When is containment decidable for probabilistic automata?

Laure Daviaud
University of Warwick

Joint work with Marcin Jurdziński, Ranko Lazić, Filip Mazowiecki, Guillermo A.Pérez and James Worrell.

Plume, Lyon, 23-04-2018

The containment problem

$$[\![\mathcal{A}]\!] \subseteq [\![\mathcal{B}]\!]$$

The containment problem

Boolean automata over Σ^*

$$[\![\mathcal{A}]\!] \subseteq [\![\mathcal{B}]\!]$$

The containment problem

Boolean automata over Σ^*

$$\underbrace{[\![\mathcal{A}]\!]}\subseteq\underbrace{[\![\mathcal{B}]\!]}$$

Languages over Σ^*

The containment problem

Boolean automata over Σ^*

$$\underbrace{[\![\mathcal{A}]\!]}_{} \subseteq \underbrace{[\![\mathcal{B}]\!]}_{}$$

Languages over Σ^*

Check whether:
$[\![\mathcal{B}]\!]^c \cap [\![\mathcal{A}]\!] = \emptyset$

The containment problem

Boolean automata over Σ^*

$$\underbrace{[\![\mathcal{A}]\!]}\subseteq\underbrace{[\![\mathcal{B}]\!]}$$

Functions: $\Sigma^* \to \mathbb{R}$

Check whether:
$[\![\mathcal{B}]\!]^c \cap [\![\mathcal{A}]\!] = \emptyset$

The containment problem

Weighted automata over Σ^*

$$\underbrace{[\![\mathcal{A}]\!]}\subseteq\underbrace{[\![\mathcal{B}]\!]}$$

Functions: $\Sigma^* \to \mathbb{R}$

Check whether:
$[\![\mathcal{B}]\!]^c \cap [\![\mathcal{A}]\!] = \emptyset$

The containment problem

Weighted automata over Σ^*

$$\underbrace{[\![\mathcal{A}]\!]}\ \leqslant\ \underbrace{[\![\mathcal{B}]\!]}$$

Functions: $\Sigma^* \to \mathbb{R}$

Check whether:
$[\![\mathcal{B}]\!]^c \cap [\![\mathcal{A}]\!] = \emptyset$

The containment problem

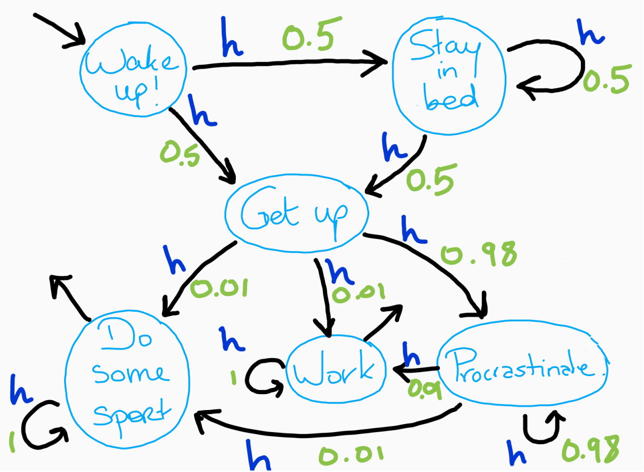
Weighted automata over Σ^*

$$\underbrace{[\![\mathcal{A}]\!]}\leqslant\underbrace{[\![\mathcal{B}]\!]}$$

Functions: $\Sigma^* \to \mathbb{R}$

Check whether:
$[\![\mathcal{B}]\!] - [\![\mathcal{A}]\!] \geqslant 0$

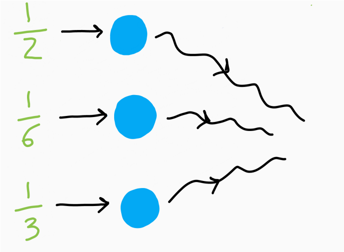
Probabilistic automata

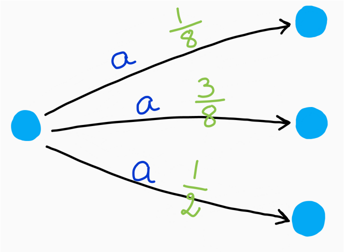


What is the probability that after 8 hours I have done some sport or work?

