Max-plus automata and Size-change abstraction

Laure Daviaud

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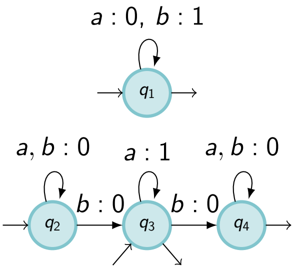
Joint work with Thomas Colcombet (LIAFA) and Florian Zuleger (Vienna University)

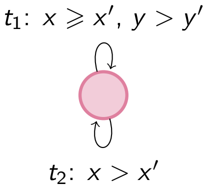
LACL, 26 janvier 2015

Max-plus automata



- automata theory
- tropical algebra

Size-change abstraction



- verification
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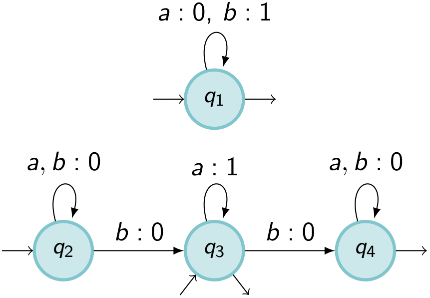
Max-plus automata

Syntax :

Non deterministic finite automaton for which each transition is labelled by a non negative integer (weight).



Semantic :

Weight of a run = sum of the weights of the transitions.

$\mathbb{A}^* \rightarrow \mathbb{N} \cup \{-\infty\}$

$w \mapsto$ Maximum of the weights of accepting runs labelled by w ($-\infty$ if no such run)

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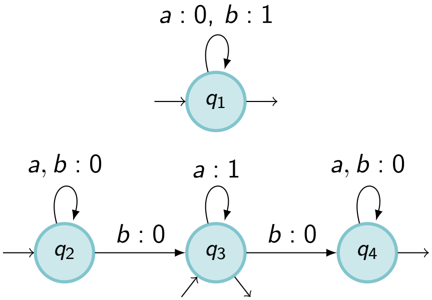

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$a^{n_0} b a^{n_1} b \cdots b a^{n_k}$

$\mapsto \max(n_0, n_1, \ldots, n_k, k)$

Weighted automata

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[Schützenberger '61]

Max-plus automata

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Semiring $(S, \oplus, \otimes)$

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$(\mathbb{N} \cup \{-\infty\}, \max, +)$

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Weights in S	Weights in $\mathbb{N}$

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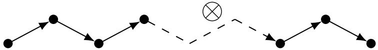

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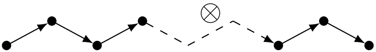

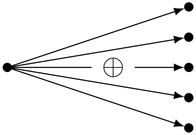

Max-plus automata

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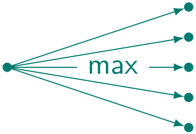

Max-plus automata: Decidability

$$f, g : \mathbb{A}^* \to \mathbb{N} \cup \{-\infty\}$$

Decidable

Undecidable

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$\uparrow$

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$$\forall w \in \mathbb{A}^*,\ f(w) \leqslant g(w)\ ?$$

[Krob '92]

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[Krob '92]

- $\forall w,\ |w| \leqslant f(w)$
- $\exists b,\ \forall w,\ |w| \leqslant f(w) + b$

Max-plus automata: main theorem

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$$n \mapsto \sup_{f(w) \leqslant n} |w|$$

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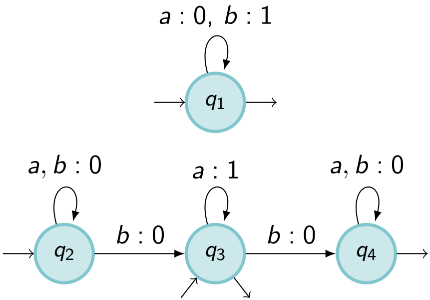
Theorem [Colcombet, D., Zuleger]

There is an algorithm with input a max-plus automaton, that computes a rational $\alpha \geqslant 1$ such that:

$$g(n) = \Theta(n^{\alpha})$$

Max-plus automata: example



$$f(a^{n_0} b a^{n_1} b \cdots a^{n_{k-1}} b a^{n_k}) = \max(n_0, n_1, \ldots, n_k, k)$$

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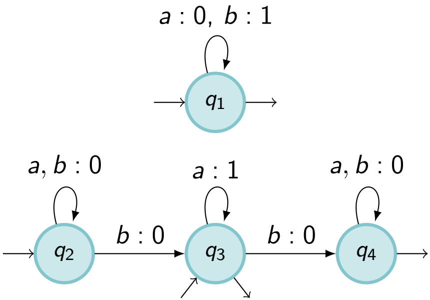

