Asymptotic Behaviour of Max-Plus and Min-Plus Automata

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based on joint works with Thomas Colcombet (LIAFA)
and Florian Zuleger (TU Vienna)

What are min-plus and max-plus automata ?

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↙

NDFA that compute functions

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Results about the behaviour of these functions

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An algebraic definition with matrices

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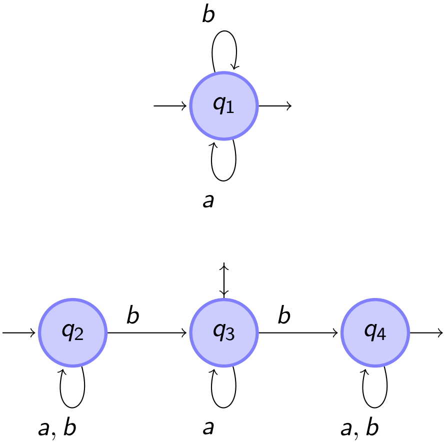
Results about the behaviour of these functions

Ideas of the proofs

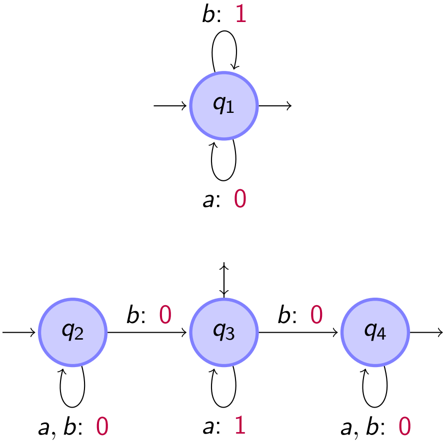
Max-Plus and Min-plus Automata

From NDFA to Min-Plus and Max-Plus Automata



Non deterministic finite automata : $A^* \to \{0, \infty\}$

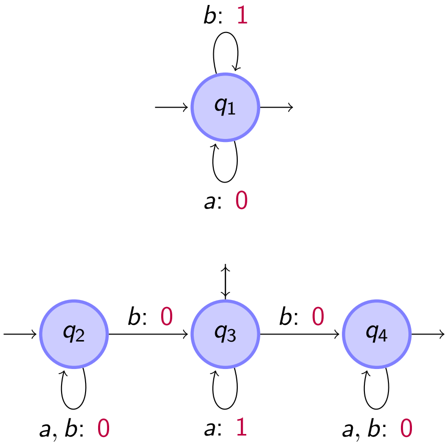
From NDFA to Min-Plus and Max-Plus Automata



Min-plus automata : $A^* \to \mathbb{N} \cup \{+\infty\}$

Max-plus automata : $A^* \to \mathbb{N} \cup \{-\infty\}$

Min-Plus and Max-Plus Automata



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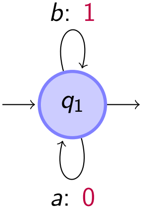
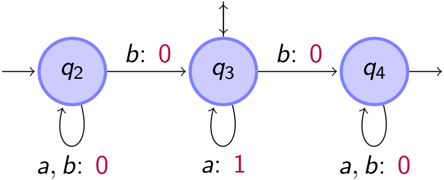


Weight of a run:
sum of the weights of the transitions



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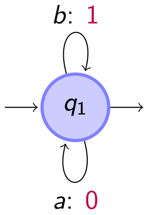

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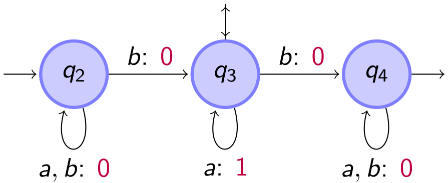

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$w \mapsto$ minimum of the weights of the accepting runs labelled by w

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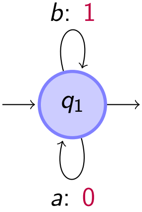

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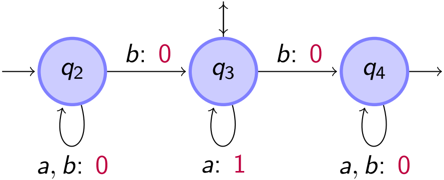