Probabilistic automata

Initial states and transitions are weighted with probability:





$[\![\mathcal{A}]\!]$: $w \mapsto$ probability to read w from an initial to a final state.

A few results

A few results

Probabilistic automata

Max-plus automata

A few results

Probabilistic automata

- Undecidable in general
  Post correspondence problem - Paz, Bertoni...

Max-plus automata

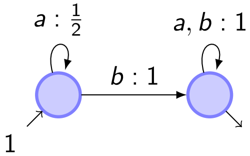
- Undecidable in general
  Diophantine equations - Krob

Notion of ambiguity

How many accepting runs are labelled by a given word?

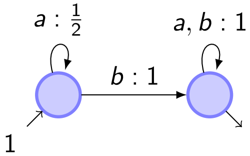
Notion of ambiguity

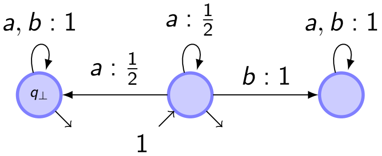
How many accepting runs are labelled by a given word?



Notion of ambiguity

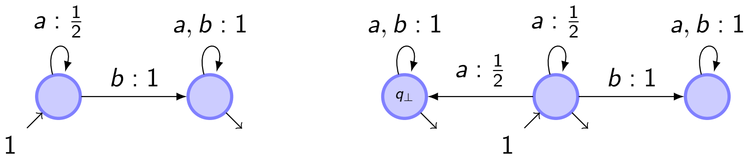
How many accepting runs are labelled by a given word?





Notion of ambiguity

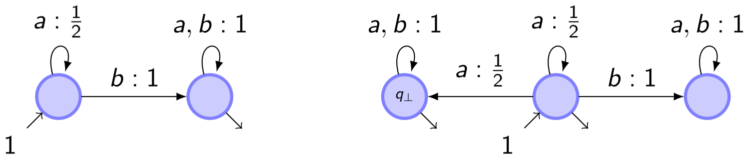
How many accepting runs are labelled by a given word?



- Unambiguous: for all words, at most 1

Notion of ambiguity

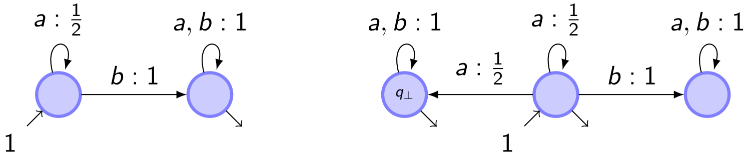
How many accepting runs are labelled by a given word?



- Unambiguous: for all words, at most 1
- Finitely ambiguous: for all words, at most k

Notion of ambiguity

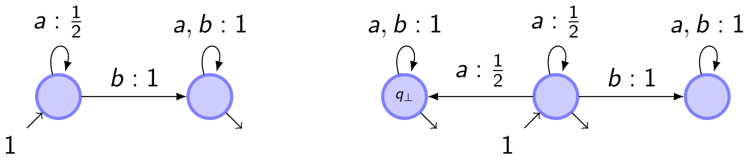
How many accepting runs are labelled by a given word?



- Unambiguous: for all words, at most 1
- Finitely ambiguous: for all words, at most k
- Linearly ambiguous: for all words w, at most $k|w|$

Notion of ambiguity

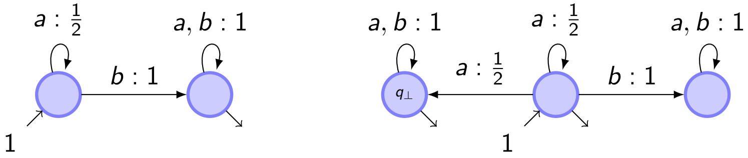
How many accepting runs are labelled by a given word?