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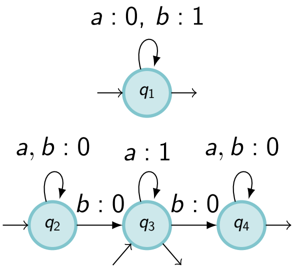
Longest word with weight at most n

$$f((a^n b)^n a^n) = n$$

Max-plus automata



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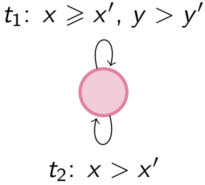

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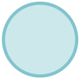
Size-change abstraction

```
Input x, y :
    while x>=0 {
        y--;
        if y=0 {
            x--;
            y=random();
        }
    }
```



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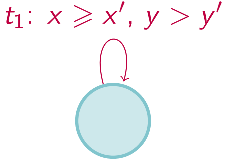

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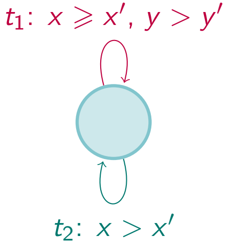

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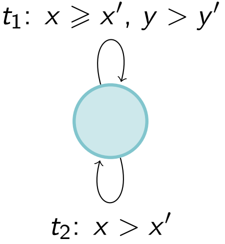

- Finite number of variables (values in $\mathbb{N}$)
- Transition: conjunction of a finite number of predicates of the form $x_i > x_j'$ or $x_i \geqslant x_j'$
- Trace: sequence of transitions and valuations compatible

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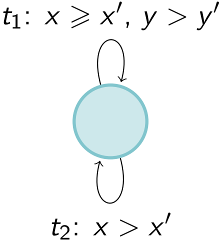

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$$(5,5) \xrightarrow{t_1} (5,4) \xrightarrow{t_2} (4,8) \xrightarrow{t_1} (3,6) \xrightarrow{t_1} (3,2) \xrightarrow{t_2} (2,10)...$$

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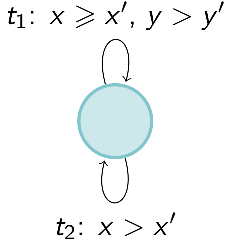

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Terminating sca: no infinite trace

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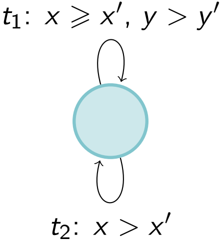

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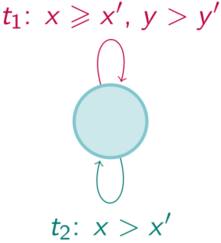
Terminating sca: no infinite trace

Theorem [Lee, Jones, Ben-Amram]

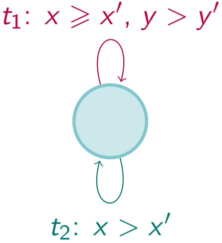
It is decidable whether a given sca instance is terminating.

Size-change abstraction: longest traces



- Terminating
- Traces of unbounded lengths

Size-change abstraction: longest traces



- Terminating
- Traces of unbounded lengths

Restriction to $[0, n]$, what is the length of the longest trace ?

$$\begin{array}{cccccccc}
 & (n, n) & \xrightarrow{t_1} & (n, n-1) & \xrightarrow{t_1} & \cdots & \xrightarrow{t_1} & (n, 0) \\
\xrightarrow{t_2} & (n-1, n) & \xrightarrow{t_1} & (n-1, n-1) & \xrightarrow{t_1} & \cdots & \xrightarrow{t_1} & (n-1, 0) \\
 & & & \cdots & & & & \\
\xrightarrow{t_2} & (0, n) & \xrightarrow{t_1} & (0, n-1) & \xrightarrow{t_1} & \cdots & \xrightarrow{t_1} & (0, 0)
\end{array}$$

Size-change abstraction: main theorem

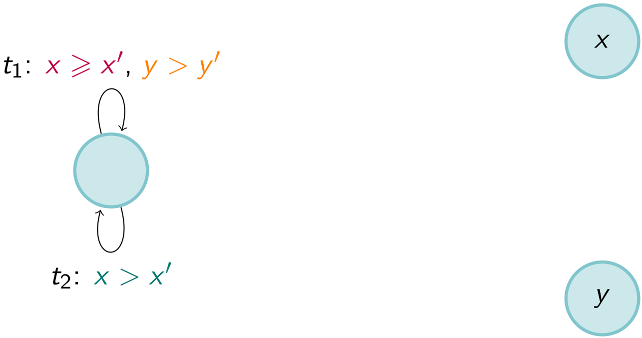
Theorem [Colcombet, D., Zuleger]

Given a terminating sca instance,
there is a computable rational $\alpha \geqslant 1$ such that

$$f = \Theta(n^{\alpha})$$

where f associates a positive integer n to the length of the longest traces if the variables are restricted to be in $[0, n]$.

