
Max-plus automata and Size-change abstraction

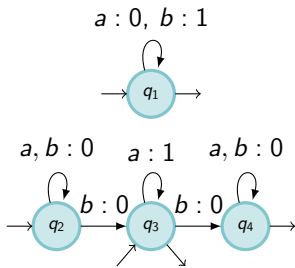
Laure Daviaud

LIF, Aix-Marseille Université

Joint work with Thomas Colcombet (LIAFA)
and Florian Zuleger (Vienna University)

LABRI, 16 décembre 2014

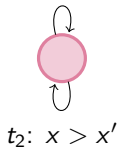
Max-plus automata



- automata theory
- tropical algebra

Size-change abstraction

$$t_1: x \geq x', y > y'$$



- verification
- program analysis

Max-plus automata

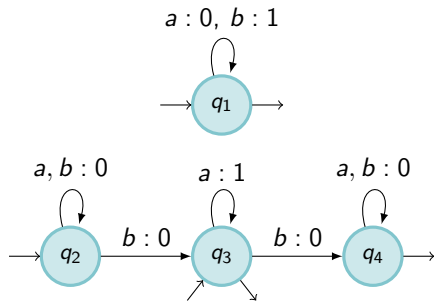
Syntax :

Non deterministic finite automaton for which each transition is labelled by a **non negative integer (weight)**.

Semantic :

Weight of a run = sum of the weights of the transitions.

A^* $\rightarrow \mathbb{N} \cup \{-\infty\}$
 w \mapsto Maximum of the weights of accepting runs labelled by w
($-\infty$ if no such run)



Max-plus automata

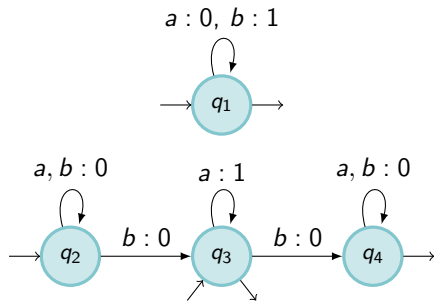
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$a^{n_0} b a^{n_1} b \dots b a^{n_k}$
 $\mapsto \max(n_0, n_1, \dots, n_k, k)$

Weighted automata

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[Schützenberger '61]

Max-plus automata

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Semiring (S, \oplus, \otimes)

Max-plus automata

$(\mathbb{N} \cup \{-\infty\}, \max, +)$

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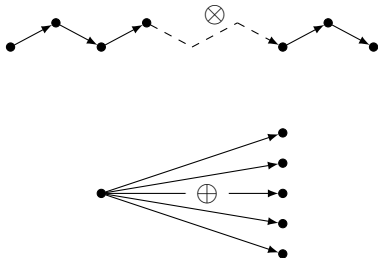


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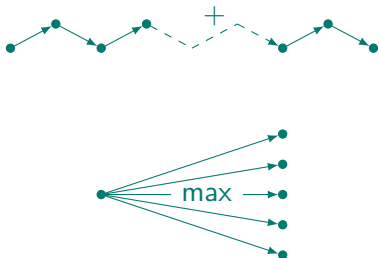
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Weights in \mathbb{N}



Max-plus automata: Decidability

$$f, g : \mathbb{A}^* \rightarrow \mathbb{N} \cup \{-\infty\}$$

Decidable

Undecidable



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$\exists w, f(w) = k ?$



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$$\forall w \in \mathbb{A}^*, f(w) \leq g(w) ?$$

[Krob '92]

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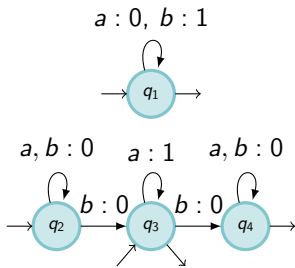
Comparison

$$\forall w \in \mathbb{A}^*, f(w) \leq g(w) ?$$

[Krob '92]

- $\forall w, |w| \leq f(w)$
- $\exists b, \forall w, |w| \leq f(w) + b$

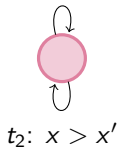
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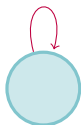
```
Input x, y :  
  while x>=0 {  
    y--;  
    if y=0 {  
      x--;  
      y=random();  
    }  
  }
```



