An introduction to the algebraic theory of rational languages

(and a little bit of topology...)

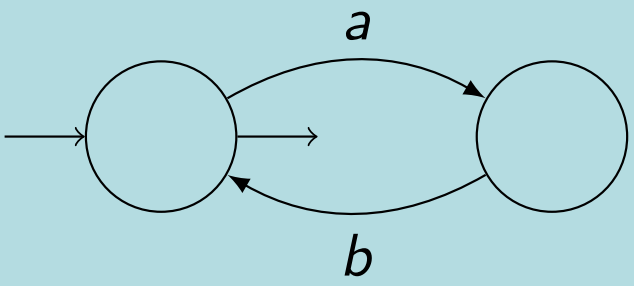
Laure Daviaud

Warsaw University

MFPS 2017

Rational languages

Automata



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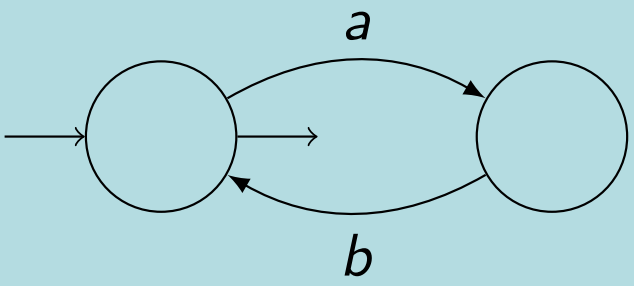

Rational languages

$(ab)^*$

Rational Expressions

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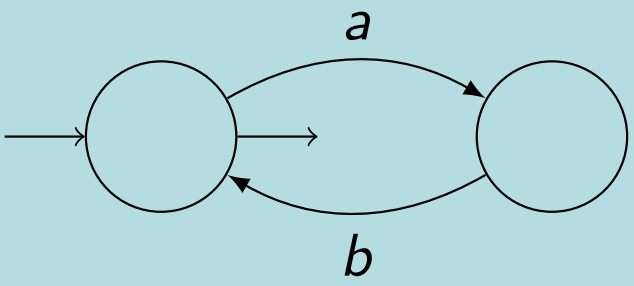

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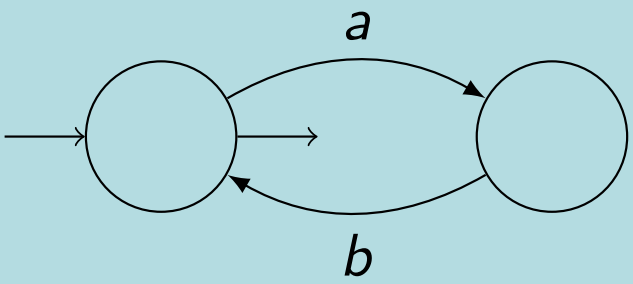

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Finite monoids

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Algebra

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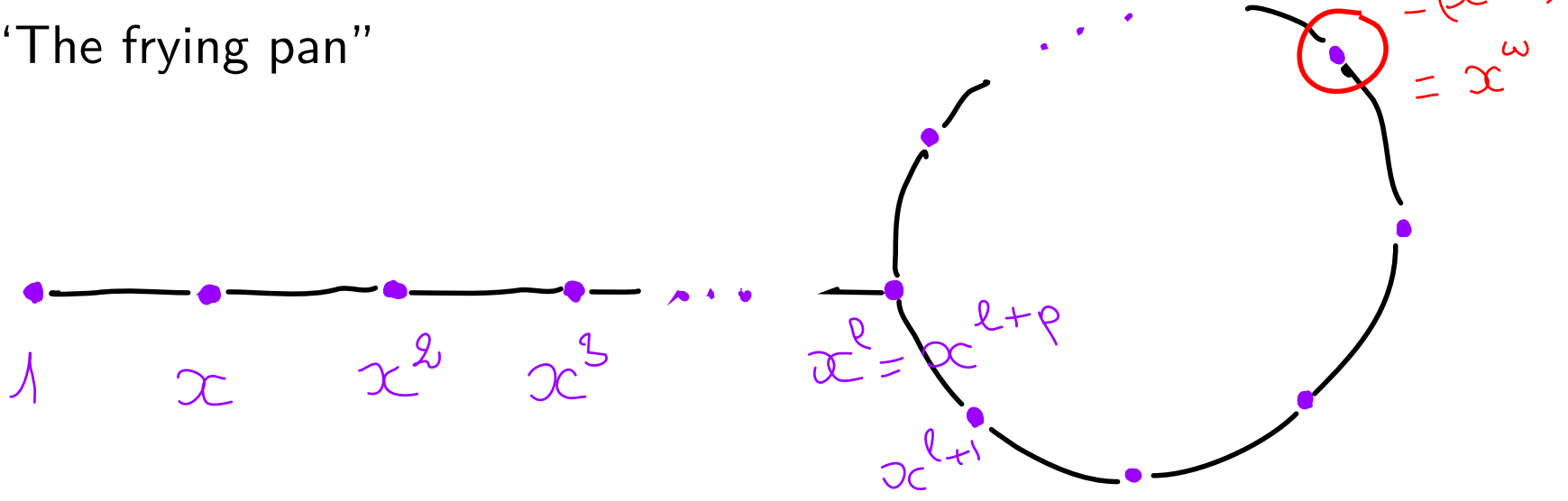
3. "The frying pan"