From sca to max-plus automata



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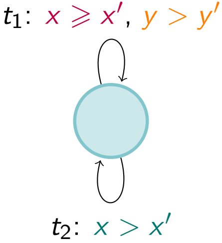

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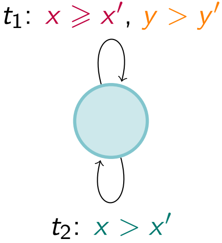

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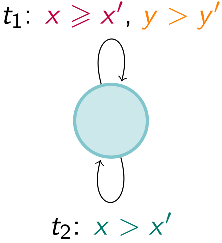

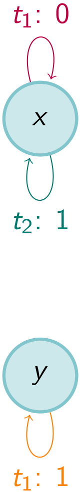

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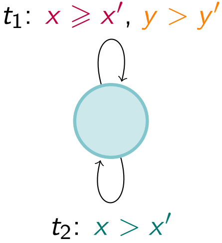

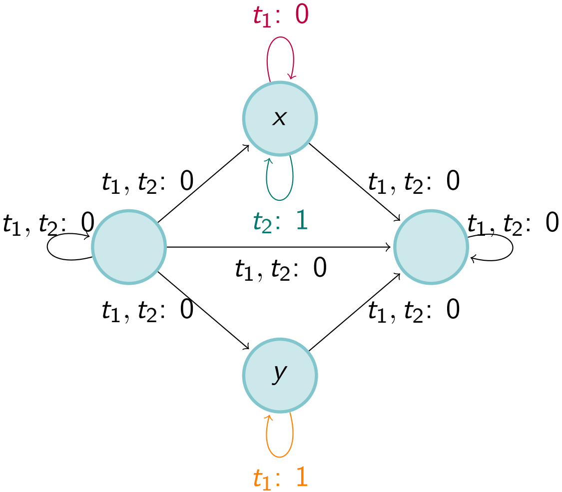

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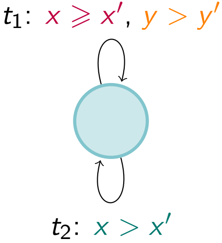

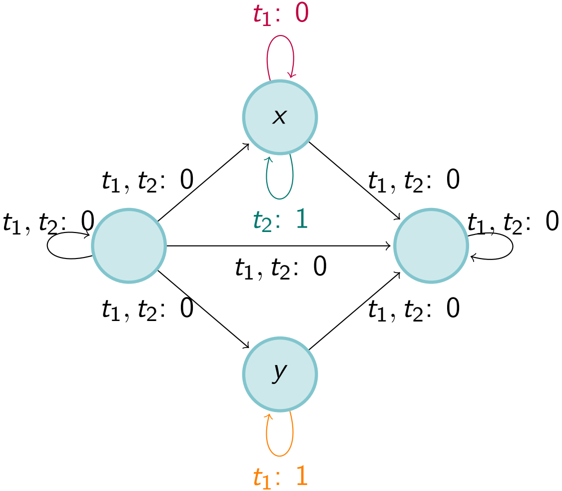

$$(2,2)\xrightarrow{t_1}(2,1)\xrightarrow{t_1}(2,0)\xrightarrow{t_2}(1,2)\xrightarrow{t_1}(1,1)\xrightarrow{t_1}(1,0)\xrightarrow{t_2}(0,2)\xrightarrow{t_1}(0,1)\xrightarrow{t_1}(0,0)$$

$$x \geqslant x' \quad x \geqslant x' \quad x > x' \quad x \geqslant x' \quad x \geqslant x' \quad x > x' \quad x \geqslant x' \quad x \geqslant x'$$

From sca to max-plus automata

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$x \geqslant x'$	$x \geqslant x'$	$x > x'$	$x \geqslant x'$	$x \geqslant x'$	$x > x'$	$x \geqslant x'$	$x \geqslant x'$
$y > y'$	$y > y'$						
			$y > y'$	$y > y'$			
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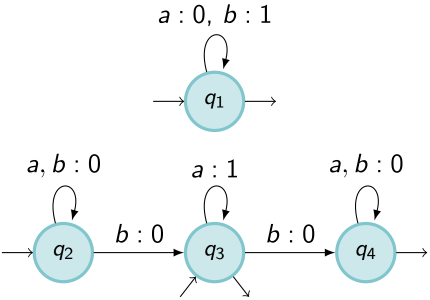
Longest trace when variables do not exceed n
$\longrightarrow$ Sequences of inequalities with at most n strict inequalities
$\longrightarrow$ Runs of weight at most n
Longest word of weight at most n

