Approximate comparison of distance automata

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Goal: compare functions computed by distance automata

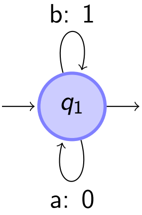
- Distance automata: definition, examples and context
- Results on comparison
- An algebraic view of distance automata
- Ideas of proofs

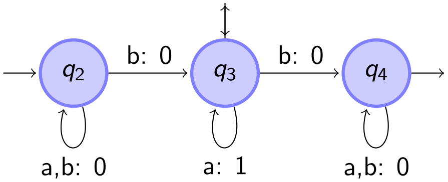
Distance automata

Distance automata

Distance automaton: Non deterministic finite automaton for which each transition is also labelled by a non-negative integer called the weight of the transition.

$$(\mathbb{A}, Q, I, T, E) \text{ with } E \subseteq (Q \times \mathbb{A} \times \mathbb{N} \times Q)$$





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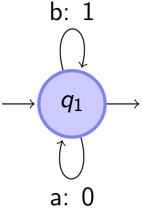
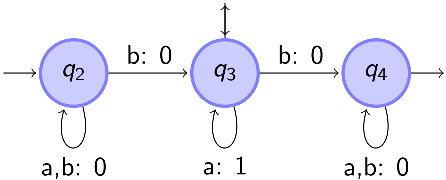


Weight of a run:
sum of the weights of the transitions



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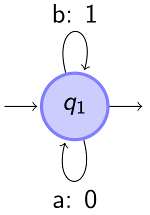
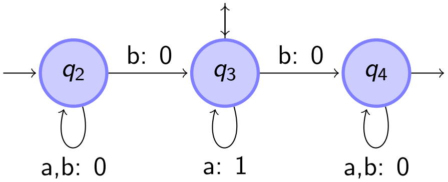


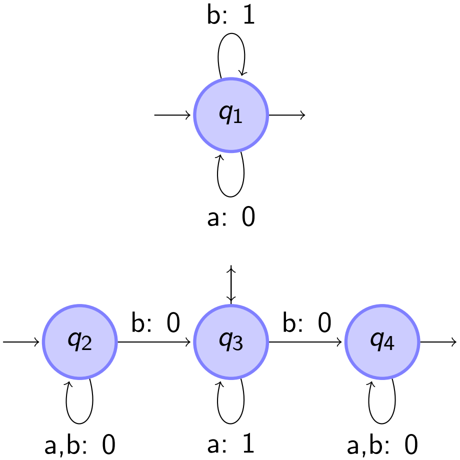


Weight of a run:
sum of the weights of the transitions

Computed function:

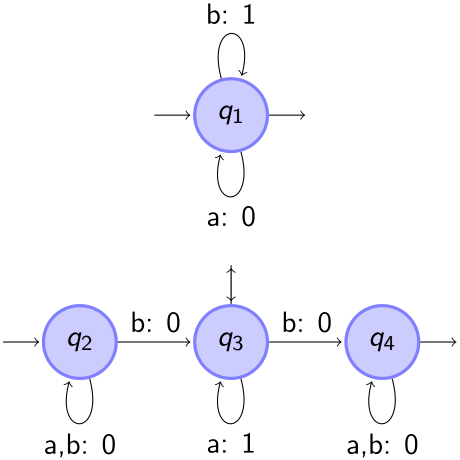
$$\begin{array}{rcl} \mathbb{A}^* & \to & \mathbb{N} \cup \{\infty\} \\ w & \mapsto & \text{minimum of the weights of the runs labelled by } w \text{ going from an initial state to a final state } (\infty \text{ if no such run}) \end{array}$$

Distance automata



Example: $a^m b a^n b a^p$

Distance automata



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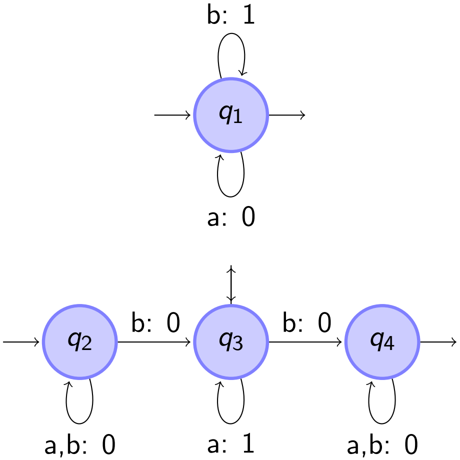
weight of run (1): 2