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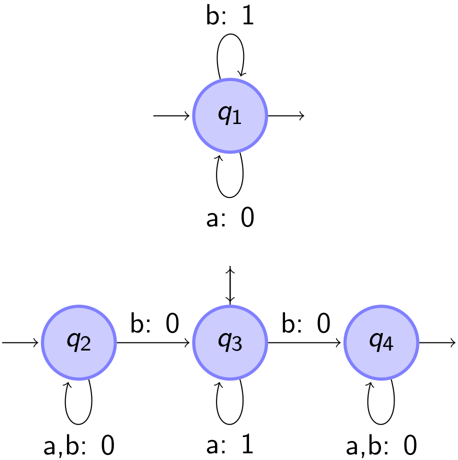
An example



$$a^m ba^n ba^p \mapsto ?$$

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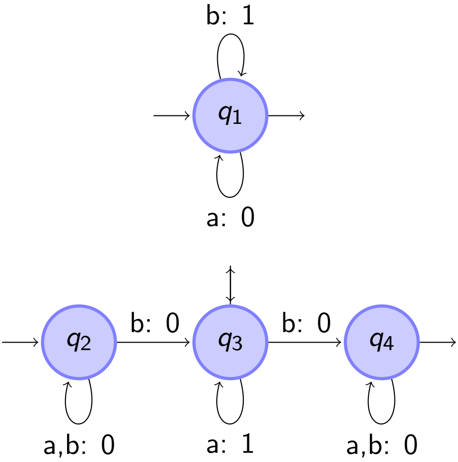
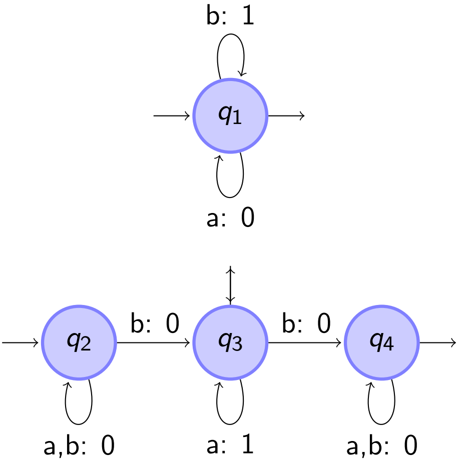


$$a^m b a^n b a^p \mapsto \min(2, m, n, p)$$

An example



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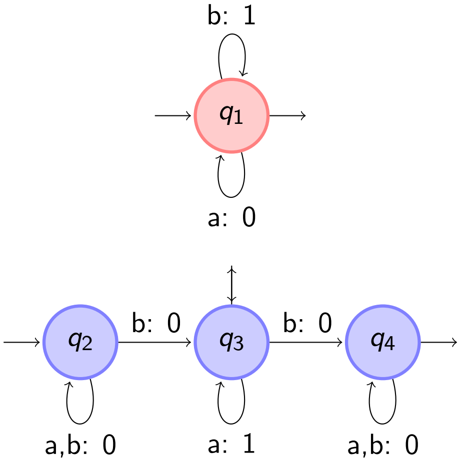

$$a^{n_0} b a^{n_1} b \cdots b a^{n_k} \mapsto ?$$

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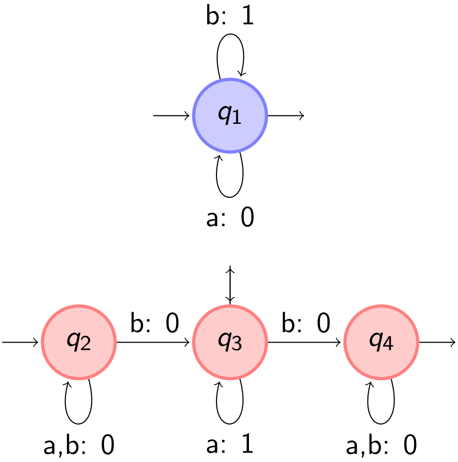

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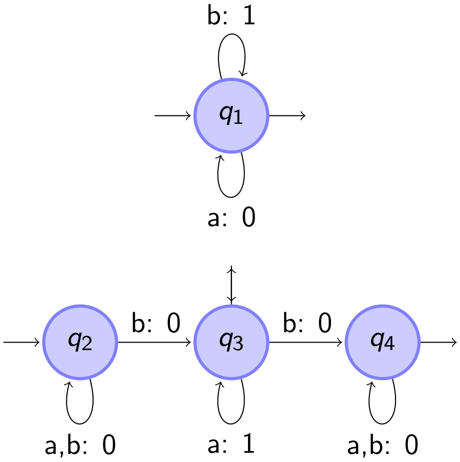