Matrix representation



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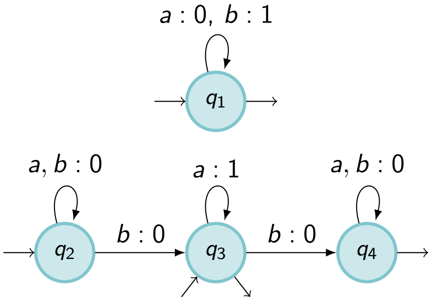

$$\begin{array}{c} \\ q_1 \\ q_2 \\ q_3 \\ q_4 \end{array} \begin{array}{c} \begin{array}{cccc} q_1 & q_2 & q_3 & q_4 \end{array} \\ \begin{pmatrix} 0 & \infty & \infty & \infty \\ \infty & 0 & \infty & \infty \\ \infty & \infty & 1 & \infty \\ \infty & \infty & \infty & 0 \end{pmatrix} \end{array} = \mu(a)$$

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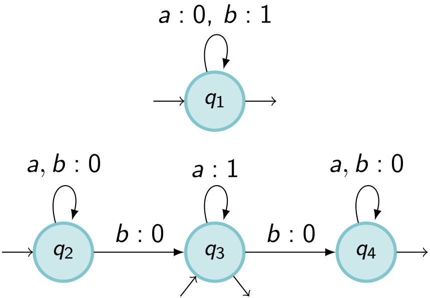

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$$\begin{pmatrix} 1 & \infty & \infty & \infty \\ \infty & 0 & 0 & \infty \\ \infty & \infty & \infty & 0 \\ \infty & \infty & \infty & 0 \end{pmatrix} = \mu(b)$$

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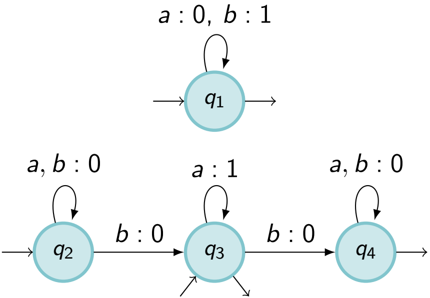

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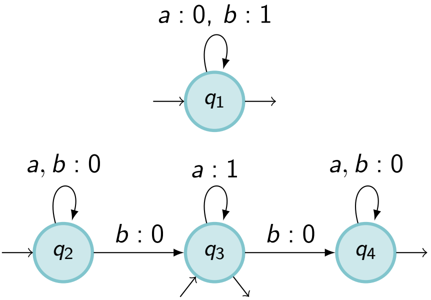
$$I = \begin{pmatrix} 0 & 0 & 0 & \infty \end{pmatrix} \qquad F = \begin{pmatrix} 0 \\ \infty \\ 0 \\ 0 \end{pmatrix}$$

Matrix representation

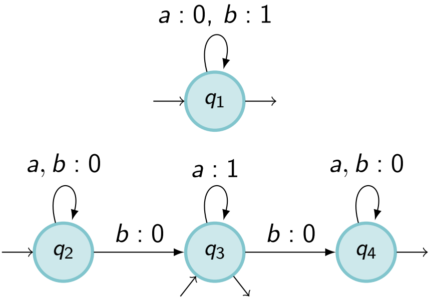


$$\begin{array}{c} \\ q_1 \\ q_2 \\ q_3 \\ q_4 \end{array}\begin{array}{c} \begin{array}{cccc} q_1 & q_2 & q_3 & q_4 \end{array} \\ \begin{pmatrix} 0 & \infty & \infty & \infty \\ \infty & 0 & \infty & \infty \\ \infty & \infty & 1 & \infty \\ \infty & \infty & \infty & 0 \end{pmatrix} \end{array} = \mu(a)$$

$$\begin{pmatrix} 1 & \infty & \infty & \infty \\ \infty & 0 & 0 & \infty \\ \infty & \infty & \infty & 0 \\ \infty & \infty & \infty & 0 \end{pmatrix} = \mu(b)$$

$$\mu(a_1 \cdots a_n) = \mu(a_1) \cdots \mu(a_n) \qquad I = \begin{pmatrix} 0 & 0 & 0 & \infty \end{pmatrix} \qquad F = \begin{pmatrix} 0 \\ \infty \\ 0 \\ 0 \end{pmatrix}$$

Matrix representation



$$\begin{array}{c} \\ q_1 \\ q_2 \\ q_3 \\ q_4 \end{array} \begin{array}{c} \begin{array}{cccc} q_1 & q_2 & q_3 & q_4 \end{array} \\ \begin{pmatrix} 0 & \infty & \infty & \infty \\ \infty & 0 & \infty & \infty \\ \infty & \infty & 1 & \infty \\ \infty & \infty & \infty & 0 \end{pmatrix} \end{array} = \mu(a)$$

$$\begin{pmatrix} 1 & \infty & \infty & \infty \\ \infty & 0 & 0 & \infty \\ \infty & \infty & \infty & 0 \\ \infty & \infty & \infty & 0 \end{pmatrix} = \mu(b)$$

$$\mu(a_1 \cdots a_n) = \mu(a_1) \cdots \mu(a_n) \qquad I = \begin{pmatrix} 0 & 0 & 0 & \infty \end{pmatrix} \qquad F = \begin{pmatrix} 0 \\ \infty \\ 0 \\ 0 \end{pmatrix}$$