Distance automata



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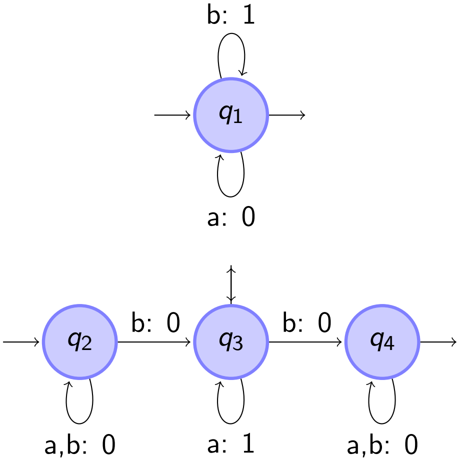
Distance automata



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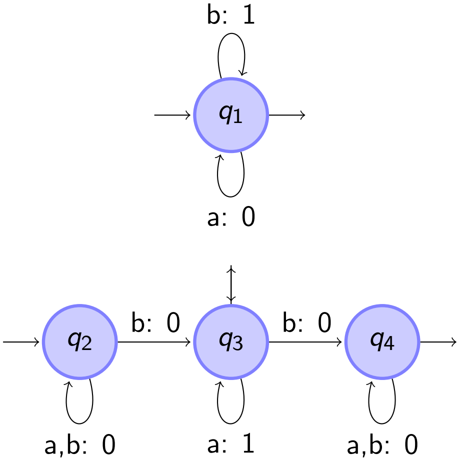

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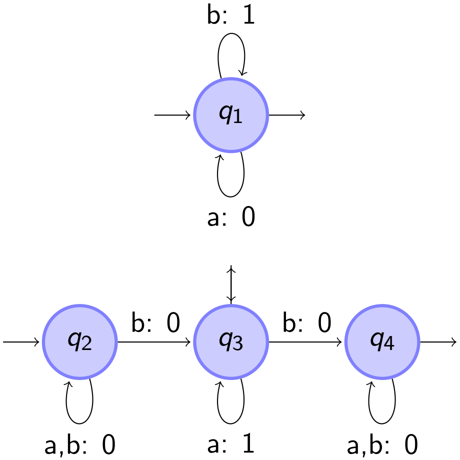

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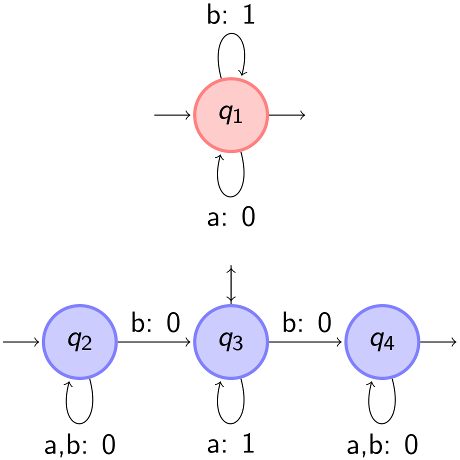

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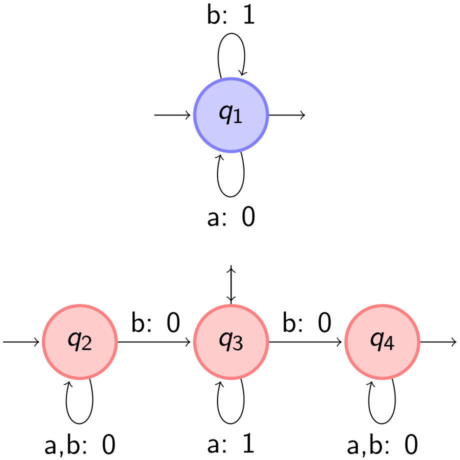