- Unambiguous: for all words, at most 1
- Finitely ambiguous: for all words, at most k
- Linearly ambiguous: for all words w, at most $k|w|$
- Quadratic: for all words w, at most $k|w|^2$

Notion of ambiguity

How many accepting runs are labelled by a given word?



- Unambiguous: for all words, at most 1
- Finitely ambiguous: for all words, at most k
- Linearly ambiguous: for all words w, at most $k|w|$
- Quadratic: for all words w, at most $k|w|^2$
- Polynomially ambiguous, exponentially ambiguous...

A few results

Probabilistic automata

- Undecidable in general
  Post correspondence problem - Paz, Bertoni...

Max-plus automata

- Undecidable in general
  Diophantine equations - Krob

A few results

Probabilistic automata

- Undecidable in general
  Post correspondence problem - Paz, Bertoni...
- Undecidable for quadratic ambiguous
  Post correspondence problem - Fijalkow-Riveros-Worrell

Max-plus automata

- Undecidable in general
  Diophantine equations - Krob
- Undecidable for linearly ambiguous
  halting problem of two-counter machines - Colcombet, Amalgor-Boker-Kupferman

A few results

Probabilistic automata

- Undecidable in general
  Post correspondence problem - Paz, Bertoni...
- Undecidable for quadratic ambiguous
  Post correspondence problem - Fijalkow-Riveros-Worrell
- Emptiness problem decidable for finitely ambiguous
  Fijalkow-Riveros-Worrell

Max-plus automata

- Undecidable in general
  Diophantine equations - Krob
- Undecidable for linearly ambiguous
  halting problem of two-counter machines - Colcombet, Amalgor-Boker-Kupferman
- Decidable for finitely ambiguous
  Filiot-Gentilini-Raskin

When is containment decidable?

$$\llbracket \mathcal{A} \rrbracket \leqslant \llbracket \mathcal{B} \rrbracket$$

Undecidable

When either $\mathcal{A}$ or $\mathcal{B}$ is at least linearly ambiguous.

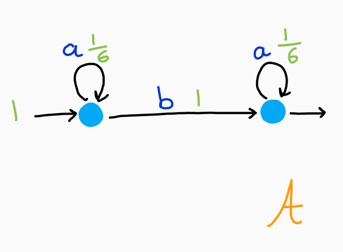
Decidable

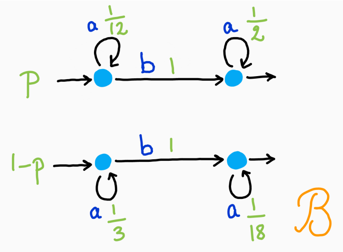
When $\mathcal{A}$ and $\mathcal{B}$ are finitely ambiguous and one is unambiguous.

Open

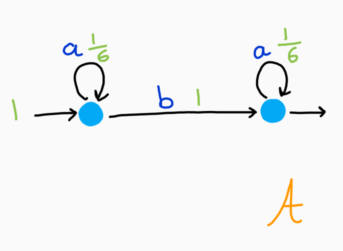
When $\mathcal{A}$ and $\mathcal{B}$ are finitely ambiguous.