$\mu(w)_{i,j} =$ maximum of weights of runs from i to j labelled by w

$$f(w) = I\mu(w)F$$

Sketch of the proof

Compare $f(w)$ and $|w|$

Sketch of the proof

Compare $\cancel{f(w)}$ $\mu(w)$ and $|w|$

Sketch of the proof

Compare ~~$f(w)$~~ $\mu(w)$ and $|w|$ $\longrightarrow$ Describe $\{(\mu(w), |w|) \mid w \in \mathbb{A}^*\}$

Sketch of the proof

Compare ~~$f(w)$~~ [$\mu(w)$] and $|w|$ ⟶ Describe $\{(\mu(w), |w|) \mid w \in \mathbb{A}^*\}$

Weighted matrices :

$$(M, \ell)$$

M: matrix; $\ell \in \mathbb{N} - \{0\}$

Sketch of the proof

Compare ~~$f(w)$~~ $\mu(w)$ and $|w|$ ⟶ Describe $\{(\mu(w), |w|) \mid w \in \mathbb{A}^*\}$

Weighted matrices :

$$(M, \ell)$$

M: matrix, $\ell \in \mathbb{N} - \{0\}$

$$(M, \ell)(M', \ell') = (MM', \ell + \ell')$$

Sketch of the proof

Compare ~~$f(w)$~~ $\mu(w)$ and $|w|$ $\longrightarrow$ Describe $\{(\mu(w), |w|) \mid w \in \mathbb{A}^*\}$

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$$(M, \ell)(M', \ell') = (MM', \ell + \ell')$$

$$(\mu(w), |w|)(\mu(w'), |w'|) = (\mu(ww'), |ww'|)$$

Sketch of the proof

Compare ~~$f(w)$~~ $\mu(w)$ and $|w|$ ⟶ Describe $\{(\mu(w), |w|) \mid w \in \mathbb{A}^*\}$

↓

Describe $\langle\{(\mu(a), 1) \mid a \in \mathbb{A}\}\rangle$

Weighted matrices :

$$(M, \ell)$$

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$$(M, \ell)(M', \ell') = (MM', \ell + \ell')$$

$$(\mu(w), |w|)(\mu(w'), |w'|) = (\mu(ww'), |ww'|)$$

Sketch of the proof

Compare ~~$f(w)$~~ $\mu(w)$ and $|w|$ → Describe $\{(\mu(w), |w|) \mid w \in \mathbb{A}^*\}$

Describe $\langle\{(\mu(a), 1) \mid a \in \mathbb{A}\}\rangle$

Approximation $\approx$

Weighted matrices :

$$(M, \ell)$$

matrix $\quad \in \mathbb{N} - \{0\}$

$$(M, \ell)(M', \ell') = (MM', \ell + \ell')$$

$$(\mu(w), |w|)(\mu(w'), |w'|) = (\mu(ww'), |ww'|)$$

Sketch of the proof

Compare ~~$f(w)$~~ $\mu(w)$ and $|w|$ → Describe $\{(\mu(w), |w|) \mid w \in \mathbb{A}^*\}$

↓

Describe $\langle\{(\mu(a), 1) \mid a \in \mathbb{A}\}\rangle$

↙ ↘

Approximation $\approx$ | Presentable sets

Weighted matrices :

(M, ℓ): M = matrix, $\ell \in \mathbb{N} - \{0\}$

$$(M, \ell)(M', \ell') = (MM', \ell + \ell')$$

$$(\mu(w), |w|)(\mu(w'), |w'|) = (\mu(ww'), |ww'|)$$

Sketch of the proof

Compare ~~$f(w)$~~ $\mu(w)$ and $|w|$ → Describe $\{(\mu(w), |w|) \mid w \in \mathbb{A}^*\}$

↓

Describe $\langle\{(\mu(a), 1) \mid a \in \mathbb{A}\}\rangle$

↙ ↘

Approximation $\approx$

Presentable sets

Weighted matrices :

$$(M, \ell)$$

matrix; $\ell \in \mathbb{N} - \{0\}$

$$(M, \ell)(M', \ell') = (MM', \ell + \ell')$$

$$(\mu(w), |w|)(\mu(w'), |w'|) = (\mu(ww'), |ww'|)$$

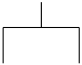
Given a finite set of weighted matrices X, compute a presentable set Y such that $\langle X\rangle \approx Y$.