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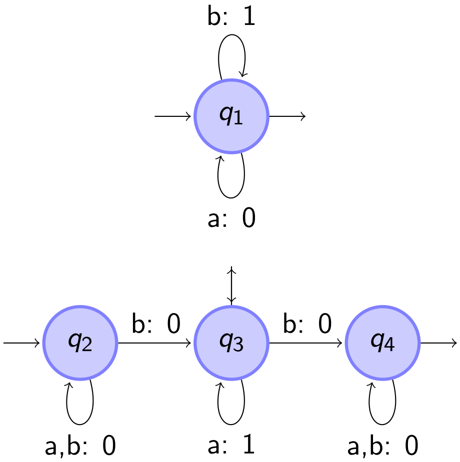
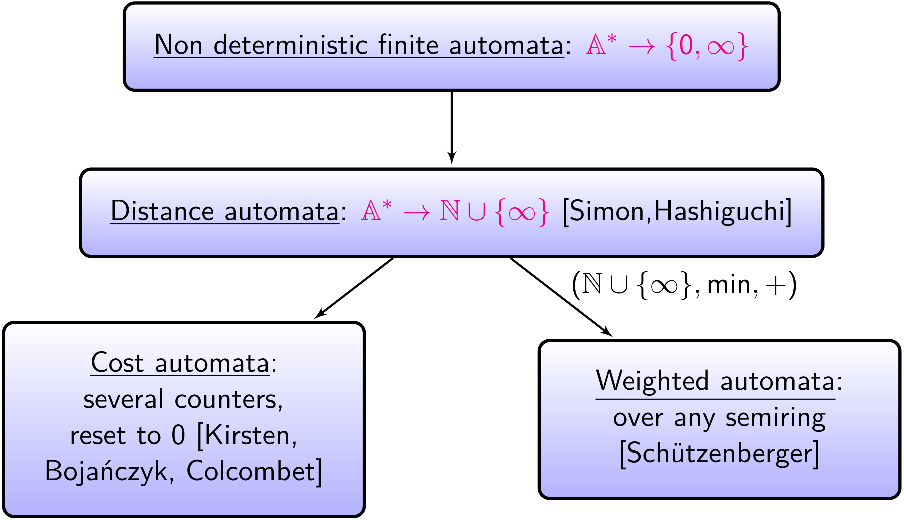


$$a^{n_0}ba^{n_1}b\cdots ba^{n_k} \mapsto \min(n_0, n_1, \ldots, n_k, k)$$

Context: different kinds of automata computing functions



How can we compare two functions given by distance automata?

Decision problems on comparison

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f, g computed by distance automata :

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Undecidable [Krob, 92]

Given f, g computed by distance automata, is $f \leqslant g$?

Even if $g = |.|$

Decidable [Colcombet, 09]

Is there a polynomial P s.t $f \leqslant P \circ g$? (context of cost functions)

Generalisation of results by Hashiguchi, Leung and Simon

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Generalisation of results by Hashiguchi, Leung and Simon

Where does the border of decidability lie?

Decision problems: the theorem of affine domination

Proposition

Given f, g computed by distance automata, the two assertions are equivalent:

1. There is a polynomial P s.t $f \leqslant P \circ g$.
2. There is an integer a s.t $f \leqslant ag + a$.

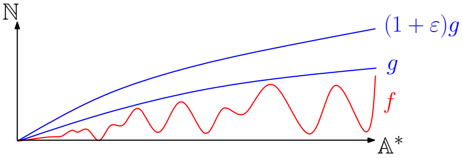
Theorem

Given f, g computed by distance automata, one can decide if there is an integer a s.t $f \leqslant ag + a$.

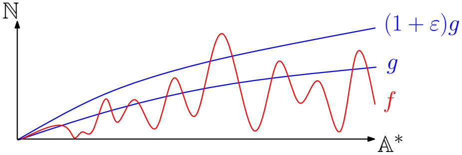
The approximate comparison of distance automata

Approximate comparison

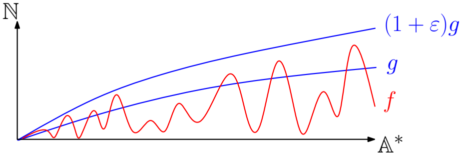
f,g computed by distance automata - Case $\varepsilon = 0$: undecidable result of Krob.



$\Longrightarrow$ YES



$\Longrightarrow$ NO



$\Longrightarrow$ YES or NO

Approximate comparison

Theorem: There is an algorithm with the following behaviour:

Input:

- f, g computed by distance automata and $\varepsilon > 0$.

Output:

- *yes* if $f \leqslant g$,
- *no* if $f \nleqslant (1+\varepsilon)g$ (i.e if $\exists$ w such that $f(w) > (1+\varepsilon)g(w)$),
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Alternative *Output:*

- *yes* if $f \leqslant (1-\varepsilon)g$,
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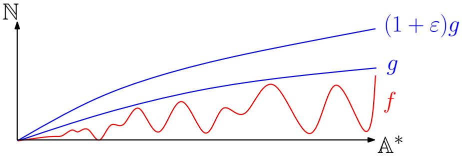
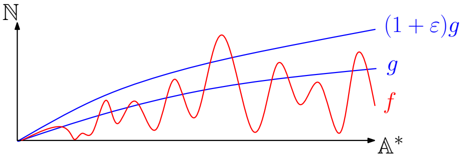
Approximate comparison