Min-plus: $a^{n_0} b a^{n_1} b \cdots b a^{n_k} \mapsto \min(n_0, n_1, \ldots, n_k, k)$

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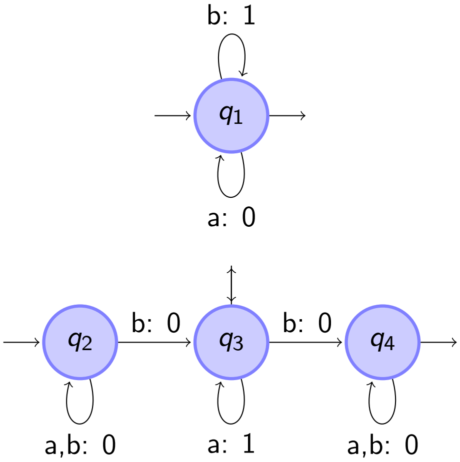

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Max-plus: $a^{n_0}ba^{n_1}b\cdots ba^{n_k} \mapsto \max(n_0, n_1, \ldots, n_k, k)$

Description of the behaviour of computed functions

Asymptotic behaviour

Undecidable: Given f, g computed by min-plus (resp. max-plus) automata, is $f(w) \leqslant g(w)$ for all words w ? [Krob, 92]

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$$h(n) = \sup_{|w| \leqslant n} f(w)$$

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Decidable: Given f computed by min-plus (resp. max-plus) automata, is f bounded ? [max: easy, min: Hashiguchi, 82]

Example

Min-Plus $a^{n_0} b a^{n_1} b \cdots b a^{n_k} \mapsto \min(n_0, n_1, \ldots, n_k, k)$

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What is the maximal weight for words of length at most n ?

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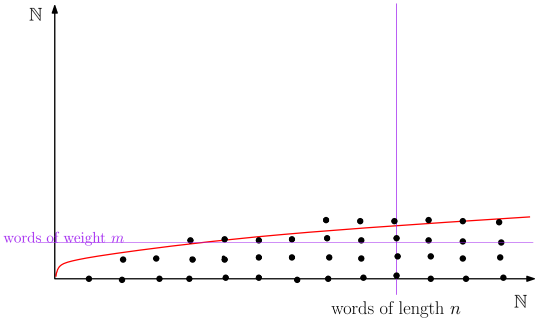
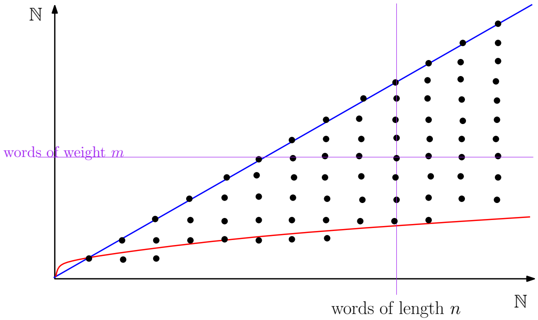


Example

Max-Plus $a^{n_0}ba^{n_1}b\cdots ba^{n_k} \mapsto \max(n_0, n_1, \ldots, n_k, k)$

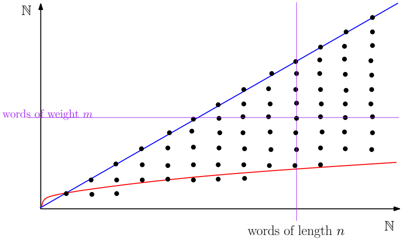
$$h(n) = \inf_{|w|\geqslant n} f(w)$$

What is the minimal weight for words of length at least n ?



Asymptotic equivalent

Max-Plus $h(n) = \inf_{|w| \geqslant n} f(w)$



Theorem: There exists effectively $\alpha \in (\mathbb{Q} \cap [0,1]) \cup \{-\infty\}$ such that:

$$h(n) = \Theta(n^{\alpha})$$

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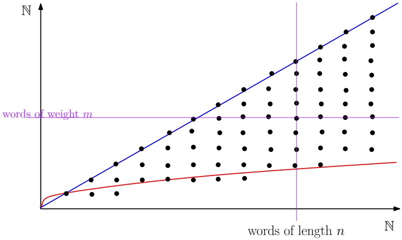

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Min-Plus $h(n) = \sup_{|w| \leqslant n} f(w)$

$h_1 \leqslant h \leqslant h_2$ with $h_1(n) = O(n^{\frac{1}{p+1}})$, $h_2 = O(n^{\frac{1}{p}})$ for some integer p [Simon, 90].

