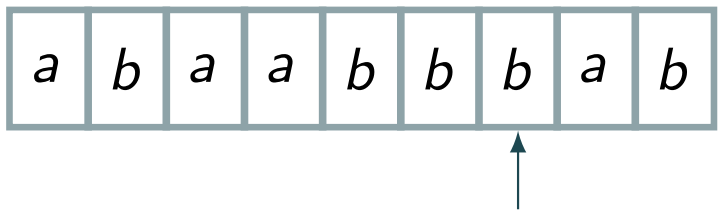

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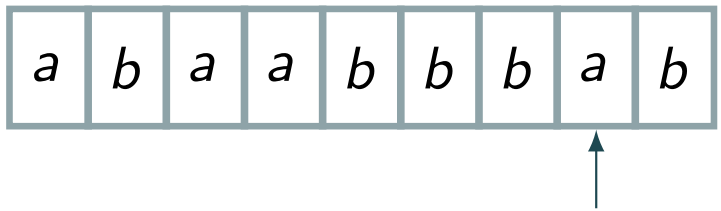

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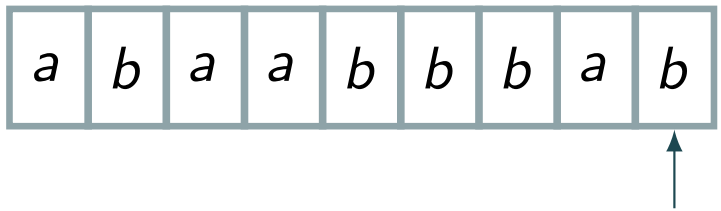

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return XYZ

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a	b	a	a	b	b	b	a	b

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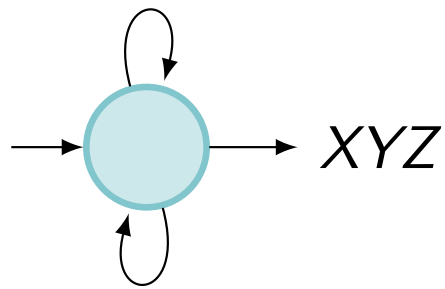
return XYZ

$a : X := Xa \quad Y := aY \quad Z := Za$



$b : X := Xb \quad Y := bY \quad Z := Z$

Copyless:

$\begin{cases} X := XY \\ Y := ZX \end{cases}$

MSO-transductions [Engelfriet-Hoogeboom]

$$w \quad \mapsto \quad w\overline{w}a^{|w|_a}$$

logical structure on words
$<, P_a, P_b$

$$a \longrightarrow b \longrightarrow a \longrightarrow a \longrightarrow b \longrightarrow b \longrightarrow b \longrightarrow a \longrightarrow b$$

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- Select positions in the output with MSO-formulas