Size-change abstraction

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$t_1: x \geq x', y > y'$



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Input x, y :

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while  $x \geq 0$  {
```

```
   $y--$ ;
```

```
  if  $y=0$  {
```

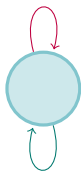
```
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```

```
     $y=\text{random}()$ ;
```

```
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```

```
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```

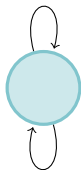
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$t_2: x > x'$

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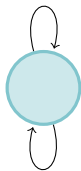


$t_2: x > x'$

- Finite number of variables (values in \mathbb{N})
- Transition: conjunction of a finite number of predicates of the form $x_i > x'_j$ or $x_i \geq x'_j$
- Trace: sequence of transitions and valuations compatible

Size-change abstraction

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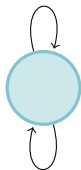
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$$(5, 5) \xrightarrow{t_1} (5, 4) \xrightarrow{t_2} (4, 8) \xrightarrow{t_1} (3, 6) \xrightarrow{t_1} (3, 2) \xrightarrow{t_2} (2, 10) \dots$$

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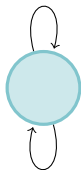
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Terminating sca: no infinite trace

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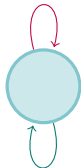
Terminating sca: no infinite trace

— Theorem [Lee, Jones, Ben-Amram] —

It is decidable whether a given sca instance is terminating.

Size-change abstraction: longest traces

$t_1: x \geq x', y > y'$

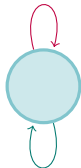


$t_2: x > x'$

- Terminating
- Traces of unbounded lengths

Size-change abstraction: longest traces

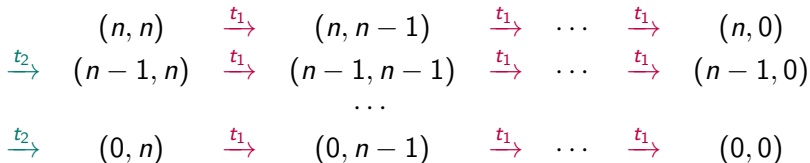
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- Terminating
- Traces of unbounded lengths

Restriction to $[0, n]$, what is the length of the longest trace ?



Size-change abstraction: main theorem

— Theorem [Colcombet, D., Zuleger] —

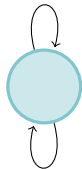
Given a terminating sca instance,
there is a computable rational $\beta \geq 1$ such that

$$f = \Theta(n^\beta)$$

where f associates a positive integer n to the length of the longest traces if the variables are restricted to be in $[0, n]$.

From sca to max-plus automata

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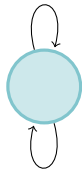


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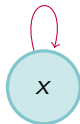
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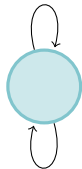
$t_2: x > x'$

$t_1: 0$



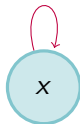
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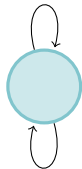
$t_1: 0$



$t_1: 1$

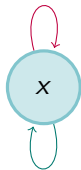
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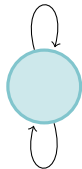
$t_2: 1$



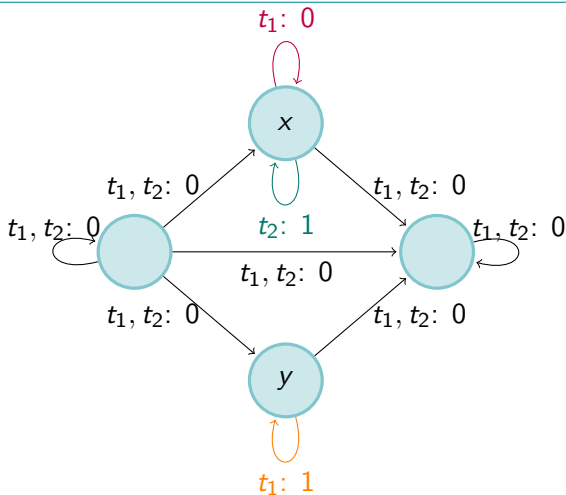
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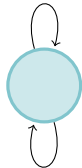