Every element $x \in M$ has a unique idempotent power $x^\omega = x^{|M|!}$

Recognisability by monoids

Theorem [Kleene]

A language is rational if and only if it is recognised by a finite monoid

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A language L of A^* is recognised by M if there is a morphism $\varphi : A^* \to M$ and $P \subseteq M$ such that $L = \varphi^{-1}(P)$.

$$\begin{array}{ccc} & \overset{\varphi}{} & \\ A^* & \longrightarrow & M \\ \cup\mid & & \cup\mid \\ \varphi^{-1}(P) = L & \dashrightarrow & P \end{array}$$

A canonical object: the syntactic monoid

. Three equivalent definitions .

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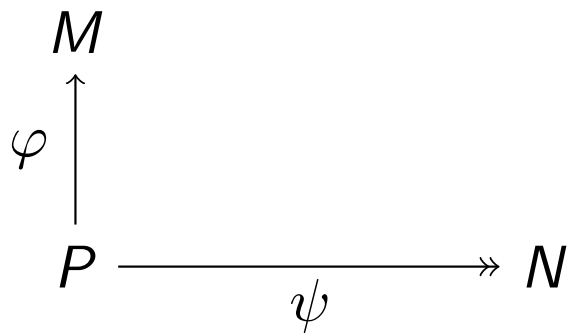

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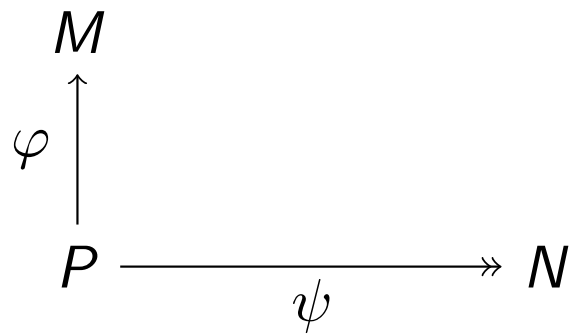

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where $\sim_L$ is the equivalence relation of finite index:
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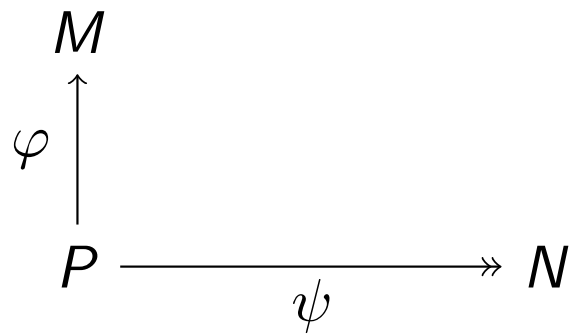

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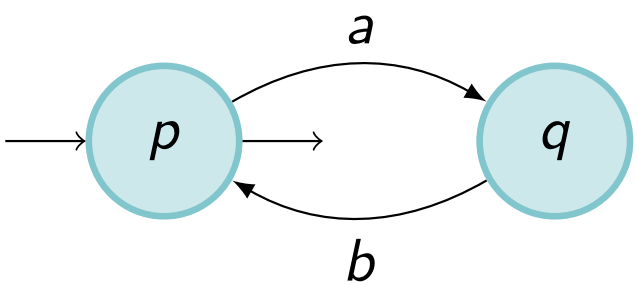

	p	q

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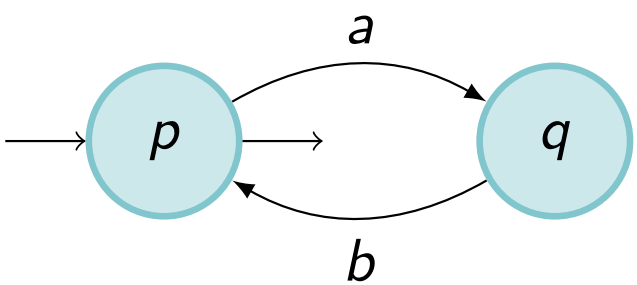