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FO

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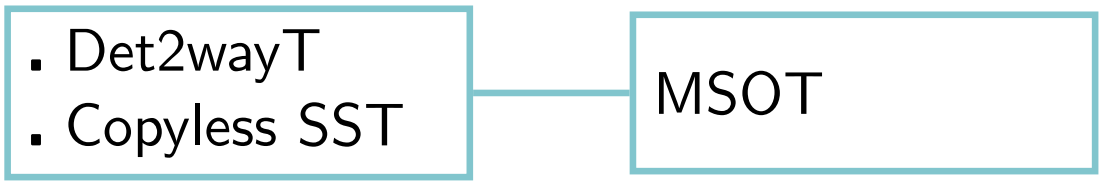
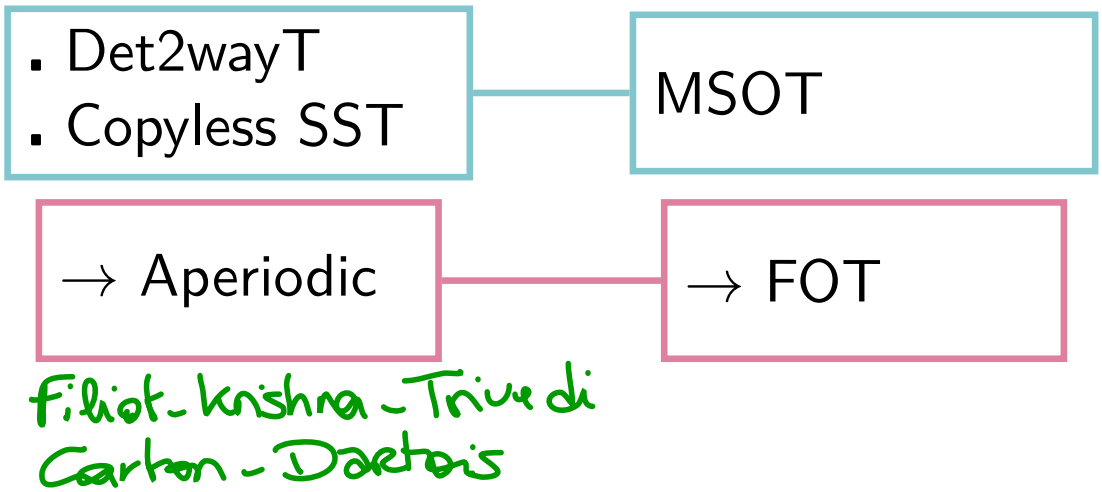
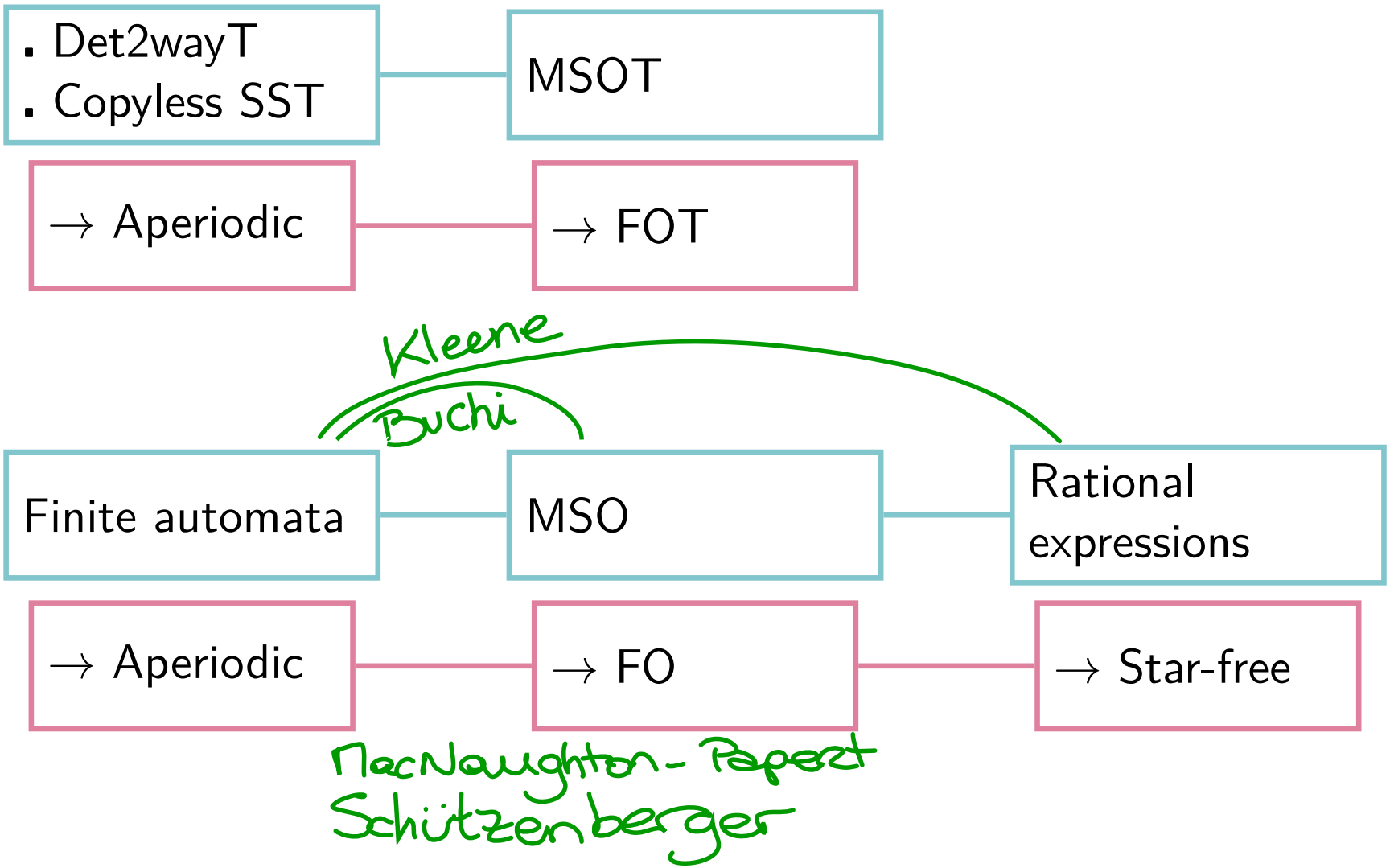
FO