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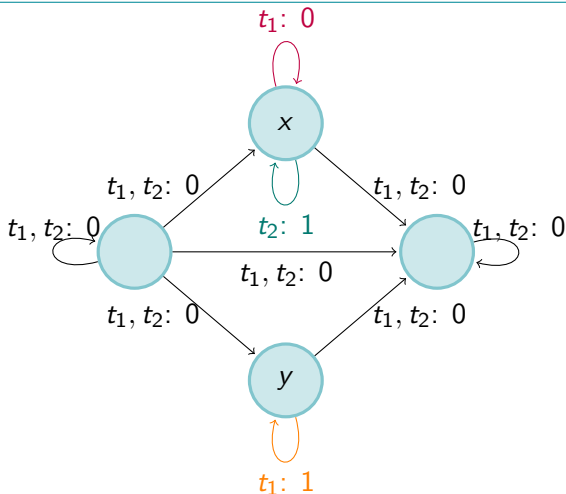


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$$x \geq x' \quad x \geq x' \quad x > x' \quad x \geq x' \quad x \geq x' \quad x > x' \quad x \geq x' \quad x \geq x'$$

From sca to max-plus automata

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$$\begin{array}{cccccccc} x \geq x' & x \geq x' & x > x' & x \geq x' & x \geq x' & x > x' & x \geq x' & x \geq x' \\ y > y' & y > y' & & & & & & \\ & & y > y' & y > y' & & & & \\ & & & & & & y > y' & y > y' \end{array}$$

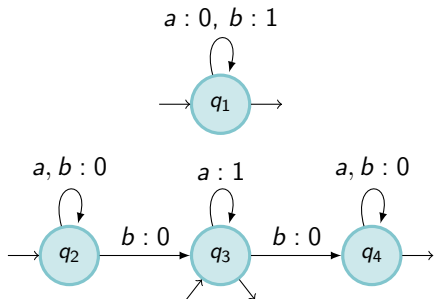
Longest trace when variables do not exceed n

→ Sequences of inequalities with at most n strict inequalities

→ Runs of weight at most n

Longest word of weight at most n

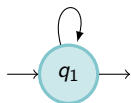
Max-plus automata: example



$$f(a^{n_0} b a^{n_1} b \dots a^{n_{k-1}} b a^{n_k}) \\ = \max(n_0, n_1, \dots, n_k, k)$$

Max-plus automata: example

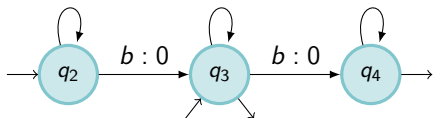
$a : 0, b : 1$



$a, b : 0$

$a : 1$

$a, b : 0$



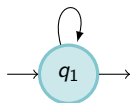
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$$f((a^n b)^n a^n) = n$$

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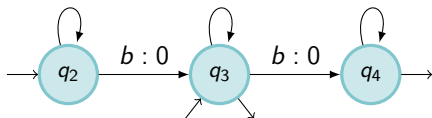


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$a, b : 0$

$a : 1$

$a, b : 0$



Longest word with weight at most n

$$f((a^n b)^n a^n) = n$$

→ Smallest weight of words of length at least n

$$g : \mathbb{N} \rightarrow \mathbb{N} \cup \{-\infty\} \\ n \mapsto \min_{|w| \geq n} f(w)$$

Max-plus automata: main theorem

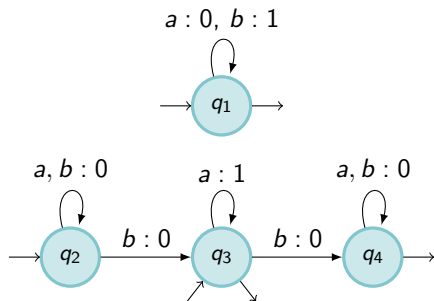
$$g : \mathbb{N} \rightarrow \mathbb{N} \cup \{-\infty\}$$
$$n \mapsto \min_{|w| \geq n} f(w)$$