The factorisation forest theorem [Simon]

$$(M_1, \ell_1)(M_2, \ell_2)(M_3, \ell_3)(M_4, \ell_4)(M_5, \ell_5)(M_6, \ell_6)(M_7, \ell_7) \cdots\cdots (M_k, \ell_k)$$

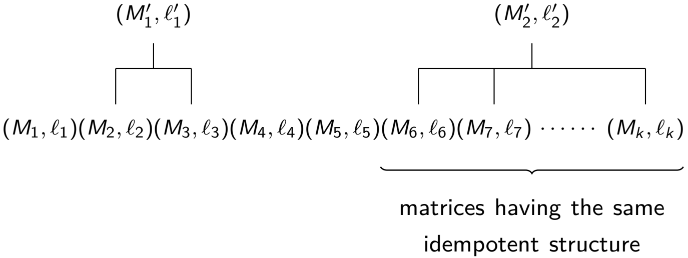
The factorisation forest theorem [Simon]

$$(M'_1, \ell'_1)$$

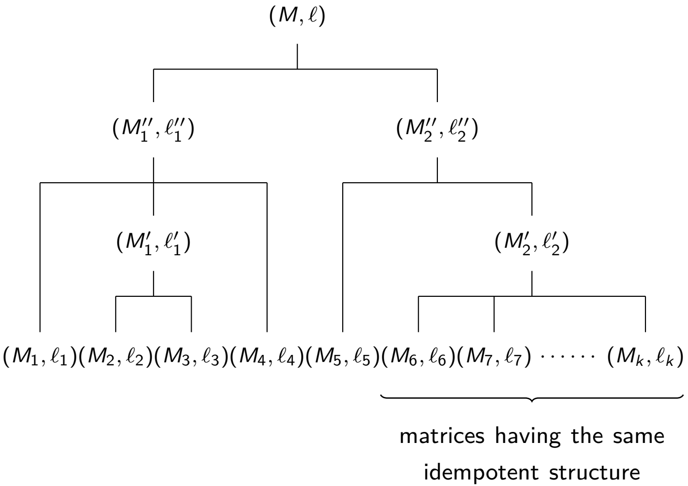


$$(M_1, \ell_1)(M_2, \ell_2)(M_3, \ell_3)(M_4, \ell_4)(M_5, \ell_5)(M_6, \ell_6)(M_7, \ell_7) \cdots\cdots (M_k, \ell_k)$$

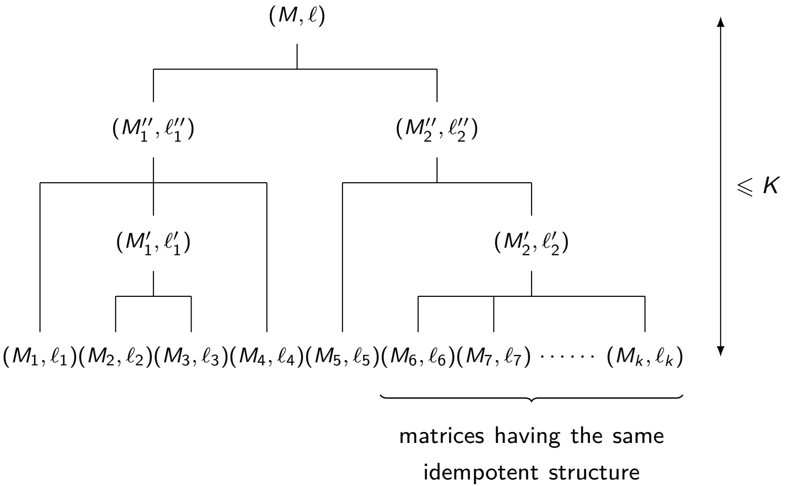
The factorisation forest theorem [Simon]



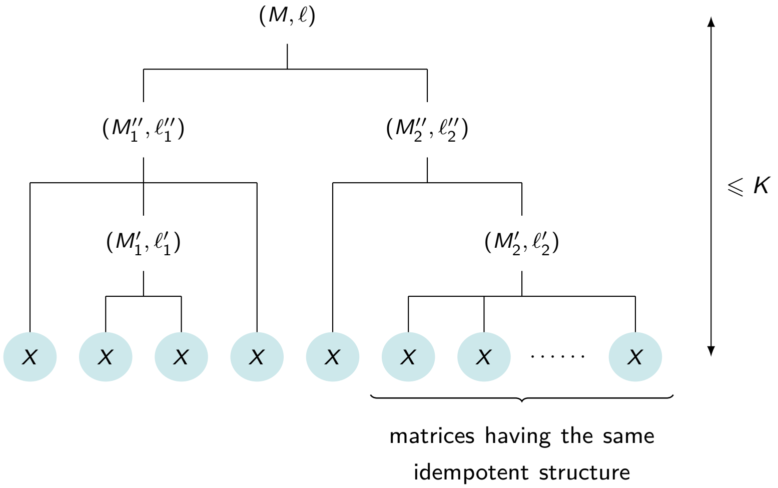
The factorisation forest theorem [Simon]



