When is containment decidable for probabilistic automata?

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The containment problem

\[ A \subseteq B \]
The containment problem

$[A] \subseteq [B]$
The containment problem

Languages over $\Sigma^*$

$[A] \subseteq [B]$

Boolean automata over $\Sigma^*$

Functions:

$\Sigma^* \rightarrow \mathbb{R}$

Weighted automata

Check whether:

$[B] \cap [A] = \emptyset$

Check whether:

$[B] - [A] \geq 0$

The containment problem
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Boolean automata over $\Sigma^*$

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Check whether:

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Boolean automata over \( \Sigma^* \)

Functions: \( \Sigma^* \rightarrow \mathbb{R} \)

Check whether:
\[ [B]^c \cap [A] = \emptyset \]
The containment problem

\[ \mathcal{A} \subseteq \mathcal{B} \]

Weighted automata over \( \Sigma^* \)

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Functions: $\Sigma^* \rightarrow \mathbb{R}$

Weighted automata over $\Sigma^*$

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$[B]^c \cap [A] = \emptyset$
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Weightsided automata

over \( \Sigma^* \)

Functions: \( \Sigma^* \rightarrow \mathbb{R} \)

\([A] \preceq [B]\)

Check whether:

\([B] - [A] \geq 0\)
What is the probability that after 8 hours I have done some sport or work?
Initial states and transitions are weighted with probability:

$[A]:$ $w \mapsto$ probability to read $w$ from an initial to a final state.
A few results
A few results

Probabilistic automata

Max-plus automata
A few results

**Probabilistic automata**

- Undecidable in general
  - Post correspondence problem - Paz, Bertoni...

**Max-plus automata**

- Undecidable in general
  - Diophantine equations - Krob
Notion of ambiguity

How many accepting runs are labelled by a given word?
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Unambiguous: for all words, at most 1
Finitely ambiguous: for all words, at most $k$
Linearly ambiguous: for all words $w$, at most $k|w|
Quadratic: for all words $w$, at most $k|w|^2$
Polynomially ambiguous, exponentially ambiguous...
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- Undecidable for quadratic ambiguous
  Post correspondence problem - Fijalkow-Riveros-Worrell

Max-plus automata

- Undecidable in general
  Diophantine equations - Krob

- Undecidable for linearly ambiguous
  Halting problem of two-counter machines - Colcombet, Amalgor-Boker-Kupferman
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- Emptiness problem decidable for finitely ambiguous
  Fijalkow-Riveros-Worrell

Max-plus automata

- Undecidable in general
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- Undecidable for linearly ambiguous
  Halting problem of two-counter machines - Colcombet, Amalgor-Boker-Kupferman

- Decidable for finitely ambiguous
  Filiot-Gentilini-Raskin
When is containment decidable?

\[ \mathcal{A} \subseteq \mathcal{B} \]

### Undecidable
- When either \( \mathcal{A} \) or \( \mathcal{B} \) is at least linearly ambiguous.

### Decidable
- When \( \mathcal{A} \) and \( \mathcal{B} \) are finitely ambiguous and one is unambiguous.

### Open
- When \( \mathcal{A} \) and \( \mathcal{B} \) are finitely ambiguous.
Are there positive integers \( x \) and \( y \) such that:

\[
p \cdot \left( \frac{1}{12} \right)^x \cdot \left( \frac{1}{18} \right)^y + \left( 1 - p \right) \cdot \left( \frac{1}{3} \right)^x \cdot \left( \frac{1}{18} \right)^y < \left( \frac{1}{5} \right)^x \cdot \left( \frac{1}{15} \right)^y
\]

Equivalently:

\[
e \log(p) - x \log(2) + y \log(3) + e \log(1 - p) + x \log(2) - y \log(3) < \frac{19}{15}
\]
Are there positive integers \( x \) and \( y \) such that:

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Are there positive integers $x$ and $y$ such that:

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Equivalently:

$$e^{\log(p) - x \log(2) + y \log(3)} + e^{\log(1-p) + x \log(2) - y \log(3)} < 1$$
Decidability: one example

\[ e^{\log(p) - x \log(2) + y \log(3)} + e^{\log(1-p) + x \log(2) - y \log(3)} < 1 \]
Decidability: one example

\[ e^{\log(p) - x \log(2) + y \log(3)} + e^{\log(1-p) + x \log(2) - y \log(3)} < 1 \]

Are there positive integers \( x, y \) s.t:

- \( e^u + e^v < 1 \) where:
  - \( u = \log(p) - x \log(2) + y \log(3) \)
  - \( v = \log(1 - p) + x \log(2) - y \log(3) \)
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- \(e^u + e^v < 1\) where:
  - \(u = \log(p) - x \log(2) + y \log(3)\)
  - \(v = \log(1 - p) + x \log(2) - y \log(3)\)

\[ \rightarrow \text{YES if and only if } p \neq \frac{1}{2}. \]
Decidability: translating the problem

Is there a word $w$ such that $[A](w) > [B](w)$?
Decidability: translating the problem

Is there a word $w$ such that $\mathcal{[A]}(w) > \mathcal{[B]}(w)$?