— Theorem [Colcombet, D., Zuleger] —

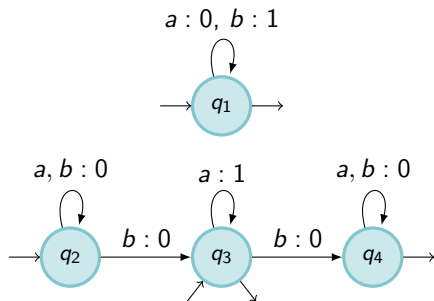
There is an algorithm with input a max-plus automaton, that computes a rational α (in $[0, 1] \cup \{-\infty\}$) such that:

$$g(n) = \Theta(n^\alpha)$$

Matrix representation

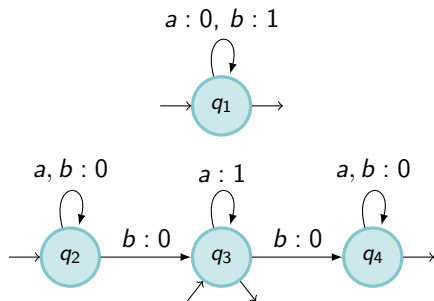


Matrix representation



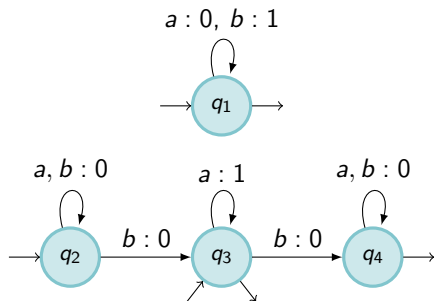
$$\begin{matrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{matrix} \begin{pmatrix} q_1 & q_2 & q_3 & q_4 \\ 0 & \infty & \infty & \infty \\ \infty & 0 & \infty & \infty \\ \infty & \infty & 1 & \infty \\ \infty & \infty & \infty & 0 \end{pmatrix} = \mu(a)$$

Matrix representation



$$\begin{matrix} & q_1 & q_2 & q_3 & q_4 \\ q_1 & \begin{pmatrix} 0 & \infty & \infty & \infty \\ \infty & 0 & \infty & \infty \\ \infty & \infty & 1 & \infty \\ \infty & \infty & \infty & 0 \end{pmatrix} & & & \\ q_2 & & & & \\ q_3 & & & & \\ q_4 & & & & \end{matrix} = \mu(a)$$
$$\begin{pmatrix} 1 & \infty & \infty & \infty \\ \infty & 0 & 0 & \infty \\ \infty & \infty & \infty & 0 \\ \infty & \infty & \infty & 0 \end{pmatrix} = \mu(b)$$

Matrix representation

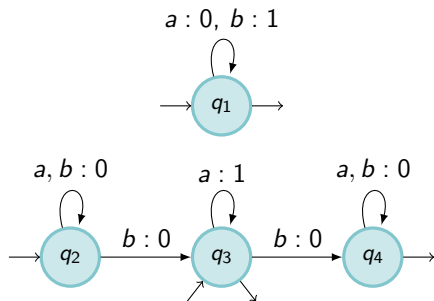


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$$I = (0 \quad 0 \quad 0 \quad \infty) \quad F = \begin{pmatrix} 0 \\ \infty \\ 0 \\ 0 \end{pmatrix}$$

Matrix representation

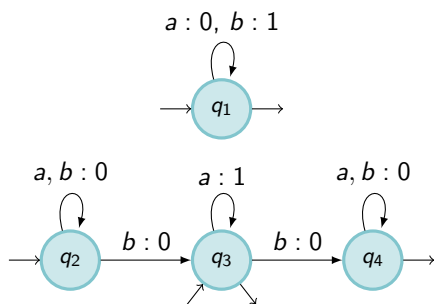


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$$\mu(a_1 \cdots a_n) = \mu(a_1) \cdots \mu(a_n) \quad I = (0 \quad 0 \quad 0 \quad \infty) \quad F = \begin{pmatrix} 0 \\ \infty \\ 0 \\ 0 \end{pmatrix}$$

Matrix representation



$$\begin{matrix} & q_1 & q_2 & q_3 & q_4 \\ q_1 & 0 & \infty & \infty & \infty \\ q_2 & \infty & 0 & \infty & \infty \\ q_3 & \infty & \infty & 1 & \infty \\ q_4 & \infty & \infty & \infty & 0 \end{matrix} = \mu(a)$$

$$\begin{pmatrix} 1 & \infty & \infty & \infty \\ \infty & 0 & 0 & \infty \\ \infty & \infty & \infty & 0 \\ \infty & \infty & \infty & 0 \end{pmatrix} = \mu(b)$$

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$\mu(w)_{i,j}$ = maximum of weights of runs from i to j labelled by w

$$f(w) = I\mu(w)F$$

Sketch of the proof

Compare $f(w)$ and $|w|$

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Compare $\mu(w)$ and $|w|$

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Compare $\mu(w)$ and $|w|$ \longrightarrow Describe $\{(\mu(w), |w|) \mid w \in \mathbb{A}^*\}$

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Presentable sets

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Approximation \approx Presentable sets

Given a finite set of weighted matrices X , compute a **presentable** set Y such that $\langle X \rangle \approx Y$.