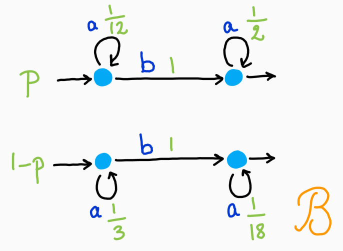
Decidability: one example





Decidability: one example

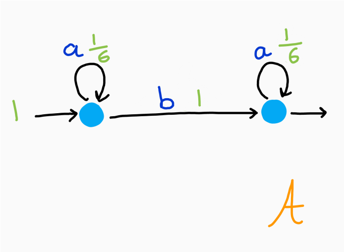


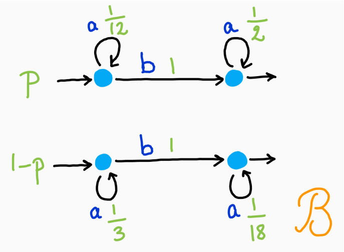


Are there positive integers x and y such that:

$$p \cdot \left(\frac{1}{12}\right)^x \cdot \left(\frac{1}{2}\right)^y + (1-p) \cdot \left(\frac{1}{3}\right)^x \cdot \left(\frac{1}{18}\right)^y < \left(\frac{1}{6}\right)^x \cdot \left(\frac{1}{6}\right)^y$$

Decidability: one example





Are there positive integers x and y such that:

$$p \cdot \left(\frac{1}{12}\right)^x \cdot \left(\frac{1}{2}\right)^y + (1-p) \cdot \left(\frac{1}{3}\right)^x \cdot \left(\frac{1}{18}\right)^y < \left(\frac{1}{6}\right)^x \cdot \left(\frac{1}{6}\right)^y$$

Equivalently:

$$e^{\log(p) - x\log(2) + y\log(3)} + e^{\log(1-p) + x\log(2) - y\log(3)} < 1$$

Decidability: one example

$$e^{\log(p)-x\log(2)+y\log(3)} + e^{\log(1-p)+x\log(2)-y\log(3)} < 1$$

Decidability: one example

$$e^{\log(p)-x\log(2)+y\log(3)} + e^{\log(1-p)+x\log(2)-y\log(3)} < 1$$

Are there positive integers x, y s.t:

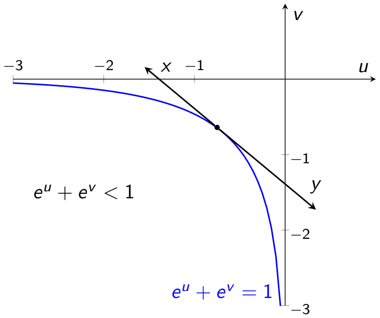
- $e^u + e^v < 1$ where:
- $u = \log(p) - x\log(2) + y\log(3)$
- $v = \log(1-p) + x\log(2) - y\log(3)$

Decidability: one example

$$e^{\log(p)-x\log(2)+y\log(3)}+e^{\log(1-p)+x\log(2)-y\log(3)}<1$$

Are there positive integers x, y s.t:

- $e^u + e^v < 1$ where:
- $u = \log(p) - x\log(2) + y\log(3)$
- $v = \log(1-p) + x\log(2) - y\log(3)$

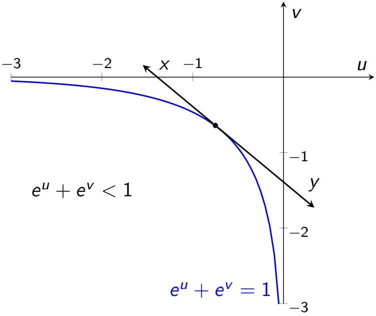


Decidability: one example

$$e^{\log(p)-x\log(2)+y\log(3)} + e^{\log(1-p)+x\log(2)-y\log(3)} < 1$$

Are there positive integers x, y s.t:

- $e^u + e^v < 1$ where:
- $u = \log(p) - x\log(2) + y\log(3)$
- $v = \log(1-p) + x\log(2) - y\log(3)$