$\Longrightarrow$ YES



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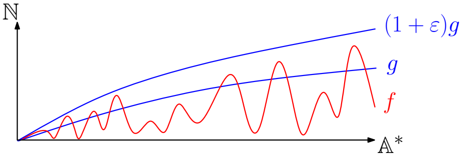

$\Longrightarrow$ YES or NO

YES $\Longrightarrow f \leqslant (1+\varepsilon)g$ NO $\Longrightarrow f \nleqslant g$

An algebraic view of distance automata

Matrices over semirings

- $(\mathbb{N} \cup \{\infty\}, \min, +)$
  Let n be an integer:
  $\min(n, \infty) = \min(\infty, n) = n$ $\quad$ $\infty + n = n + \infty = \infty$
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- Γ_n: set of $n \times n$ matrices with coefficients in $\mathbb{N} \cup \{\infty\}$

$$(M \otimes N)_{i,j} = \min_{k=1..n} \{M_{i,k} + N_{k,j}\}$$

Prop: $(\Gamma_n, \otimes)$ is a semigroup

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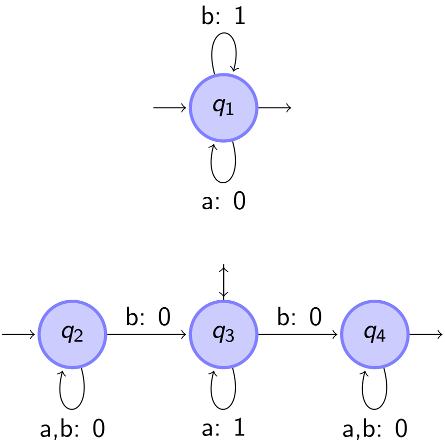
- I row-vector of size n: $(I \otimes M)_i = \min_{k=1..n}\{I_k + M_{k,i}\}$
  F column-vector of size n: $(M \otimes F)_i = \min_{k=1..n}\{M_{i,k} + F_k\}$

Distance automata: an algebraic view



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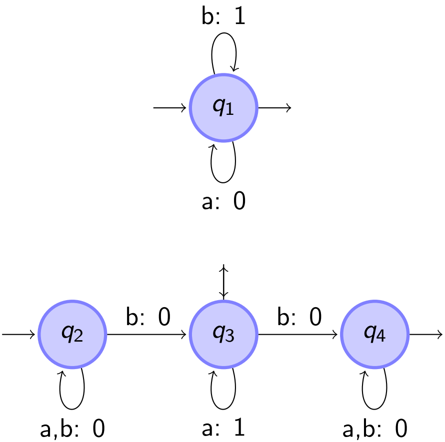

$$\mu(a) = \begin{pmatrix} 0 & \infty & \infty & \infty \\ \infty & 0 & \infty & \infty \\ \infty & \infty & 1 & \infty \\ \infty & \infty & \infty & 0 \end{pmatrix}$$

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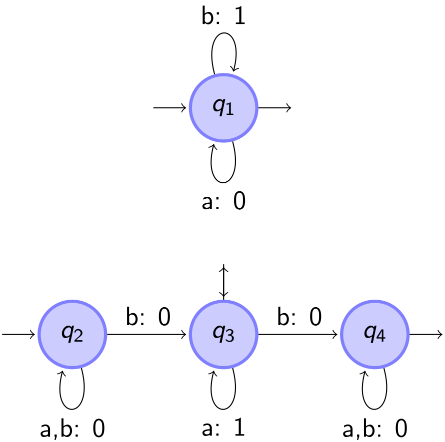

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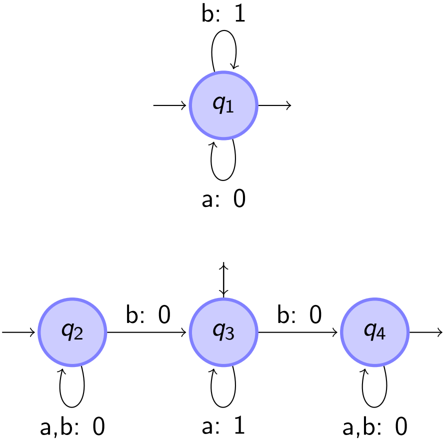

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$$\mu(a_1 a_2 \cdots a_k) = \mu(a_1) \otimes \mu(a_2) \otimes \cdots \otimes \mu(a_k)$$