The factorisation forest theorem [Simon]

$$(M_1, \ell_1)(M_2, \ell_2)(M_3, \ell_3)(M_4, \ell_4)(M_5, \ell_5)(M_6, \ell_6)(M_7, \ell_7) \cdots \cdots (M_k, \ell_k)$$

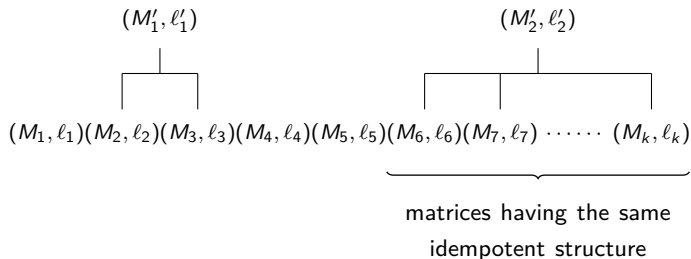
The factorisation forest theorem [Simon]

(M'_1, ℓ'_1)

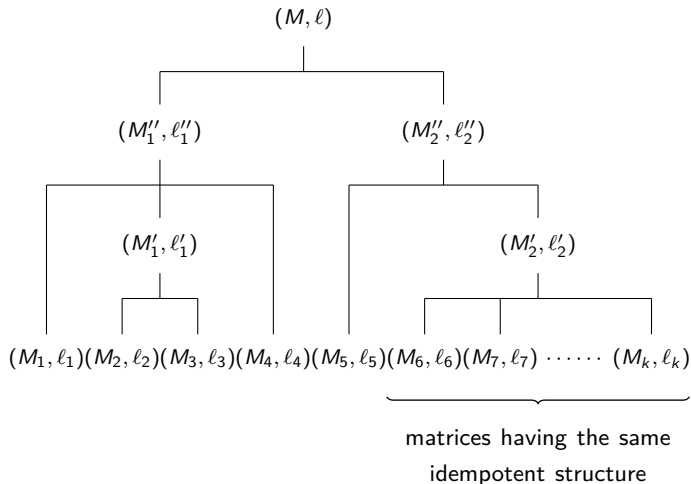


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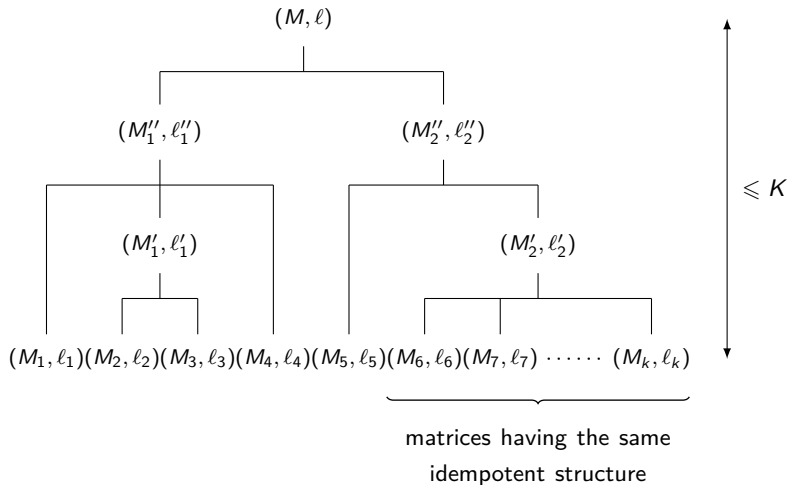
The factorisation forest theorem [Simon]