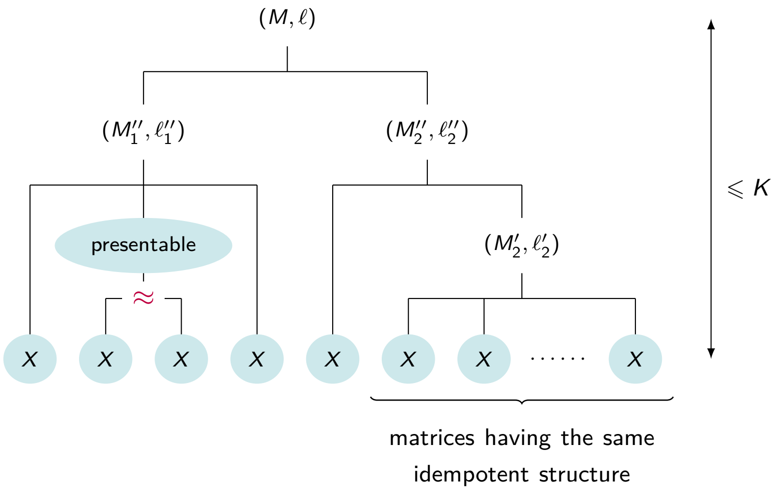
The factorisation forest theorem [Simon]



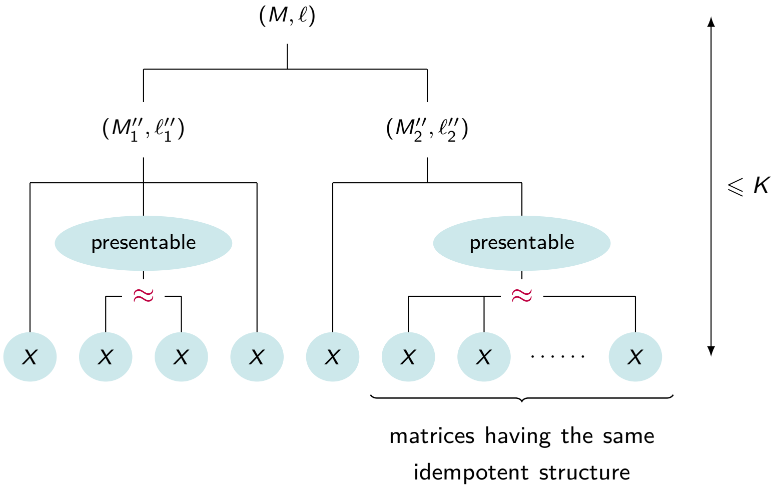
The factorisation forest theorem [Simon]



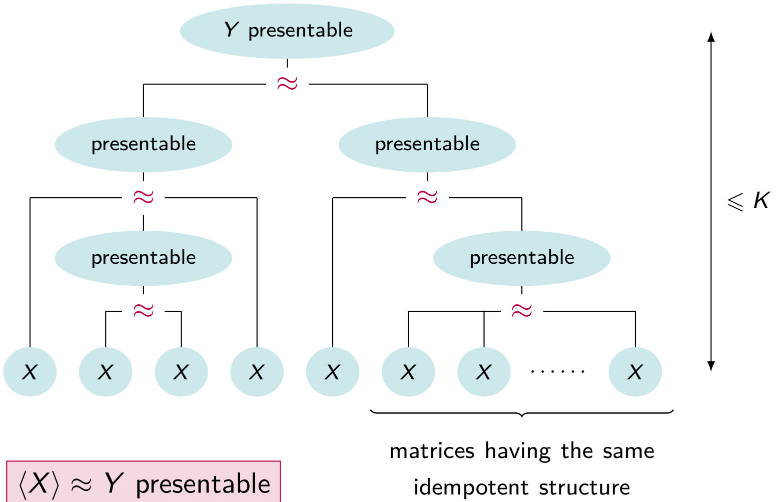
The factorisation forest theorem [Simon]



The factorisation forest theorem [Simon]



The factorisation forest theorem [Simon]



$\langle X \rangle \approx Y$ presentable

Approximation

$$(M, \ell) \preccurlyeq_a (N, k) \quad \text{if} \quad \begin{array}{l} \bullet\ M \leqslant aN \\ \bullet\ k \leqslant a\ell \\ \bullet\ \overline{M} = \overline{N} \end{array}$$