Distance automata: an algebraic view

For a distance automaton with n states:
μ is a morphism, defined on letters

$$\begin{array}{ccc} \mathbb{A}^* & \to & \Gamma_n \\ w & \mapsto & \mu(w) \end{array}$$

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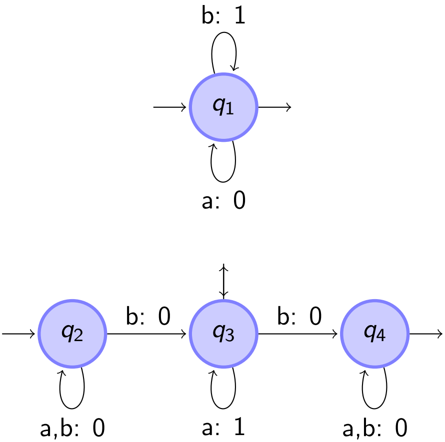

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$$I = (0, 0, 0, \infty) \qquad F = \begin{pmatrix} 0 \\ \infty \\ 0 \\ 0 \end{pmatrix}$$

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- I: vector of initial states, F: vector of final states

$$f(w) = I \otimes \mu(w) \otimes F$$

Some ideas of the proof

Ideas of the proof

First step: reduce the problem to $g = |.|$

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Compare f and $|.|$?

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Operation: $(M, \ell) \otimes (N, k) = (M \otimes N, \ell + k)$
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Start with pairs $(\mu(a), 1)$ for $a \in \mathbb{A}$ and close by product.

It gives an infinite set $\Longrightarrow$ approximate it by a set in some sense "finite".

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- approximation ?
- finite presentation ?

Ideas of the proof

Approximation:

Finite presentation:

Ideas of the proof

Approximation: $\varepsilon > 0$

$$(M,\ell) \preccurlyeq_{\varepsilon} (N,k) \quad \text{if} \quad \begin{array}{l} M \leqslant N + \varepsilon\ell \\ \ell \geqslant k \\ \widetilde{M} = \widetilde{N} \end{array}$$

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$X \preccurlyeq_\varepsilon Y$ if for all $(M,\ell) \in X$, there is $(N,k) \in Y$ s.t $(M,\ell) \preccurlyeq_\varepsilon (N,k)$

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A set is finitely presented if it is a finite union of singleton sets, and of sets of the form $\{(kM, k) : \ k \geqslant \ell\}$ for some ℓ

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Theorem

Given $\varepsilon > 0$ and a finitely presented set X, there is a computable finitely presented set Y such that $< X > \approx_\varepsilon Y$.

Conclusion

Conclusion and further questions

- Given f, g computed by distance automata
  - is $f \leqslant g$? $\rightarrow$ undecidable [Krob]
  - is there a polynomial P s.t $f \leqslant P \circ g$? $\rightarrow$ decidable [Colcombet]
- Our contribution
  - is there an integer a s.t $f \leqslant ag + a$? $\rightarrow$ decidable
  - algorithm of approximate comparison - EXPSPACE (the problem is PSPACE-hard)
- Our algorithm approximates the joint spectral radius in $(\mathbb{N} \cup \{+\infty\}, \min, +)$.
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- Other kinds of asymptotic behaviours

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  - is there an integer a s.t $f \leqslant ag + a$? $\rightarrow$ decidable
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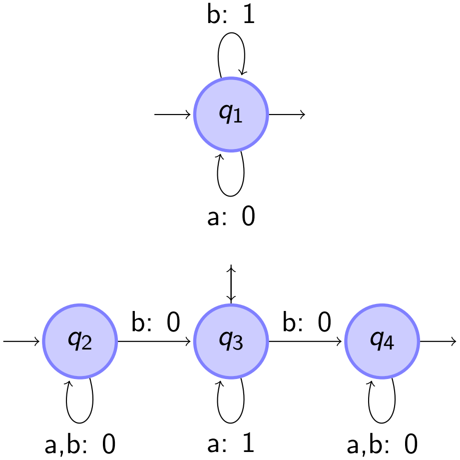
Conclusion and further questions

$$f : a^{n_0} b a^{n_1} b \cdots b a^{n_k} \mapsto \min(n_0, n_1, \ldots, n_k, k)$$



$$f((a^n b)^n a^n) = n$$

$$g : n \mapsto \max_{|w|=n} \{f(w)\}$$

$$g(n) \sim \sqrt{n}$$