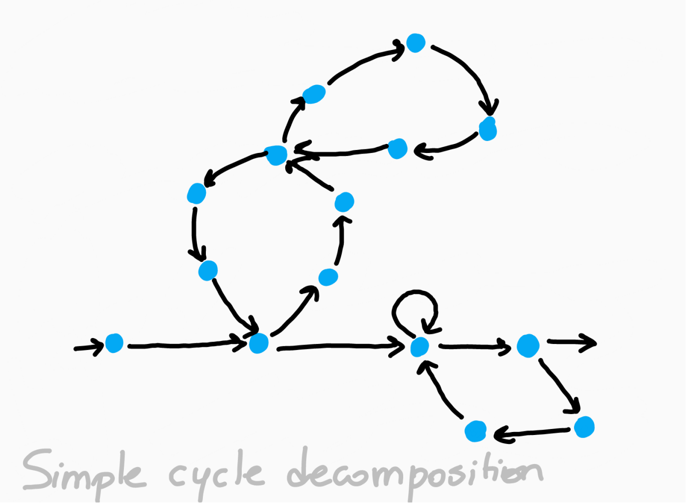
$\longrightarrow$ YES if and only if $p \neq \frac{1}{2}$.

Decidability: translating the problem

Is there a word w such that $[\![\mathcal{A}]\!](w) > [\![\mathcal{B}]\!](w)$?

Decidability: translating the problem

Is there a word w such that $[\![\mathcal{A}]\!](w) > [\![\mathcal{B}]\!](w)$?



Decidability: translating the problem

Is there a word w such that $[\![\mathcal{A}]\!](w) > [\![\mathcal{B}]\!](w)$?

Given $\mathcal{A}$ (k-ambiguous) and $\mathcal{B}$ (ℓ-ambiguous), one can compute:

- a positive integer n,
- a finite set of tuples $(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s})$ with
  - $\boldsymbol{p}$ in $\mathbb{Q}^{k}_{>0}$, $\boldsymbol{r}$ in $\mathbb{Q}^{\ell}_{>0}$, $\boldsymbol{q}$ in $\mathbb{Q}^{k\times n}_{>0}$, $\boldsymbol{s}$ in $\mathbb{Q}^{\ell\times n}_{>0}$,

such that for one of those tuples, there exist $\boldsymbol{x} \in \mathbb{N}^n$ such that:

$$\sum_{i=1}^{k} p_i q_{i,1}^{x_1} \dots q_{i,n}^{x_n} > \sum_{i=1}^{\ell} r_i s_{i,1}^{x_1} \dots s_{i,n}^{x_n}$$

if and only if there exist a word w such that $[\![\mathcal{A}]\!](w) > [\![\mathcal{B}]\!](w)$.

Decidability: first case

$[\![\mathcal{A}]\!] \leqslant [\![\mathcal{B}]\!]$ when $\mathcal{B}$ is unambiguous.

Decidability: first case

$[\![\mathcal{A}]\!] \leqslant [\![\mathcal{B}]\!]$ when $\mathcal{B}$ is unambiguous.

Is there $\boldsymbol{x} \in \mathbb{N}^n$ such that:

$$\sum_{i=1}^{k} p_i q_{i,1}^{x_1} \dots q_{i,n}^{x_n} > r s_1^{x_1} \dots s_n^{x_n}$$

Decidability: first case

$[\![\mathcal{A}]\!] \leqslant [\![\mathcal{B}]\!]$ when $\mathcal{B}$ is unambiguous.

Is there $\boldsymbol{x} \in \mathbb{N}^n$ such that:

$$\sum_{i=1}^{k} p_i q_{i,1}^{x_1} \dots q_{i,n}^{x_n} > r s_1^{x_1} \dots s_n^{x_n}$$

- First case: there is i, j such that $q_{i,j} > s_j$
- Second case: for all i, j, $q_{i,j} \leqslant s_j$

Decidability: second case

Much more difficult!

Theorem

Determining whether $[\![\mathcal{A}]\!] \leqslant [\![\mathcal{B}]\!]$ is decidable when $\mathcal{A}$ is unambiguous and $\mathcal{B}$ is finitely ambiguous, assuming Schanuel's conjecture is true.

Decidability: second case

Much more difficult!

