Joint spectral radius: the power of automata

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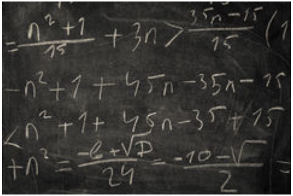
joint work with Pierre Guillon and Glenn Merlet
(I2M, Marseille)

Louvain, 12th July 2016

Two points of view...



$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \quad \begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{bmatrix}$$

$$\begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix} \quad \begin{bmatrix}0 & 1 & 0\\1 & -1 & 1\\0 & 1 & 0\end{bmatrix} \quad \begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\1 & 0 & 0\end{bmatrix}$$

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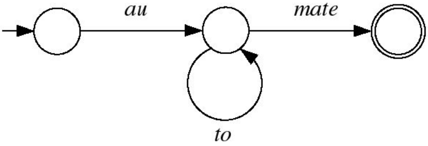
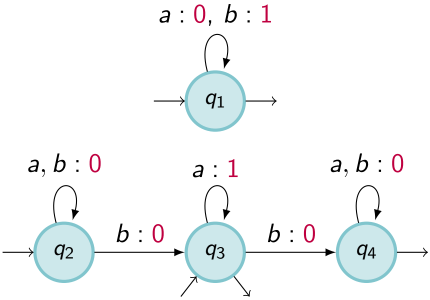


The model under study: Max-plus automata



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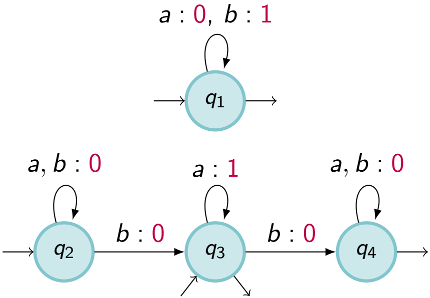

Syntax :

Non deterministic finite automaton for which each transition is labelled by a non negative integer (weight).

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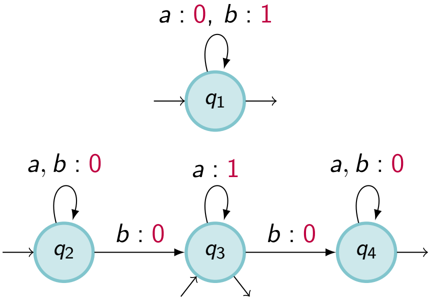

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Semantic :

Weight of a run = sum of the weights of the transitions.

$$\begin{array}{rcl} \mathbb{A}^* & \to & \mathbb{N} \cup \{-\infty\} \\ w & \mapsto & \text{Maximum of the weights of accepting runs labelled by } w \\ & & (-\infty \text{ if no such run}) \end{array}$$

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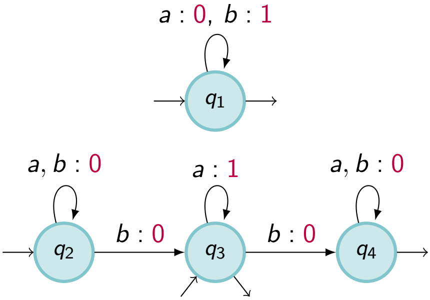

$a^{n_0} b a^{n_1} b \cdots b a^{n_k}$

$\mapsto \max(n_0, n_1, \ldots, n_k, k)$

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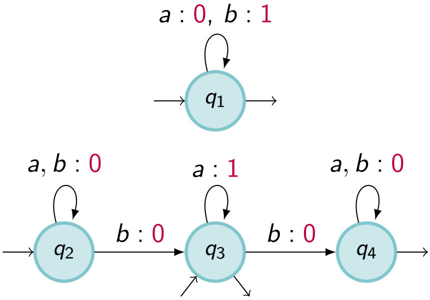

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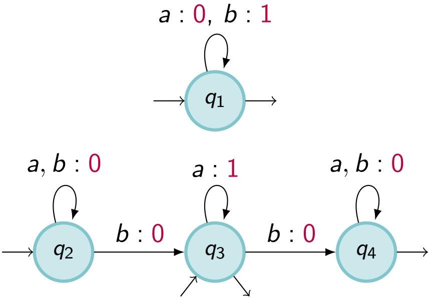

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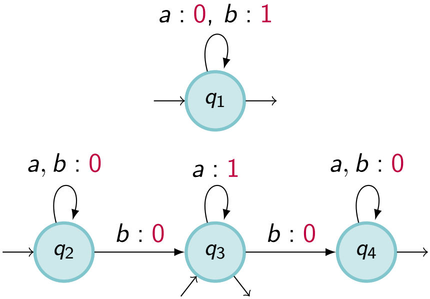