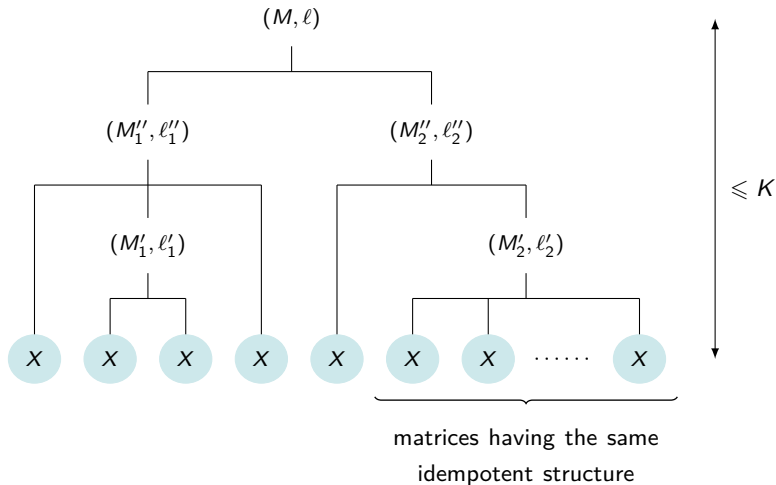
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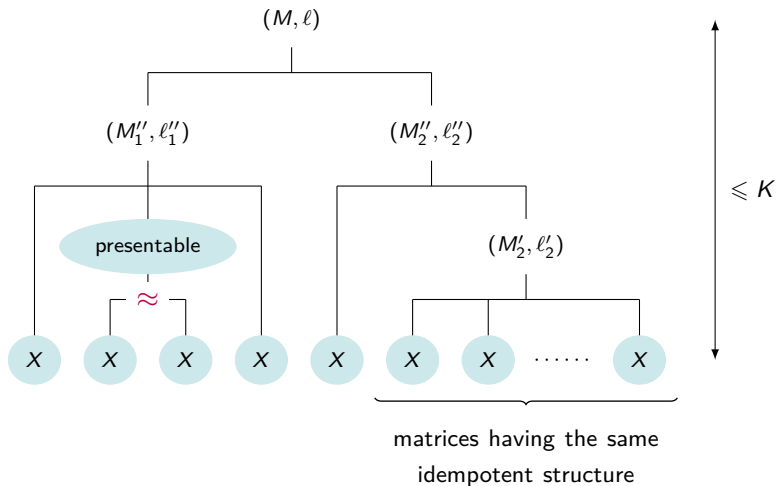
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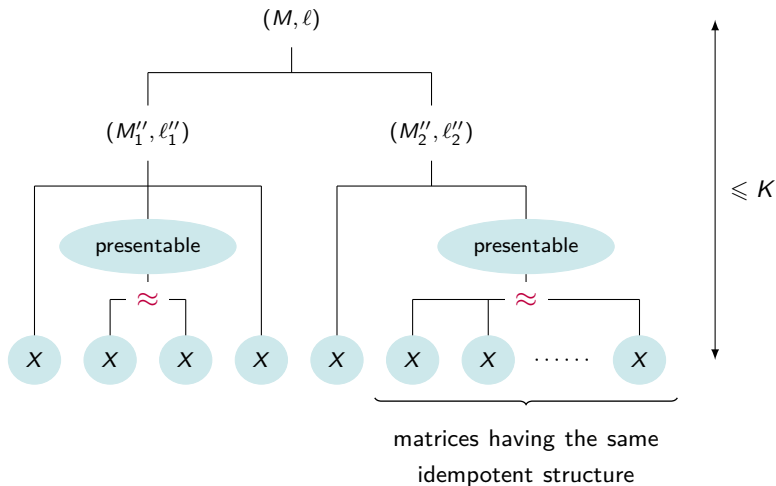
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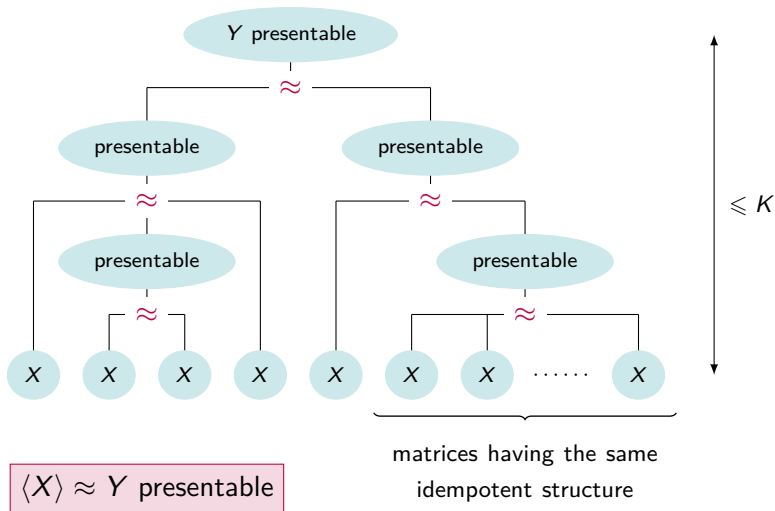
The factorisation forest theorem [Simon]