FO

Is the example FO?


Det2wayT
Copyless SST
MSOT


. Det2wayT
. Copyless SST
MSOT
→ Aperiodic
→ FOT
Filiot-Krishna-Trivedi
Carton-Dartois


. Det2wayT
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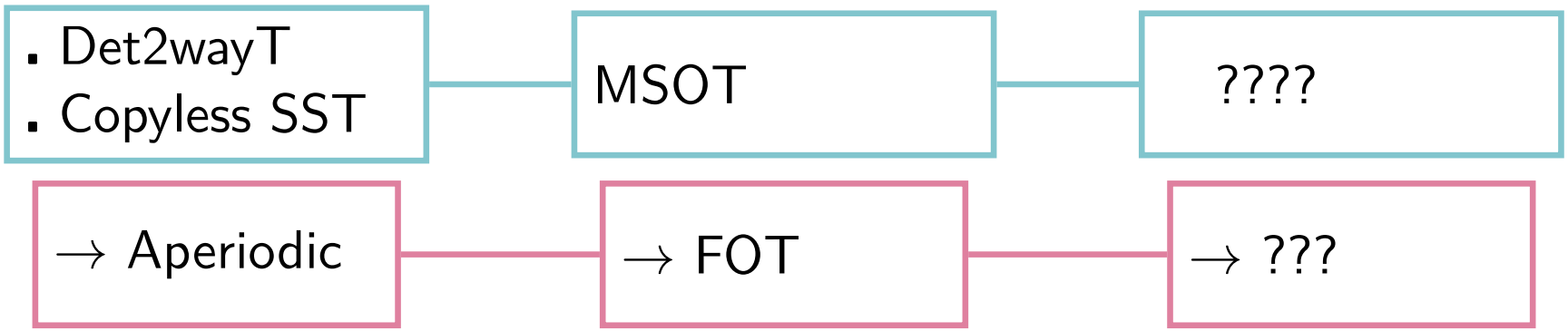
Kleene
Buchi
Finite automata
MSO
Rational expressions
→ Aperiodic
→ FO
→ Star-free
MacNaughton-Papert
Schützenberger


. Det2wayT
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????
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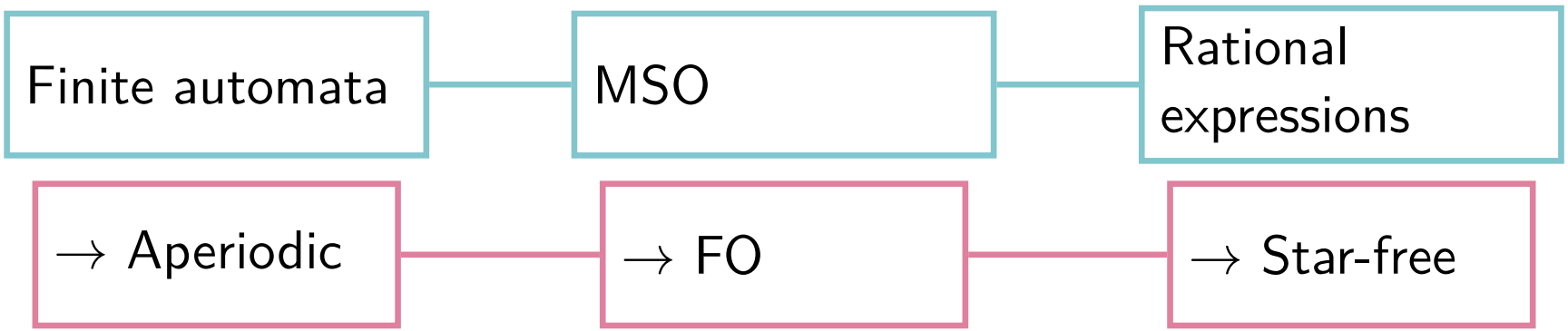
→ ???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Functions on a type system

$\longrightarrow$ Functions on lists, lists of lists, pairs of lists...