Theorem

Determining whether $[\![\mathcal{A}]\!] \leqslant [\![\mathcal{B}]\!]$ is decidable when $\mathcal{A}$ is unambiguous and $\mathcal{B}$ is finitely ambiguous, assuming Schanuel's conjecture is true.

Is there $\boldsymbol{x} \in \mathbb{N}^n$ such that:

$$\sum_{i=1}^{k} p_i q_{i,1}^{x_1} \dots q_{i,n}^{x_n} < 1$$

Decidability: second case

Much more difficult!

Theorem

Determining whether $[\![\mathcal{A}]\!] \leqslant [\![\mathcal{B}]\!]$ is decidable when $\mathcal{A}$ is unambiguous and $\mathcal{B}$ is finitely ambiguous, assuming Schanuel's conjecture is true.

Is there $\boldsymbol{x} \in \mathbb{N}^n$ such that:

$$\sum_{i=1}^{k} p_i q_{i,1}^{x_1} \dots q_{i,n}^{x_n} < 1$$

- semi-decidable to find such $\boldsymbol{x}$

Decidability: second case

Much more difficult!

Theorem

Determining whether $[\![\mathcal{A}]\!] \leqslant [\![\mathcal{B}]\!]$ is decidable when $\mathcal{A}$ is unambiguous and $\mathcal{B}$ is finitely ambiguous, assuming Schanuel's conjecture is true.

Is there $\boldsymbol{x} \in \mathbb{N}^n$ such that:

$$\sum_{i=1}^{k} p_i q_{i,1}^{x_1} \dots q_{i,n}^{x_n} < 1$$

- semi-decidable to find such $\boldsymbol{x}$
- if there is no such $\boldsymbol{x}$, there is a non-zero vector $\boldsymbol{d} \in \mathbb{Z}^n$ and

$a, b \in \mathbb{Z}$ such that $\{\boldsymbol{d}^\top \boldsymbol{y} \mid \boldsymbol{y} \text{ is a real solution }\} \subseteq [a, b]$

$\rightarrow$ decrease the dimension by 1

Undecidability

Proposition

Given a two-counter machine, one can construct two linearly ambiguous probabilistic automata $\mathcal{A}$ and $\mathcal{B}$, such that the machine halts if and only if there exists a word w such that $[\![\mathcal{A}]\!](w) \leqslant [\![\mathcal{B}]\!](w)$.

Undecidability

Proposition

Given a two-counter machine, one can construct two linearly ambiguous probabilistic automata $\mathcal{A}$ and $\mathcal{B}$, such that the machine halts if and only if there exists a word w such that $[\![\mathcal{A}]\!](w) \leqslant [\![\mathcal{B}]\!](w)$.

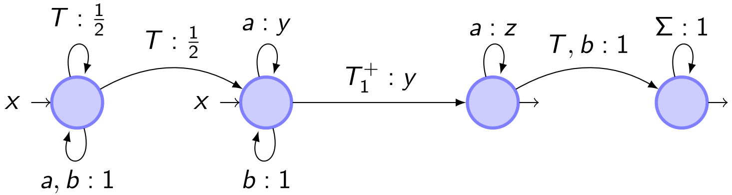
$\longrightarrow$ Simulate an execution with a word: $a^n b^m t_1 a^{n+1} b^m t_2 a^{n+1} b^{m'}$

Undecidability

Proposition

Given a two-counter machine, one can construct two linearly ambiguous probabilistic automata $\mathcal{A}$ and $\mathcal{B}$, such that the machine halts if and only if there exists a word w such that $[\![\mathcal{A}]\!](w) \leqslant [\![\mathcal{B}]\!](w)$.

$\longrightarrow$ Simulate an execution with a word: $a^n b^m t_1 a^{n+1} b^m t_2 a^{n+1} b^{m'}$



Conclusion

Containment problem
for finitely ambiguous probabilistic automata?