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Approximation

$$g : \mathbb{N} \rightarrow \mathbb{N} \cup \{-\infty\}$$
$$n \mapsto \min_{|w| \geq n} f(w)$$

$$(M, \ell) \preceq_a (N, k) \quad \text{if}$$

- $M \leq aN$
- $k \leq a\ell$
- $\overline{M} = \overline{N}$

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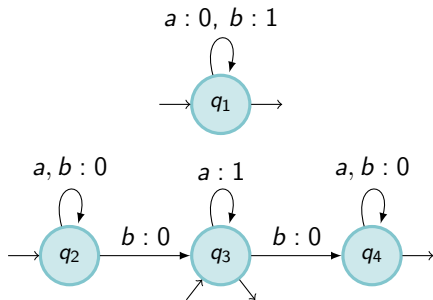
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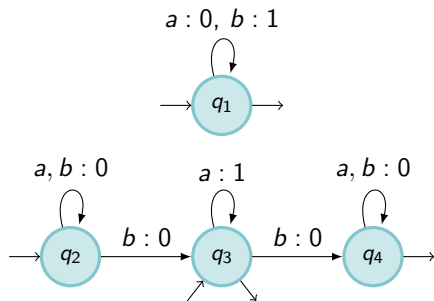
$X \approx Y$ if there is a such that $X \approx_a Y$

Presentable sets



$$(a^n b)^n a^n \mapsto n$$

Presentable sets

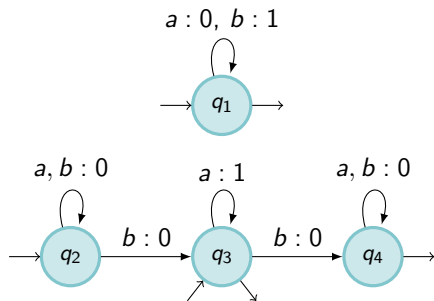


$$(a^n b)^n a^n \mapsto n$$

$$\left(\left(\begin{array}{cccc} n & \infty & \infty & \infty \\ \infty & 0 & n & n \\ \infty & \infty & \infty & n \\ \infty & \infty & \infty & 0 \end{array} \right), n^2 \right)$$

$$n \geq 1$$

Presentable sets

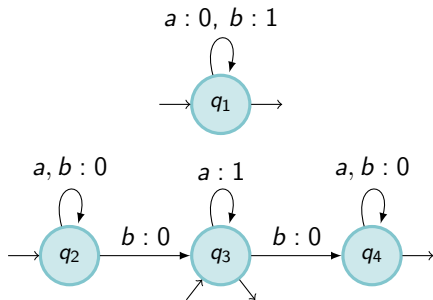


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$$\left(\begin{pmatrix} n^{\frac{1}{2}} & \infty & \infty & \infty \\ \infty & 0 & n^{\frac{1}{2}} & n^{\frac{1}{2}} \\ \infty & \infty & \infty & n^{\frac{1}{2}} \\ \infty & \infty & \infty & 0 \end{pmatrix}, n \right)$$

$$n \geq 1$$

Presentable sets



$$(a^+b)^+a^+ \rightsquigarrow (a^\lambda b)^{1-\lambda}a^\lambda$$

$$\left(\begin{pmatrix} n^{1-\lambda} & \infty & \infty & \infty \\ \infty & 0 & n^\lambda & n^\lambda \\ \infty & \infty & \infty & n^\lambda \\ \infty & \infty & \infty & 0 \end{pmatrix}, n \right)$$

$$n \geq 1, \lambda \in [0, 1]$$

Conclusion and further questions

- More precise description of functions computed by max-plus automata
- Min-plus automata
- Complexity EXPSPACE, PSPACE ongoing work
- Other applications ?