Ratio function-length

Min-Plus

$$h(n) = \sup_{|w| \leqslant n} f(w)$$

$$r = \sup_{n} \frac{h(n)}{n}$$

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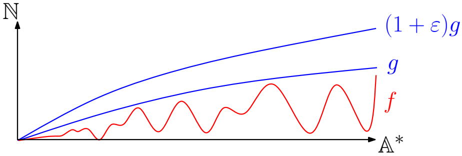
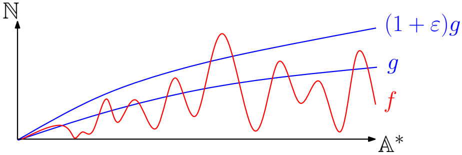
Theorem: There is an algorithm that, given a min-plus automaton (resp. a max-plus automaton), and $\varepsilon > 0$, computes r up to ε.

Approximate comparison

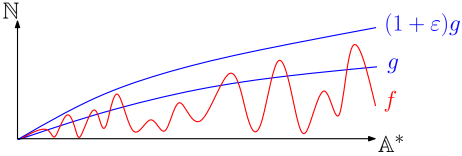
f,g computed by min-plus automata - Case $\varepsilon = 0$: undecidable result of Krob.



$\Longrightarrow$ YES



$\Longrightarrow$ NO

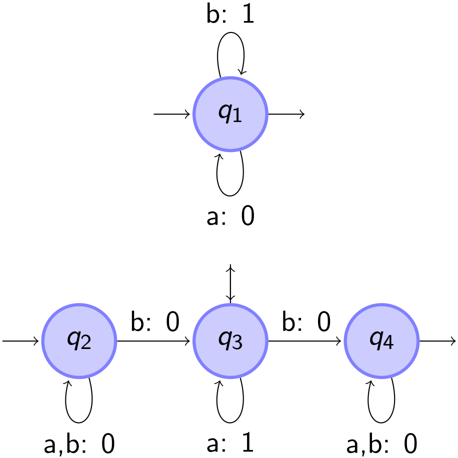


$\Longrightarrow$ YES or NO

Algebraic definition

Automata: an algebraic view

Min-Plus



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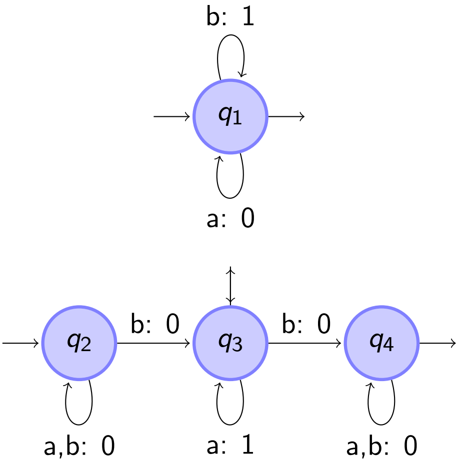

Semiring: $(\mathbb{N} \cup \{+\infty\}, \min, +)$

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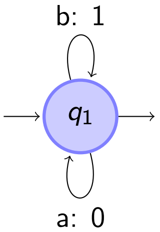

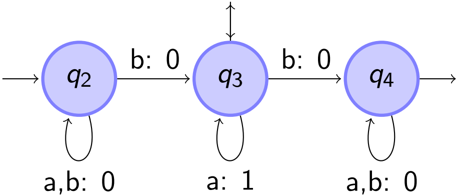

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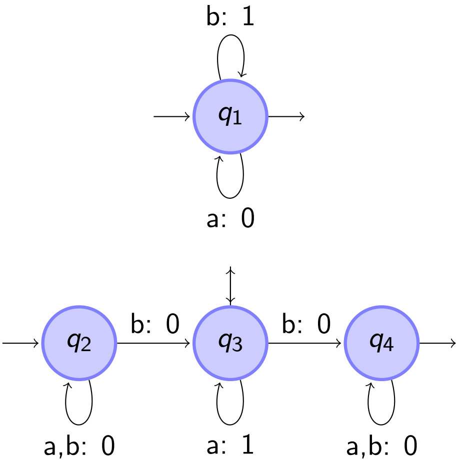