Approximation

$(M,\ell) \preccurlyeq_a (N,k)$ if

- $M \leqslant aN$
- $k \leqslant a\ell$
- $\overline{M} = \overline{N}$

$X \preccurlyeq_a Y$ if for all $(N,k) \in Y$, there is $(M,\ell) \in X$ such that $(M,\ell) \preccurlyeq_a (N,k)$

Approximation

$(M, \ell) \preccurlyeq_a (N, k)$ if

- $M \leqslant aN$
- $k \leqslant a\ell$
- $\overline{M} = \overline{N}$

$X \preccurlyeq_a Y$ if for all $(N, k) \in Y$, there is $(M, \ell) \in X$ such that $(M, \ell) \preccurlyeq_a (N, k)$

$X \approx_a Y$ if $X \preccurlyeq_a Y$ and $Y \preccurlyeq_a X$

Approximation

$(M, \ell) \preccurlyeq_a (N, k)$ if

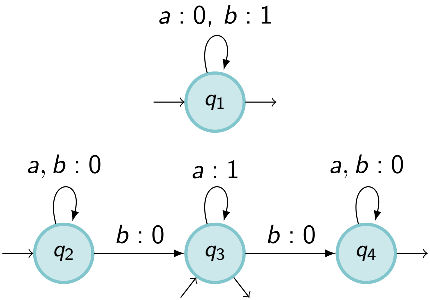
- $M \leqslant aN$
- $k \leqslant a\ell$
- $\overline{M} = \overline{N}$

$X \preccurlyeq_a Y$ if for all $(N, k) \in Y$, there is $(M, \ell) \in X$ such that $(M, \ell) \preccurlyeq_a (N, k)$

$X \approx_a Y$ if $X \preccurlyeq_a Y$ and $Y \preccurlyeq_a X$

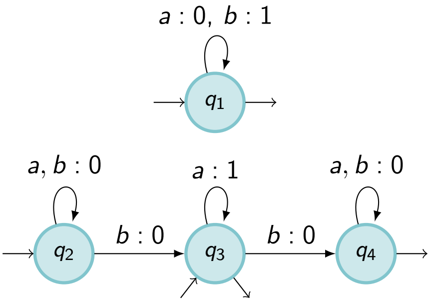
$X \approx Y$ if there is a such that $X \approx_a Y$

Presentable sets



$(a^n b)^n a^n \mapsto n$

Presentable sets

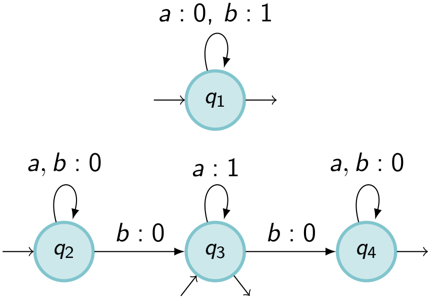


$(a^n b)^n a^n \mapsto n$

$$\left(\begin{pmatrix} n & \infty & \infty & \infty \\ \infty & 0 & n & n \\ \infty & \infty & \infty & n \\ \infty & \infty & \infty & 0 \end{pmatrix}, n^2 \right)$$

$$n \geqslant 1$$

Presentable sets

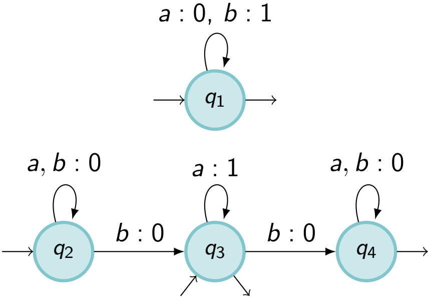


$$(a^n b)^n a^n \mapsto n$$

$$\left(\begin{pmatrix} n^{\frac{1}{2}} & \infty & \infty & \infty \\ \infty & 0 & n^{\frac{1}{2}} & n^{\frac{1}{2}} \\ \infty & \infty & \infty & n^{\frac{1}{2}} \\ \infty & \infty & \infty & 0 \end{pmatrix}, n \right)$$

$$n \geqslant 1$$

Presentable sets



$$(a^+b)^+a^+ \rightsquigarrow (a^\lambda b)^{1-\lambda} a^\lambda$$

$$\left(\begin{pmatrix} n^{1-\lambda} & \infty & \infty & \infty \\ \infty & 0 & n^\lambda & n^\lambda \\ \infty & \infty & \infty & n^\lambda \\ \infty & \infty & \infty & 0 \end{pmatrix}, n \right)$$

$$n \geqslant 1,\ \lambda \in [0,1]$$

Conclusion and further questions

- More precise description of functions computed by max-plus automata
- Min-plus automata
- Complexity EXPSPACE, PSPACE ongoing work
- Other applications ?