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$$\mathcal{T} := \text{every one-element set} \quad | \quad \mathcal{T} + \mathcal{T} \quad | \quad \mathcal{T} \times \mathcal{T} \quad | \quad \mathcal{T}^*$$

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Basic functions:

• **projection**$_1$	• **coprojection**	• **distribute**
$\Sigma \times \Gamma \longrightarrow \Sigma$	$\Sigma \longrightarrow \Sigma + \Gamma$	$(\Sigma + \Gamma) \times \Delta \longrightarrow (\Sigma \times \Delta) + (\Gamma \times \Delta)$
$(x, y) \longmapsto x$	$x \longmapsto x$	$(x, y) \longmapsto (x, y)$

Basic functions on lists

Reverse. $[a, b, a, a, b, c] \mapsto [c, b, a, a, b, a]$

Flat. $[[a, b], [c]] \mapsto [a, b, c]$

Append. $(a, [b, b, c, a]) \mapsto [a, b, b, c, a]$

Co-append. $[a, b, b, c, a] \mapsto (a, [b, b, c, a])$

Block. $[a, b, c, d, a, a, b, c, d] \mapsto [[a, b], [c, d], [a, a, b], [c, d]]$

Operations

Disjoint union.

$$\frac{f : \Sigma \longrightarrow \Delta \qquad g : \Gamma \longrightarrow \Delta}{f + g : \Sigma + \Gamma \longrightarrow \Delta} \qquad x \mapsto \begin{cases} f(x) & \text{if } x \in \Sigma \\ g(x) & \text{if } x \in \Gamma \end{cases}$$

Composition.

$$\frac{f : \Sigma \longrightarrow \Gamma \qquad g : \Gamma \longrightarrow \Delta}{g \circ f : \Sigma \longrightarrow \Delta} \qquad x \mapsto g(f(x))$$

Map.

$$\frac{f : \Sigma \longrightarrow \Gamma}{f^* : \Sigma^* \longrightarrow \Gamma^*} \qquad [x_1, \ldots, x_n] \mapsto \begin{cases} [f(x_1), \ldots, f(x_n)] & \text{if } n > 0 \\ [] & \text{if } n = 0 \end{cases}$$

Pairing.

$$\frac{f : \Sigma \longrightarrow \Gamma \qquad g : \Sigma \longrightarrow \Delta}{(f, g) : \Sigma \longrightarrow \Gamma \times \Delta} \qquad x \mapsto (f(x), g(x))$$

First-order list functions

First-order list functions:
Smallest class of functions over any Σ from $\mathcal{T}$, which contains:

- all the constant functions,
- the functions projection, co-projection and distribute,
- the functions reverse, flat, append, co-append and block,

and which is closed under applying the operations:

- disjoint union,
- composition,
- map,
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Theorem

The first-order list functions are exactly the FO-transductions.

Let's go back to our running example

$$w \quad \mapsto \quad w\overline{w}a^{|w|_a}$$

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- Identity: $[a, b, a, a, b, b, b, a, b] \mapsto [a, b, a, a, b, b, b, a, b]$

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  Map: $[[a], [b], [a, a], [b, b, b], [a], [b]] \mapsto [[a], [], [a, a], [], [a], []]$

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  Flat: $[[a], [], [a, a], [], [a], []] \mapsto [a, a, a, a]$

$\rightarrow$ Pairing, transform into lists and flat

Extension to MSO

Regular list functions:
Same as the first-order list function, except that for every group, we add its prefix multiplication function to the base functions.

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Theorem

The regular list functions are exactly the MSO-transductions.

Conclusion

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Extensions:

- Other types, $\mathbb{N}$...?
- Non-linear operators?
- Other data structures, trees...?