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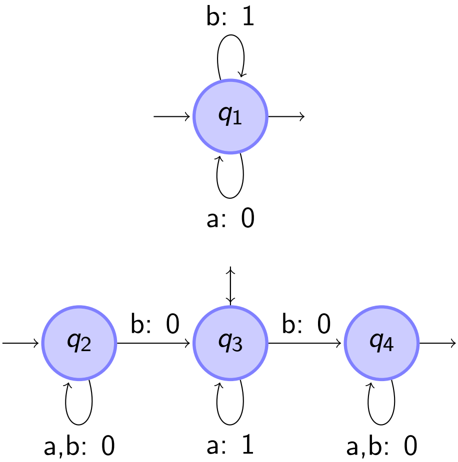

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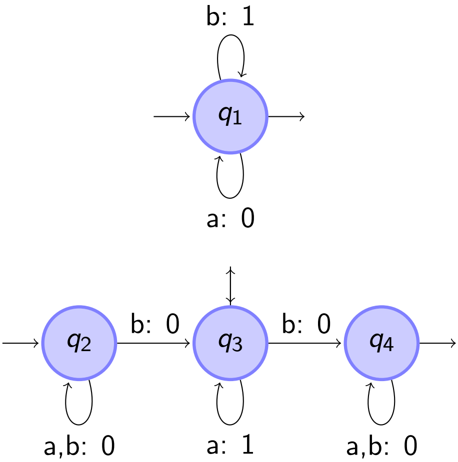

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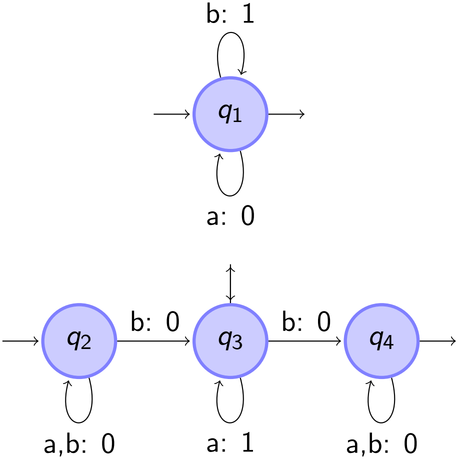

Semiring: $(\mathbb{N} \cup \{-\infty\}, \max, +)$

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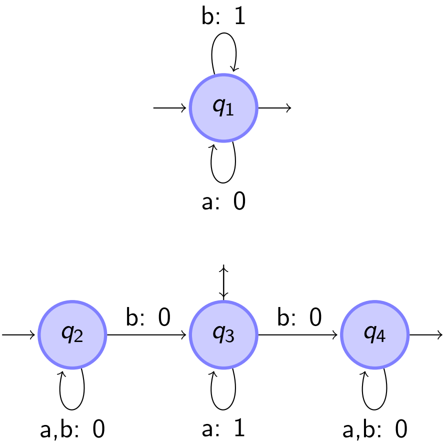

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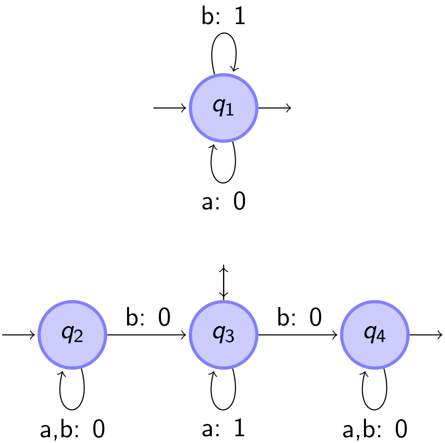

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$$f(w) = I \otimes \mu(w) \otimes F$$

Ideas of proofs

Weighted matrices

Compare $f(w)$ with $|w|$

Weighted matrices

Compare ~~$f(w)$~~ $\mu(w)$ with $|w|$

Weighted matrices

Compare ~~$f(w)$~~ $\mu(w)$ with $|w|$ $\longrightarrow$ Describe $\{(\mu(w), |w|) \mid w \in A^*\}$