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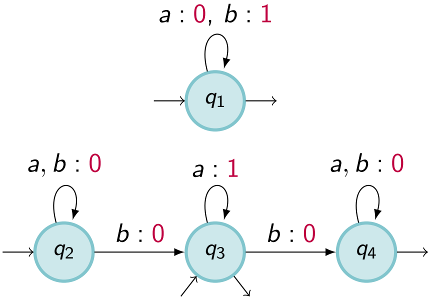

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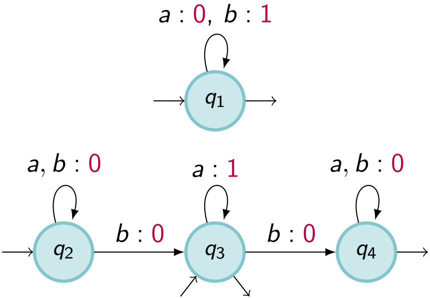

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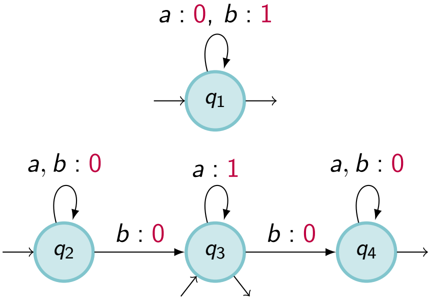

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$\mu(w)_{i,j} =$ max of the weights of the runs from i to j labelled by w

$$f(w) = I\mu(w)F$$

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Going both ways...

Joint spectral radius

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Γ: finite set of matrices of size $d \times d$

Joint spectral radius of Γ:

$$\rho(\Gamma) = \inf_{\ell > 0} \left\{ \frac{1}{\ell} \left\| M_1 \cdots M_\ell \right\|_\infty \,\middle|\, M_1, \ldots, M_\ell \in \Gamma \right\}$$

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Answer: Yes!

Undecidability

Comparison problem: Given a max-plus automaton computing a function f, do we have for all words w, $f(w) \geqslant |w|$?

Theorem [Krob, 92]

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Reduction to the problem of computing the joint spectral radius
$\rightarrow$ even with fixed weights/number of states

Approximation

Approximation problem:

Input: a max-plus automaton computing a function f and $\varepsilon > 0$

Output: a rational r such that $r - \varepsilon \leqslant \inf \left\{ \frac{f(w)}{|w|} \mid w \in A^* \right\} \leqslant r + \varepsilon$

Theorem [Colcombet, D.]

There is an algorithm that solves the approximation problem.

Consequence: There is an algorithm that approximates the joint spectral radius.

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$=$ Is there an identity on finitely generated semigroup of matrices of dimension d?

$d = 2$ [Izhakian, Margolis], $d = 3$ [Shitov], triangular [Izhakian]

Positivity

Problem:

Input: a $\mathbb{Z}$-max-plus automaton computing a function $f \geqslant 0$

Output: "yes" if f is computable by a $\mathbb{N}$-max-plus automaton, "no" otherwise

Is this problem decidable?

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Translation:

Input: a finite set of matrices Γ with coefficients in $\mathbb{Z} \cup \{-\infty\}$ such that all $(M_1 \cdots M_k)_{1,2} \geqslant 0$

Question: is there a finite set Γ' of matrices with coefficients in $\mathbb{N} \cup \{-\infty\}$ and μ a bijection $\Gamma \to \Gamma'$ such that for all $M_1, \dots, M_k \in \Gamma$:

$$(M_1 \cdots M_k)_{1,2} = (\mu(M_1) \cdots \mu(M_k))_{1,2} \text{ ?}$$

Determinisation

Problem:
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Output: "yes" if f is computable by a deterministic max-plus automaton, "no" otherwise

Is this problem decidable?

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Problem:

Input: a max-plus automaton computing a function f

Output: "yes" if f is computable by a deterministic max-plus automaton, "no" otherwise

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Translation:

Input: a finite set of matrices Γ

Question: is there a finite set Γ' of matrices with at most one finite coefficient per row and μ a bijection $\Gamma \to \Gamma'$ such that for all $M_1, \ldots, M_k \in \Gamma$:

$$(M_1 \cdots M_k)_{1,2} = (\mu(M_1) \cdots \mu(M_k))_{1,2} \text{ ?}$$

Conclusion

There are a lot of things to gain by using this connection and taking advantage of the best of both worlds.