Simple cycle decomposition
Decidability: translating the problem

Is there a word $w$ such that $\llbracket A \rrbracket(w) > \llbracket B \rrbracket(w)$?

Given $A$ ($k$-ambiguous) and $B$ ($\ell$-ambiguous), one can compute:
- a positive integer $n$,
- a finite set of tuples $(p, q, r, s)$ with
  - $p$ in $\mathbb{Q}_0^k$, $r$ in $\mathbb{Q}_0^{\ell}$, $q$ in $\mathbb{Q}_0^{k \times n}$, $s$ in $\mathbb{Q}_0^{\ell \times n}$,

such that for one of those tuples, there exist $x \in \mathbb{N}^n$ such that:

$$
\sum_{i=1}^{k} p_i q_{i,1}^{x_1} \cdots q_{i,n}^{x_n} > \sum_{i=1}^{\ell} r_i s_{i,1}^{x_1} \cdots s_{i,n}^{x_n}
$$

if and only if there exist a word $w$ such that $\llbracket A \rrbracket(w) > \llbracket B \rrbracket(w)$. 
Decidability: first case

$[A] \leq [B]$ when $B$ is unambiguous.
Decidability: first case

\[ [A] \leq [B] \text{ when } B \text{ is unambiguous.} \]

Is there \( x \in \mathbb{N}^n \) such that:

\[
\sum_{i=1}^{k} p_i q_{i,1}^{x_1} \cdots q_{i,n}^{x_n} > r s_{1}^{x_1} \cdots s_{n}^{x_n}
\]
Decidability: first case

\([A] \leq [B]\) when \(B\) is unambiguous.

Is there \(x \in \mathbb{N}^n\) such that:

\[
\sum_{i=1}^{k} p_i q_{i,1}^{x_1} \cdots q_{i,n}^{x_n} > rs_1^{x_1} \cdots s_n^{x_n}
\]

- First case: there is \(i, j\) such that \(q_{i,j} > s_j\)
- Second case: for all \(i, j\), \(q_{i,j} \leq s_j\)
Decidability: second case

Much more difficult!

**Theorem**

Determining whether \([A] \leq [B]\) is decidable when \(A\) is unambiguous and \(B\) is finitely ambiguous, assuming Schanuel’s conjecture is true.
Determining whether $[\mathcal{A}] \preceq [\mathcal{B}]$ is decidable when $\mathcal{A}$ is unambiguous and $\mathcal{B}$ is finitely ambiguous, assuming Schanuel’s conjecture is true.

Is there $x \in \mathbb{N}^n$ such that:

$$\sum_{i=1}^{k} p_i q_{i,1}^{x_1} \cdots q_{i,n}^{x_n} < 1$$

Decidability: second case
Decidability: second case

Much more difficult!

**Theorem**

Determining whether $[A] \leq [B]$ is decidable when $A$ is unambiguous and $B$ is finitely ambiguous, assuming Schanuel’s conjecture is true.

Is there $\mathbf{x} \in \mathbb{N}^n$ such that:

$$\sum_{i=1}^{k} p_i q_{i,1}^{x_1} \cdots q_{i,n}^{x_n} < 1$$

- semi-decidable to find such $\mathbf{x}$
Decidability: second case

Much more difficult!

**Theorem**

Determining whether $[A] \leq [B]$ is decidable when $A$ is unambiguous and $B$ is finitely ambiguous, assuming Schanuel’s conjecture is true.

Is there $x \in \mathbb{N}^n$ such that:

$$\sum_{i=1}^{k} p_i q_i^{x_1} \cdots q_i^{x_n} < 1$$

- semi-decidable to find such $x$
- if there is no such $x$, there is a non-zero vector $d \in \mathbb{Z}^n$ and $a, b \in \mathbb{Z}$ such that $\{d^\top y \mid y \text{ is a real solution} \} \subseteq [a, b]$
  $\rightarrow$ decrease the dimension by 1
Undecidability

Proposition

Given a two-counter machine, one can construct two linearly ambiguous probabilistic automata $A$ and $B$, such that the machine halts if and only if there exists a word $w$ such that $[A](w) \leq [B](w)$.
Given a two-counter machine, one can construct two linearly ambiguous probabilistic automata \( \mathcal{A} \) and \( \mathcal{B} \), such that the machine halts if and only if there exists a word \( w \) such that \( \llbracket \mathcal{A} \rrbracket(w) \leq \llbracket \mathcal{B} \rrbracket(w) \).

Simulate an execution with a word: \( a^n b^m t_1 a^{n+1} b^m t_2 a^{n+1} b^{m'} \)
Given a two-counter machine, one can construct two linearly ambiguous probabilistic automata $A$ and $B$, such that the machine halts if and only if there exists a word $w$ such that $[A](w) \leq [B](w)$.

→ Simulate an execution with a word: $a^n b^m t_1 a^{n+1} b^m t_2 a^{n+1} b^{m'}$
Conclusion

Containment problem for finitely ambiguous probabilistic automata?