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Weighted matrices:

$$(M, \ell)$$

M: matrix $\qquad \ell \in \mathbb{N} - \{0\}$

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$$(M,\ell)\otimes(M',\ell') = (M\otimes M', \ell+\ell')$$

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matrix ↙ ↘ $\in \mathbb{N} - \{0\}$

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Weighted matrices

Compare ~~$f(w)$~~ $\mu(w)$ with $|w|$ ⟶ Describe $\{(\mu(w), |w|) \mid w \in A^*\}$

↓

Describe $\langle\{(\mu(a), 1) \mid a \in A\}\rangle$

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M ↙ matrix ℓ ↘ $\in \mathbb{N} - \{0\}$

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$$= (\mu(w) \otimes \mu(w'), |w| + |w|')$$

Weighted matrices

Compare ~~$f(w)$~~ $\mu(w)$ with $|w|$ $\longrightarrow$ Describe $\{(\mu(w), |w|) \mid w \in A^*\}$

$\downarrow$

Describe $\langle\{(\mu(a), 1) \mid a \in A\}\rangle$

$\swarrow$

Impossible to describe precisely

$=$

Approximation $\approx$

Weighted matrices:

(M, ℓ) — M: matrix, $\ell \in \mathbb{N} - \{0\}$

$$(M,\ell)\otimes(M',\ell') = (M\otimes M', \ell+\ell')$$

$$(\mu(w), |w|) \otimes (\mu(w'), |w|') = (\mu(w) \otimes \mu(w'), |w| + |w|')$$

Weighted matrices

Compare ~~$f(w)$~~ $\mu(w)$ with $|w|$ → Describe $\{(\mu(w), |w|) \mid w \in A^*\}$

↓

Describe $\langle\{(\mu(a), 1) \mid a \in A\}\rangle$

Impossible to describe precisely = Approximation $\approx$

With nice sets = Presentable sets

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Weighted matrices:

$$(M, \ell)$$

matrix $\quad \in \mathbb{N} - \{0\}$

$$(M, \ell) \otimes (M', \ell') = (M \otimes M', \ell + \ell')$$

$$(\mu(w), |w|) \otimes (\mu(w'), |w|') = (\mu(w) \otimes \mu(w'), |w| + |w|')$$

Given a finite set X of weighted matrices, one can compute a presentable set Y of weighted matrices such that $\langle X \rangle \approx Y$.

Approximation

Theorem: There is an algorithm that, given a min-plus automaton computing a function f, and $\varepsilon > 0$, approximates, up to ε, the value:

$$\sup_{w \in A^*} \frac{f(w)}{|w|}$$

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$$(M, \ell) \preccurlyeq_\varepsilon (N, k) \quad \text{if}$$

- $M \leqslant N + \varepsilon\ell$
- $\ell \geqslant k$
- $\varphi(M) = \varphi(N)$

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$X \preccurlyeq_\varepsilon Y$ if for all $(M, \ell) \in X$, there is $(N, k) \in Y$ such that $(M, \ell) \preccurlyeq_\varepsilon (N, k)$.

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- $M \leqslant N + \varepsilon\ell$
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$X \preccurlyeq_\varepsilon Y$ if for all $(M, \ell) \in X$, there is $(N, k) \in Y$ such that $(M, \ell) \preccurlyeq_\varepsilon (N, k)$.

$X \approx_\varepsilon Y$ if $X \preccurlyeq_\varepsilon Y$ and $Y \preccurlyeq_\varepsilon X$.

Presentable sets

Theorem: There is an algorithm that, given a min-plus automaton computing a function f, and $\varepsilon > 0$, approximates, up to ε, the value:

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Presentable sets

Theorem: There is an algorithm that, given a min-plus automaton computing a function f, and $\varepsilon > 0$, approximates, up to ε, the value:

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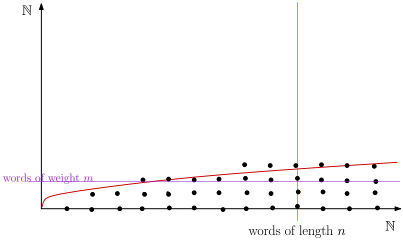
A set is finitely presented if it is a finite union of:

- singleton sets,
- sets of the form $\{(kM, k) : \ k \geqslant \ell\}$ for some ℓ.

Conclusion and further questions

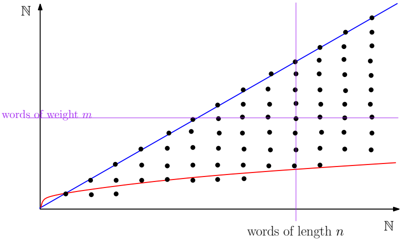
Min-plus

$$h(n) = \sup_{|w| \leqslant n} f(w)$$



Max-plus

$$h(n) = \inf_{|w| \geqslant n} f(w)$$



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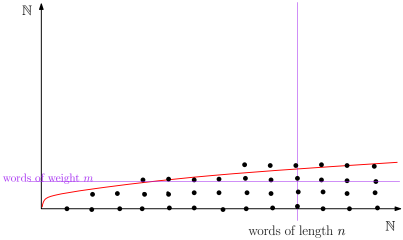

Approximation of $\sup_n \frac{h(n)}{n}$

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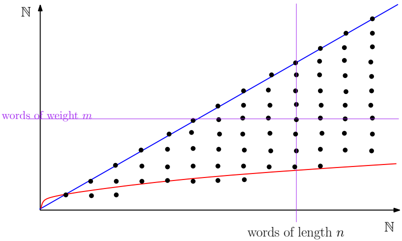

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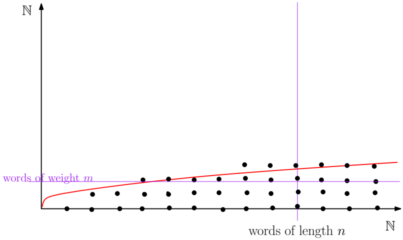

Approximation of $\sup_n \frac{h(n)}{n}$

Approximate comparison

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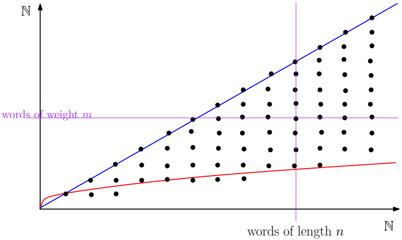


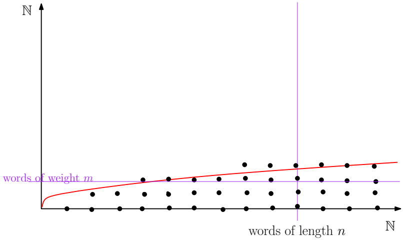
Approximation of $\inf_n \frac{h(n)}{n}$

??

Conclusion and further questions

Min-plus

$$h(n) = \sup_{|w| \leqslant n} f(w)$$



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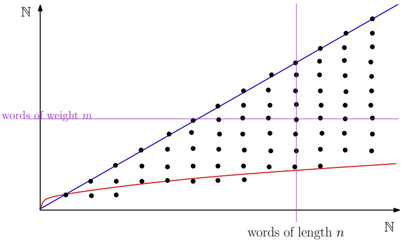


Approximation of $\inf_n \frac{h(n)}{n}$

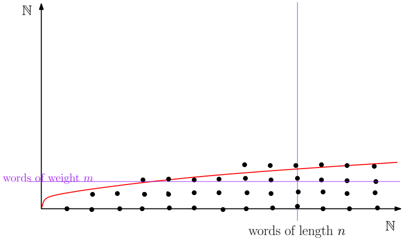
??

$$h(n) = \Theta(n^{\alpha})$$

Conclusion and further questions

Min-plus

$$h(n) = \sup_{|w| \leqslant n} f(w)$$



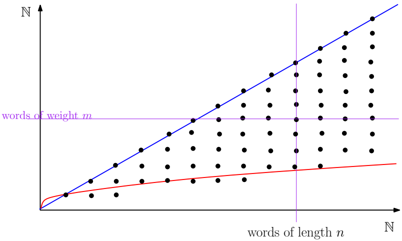
Approximation of $\sup_n \frac{h(n)}{n}$

Approximate comparison

??

Max-plus

$$h(n) = \inf_{|w| \geqslant n} f(w)$$



Approximation of $\inf_n \frac{h(n)}{n}$

??

$h(n) = \Theta(n^{\alpha})$

Describe h as cn^{α} with α rational and c up to ε