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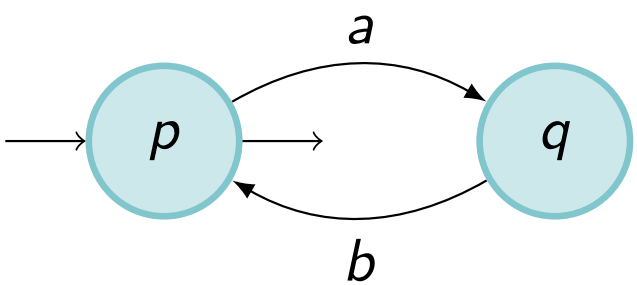

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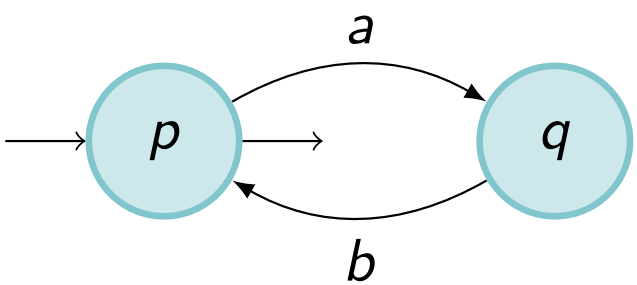

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ε	p	q
a	q	$\perp$
b	$\perp$	p

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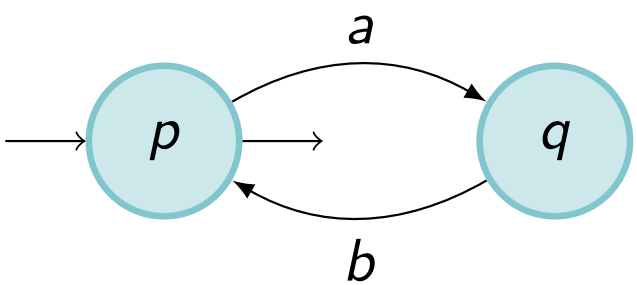

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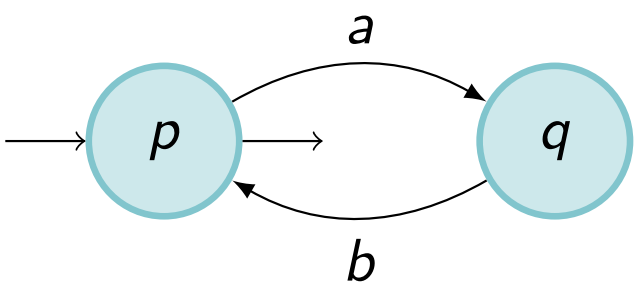

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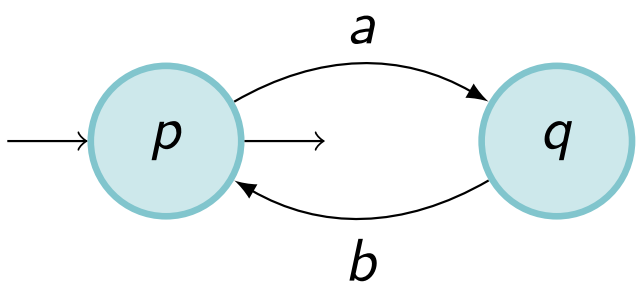

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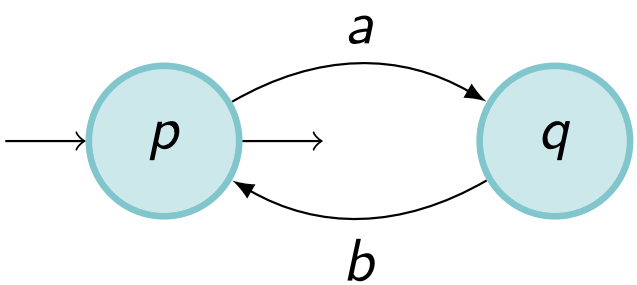

	p	q	
ε	p	q	
a	q	$\perp$	$= aba$
b	$\perp$	p	$= bab$
aa	$\perp$	$\perp$	$= bb = 0$
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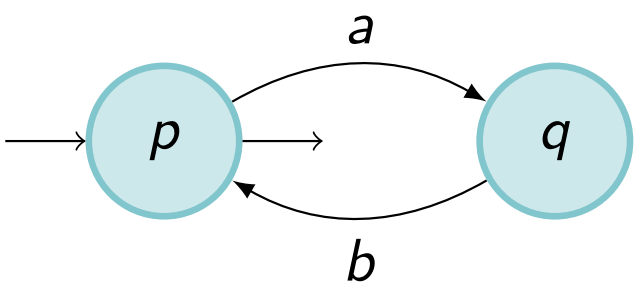

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with $aa = bb = 0$, $aba = a$, $bab = b$.

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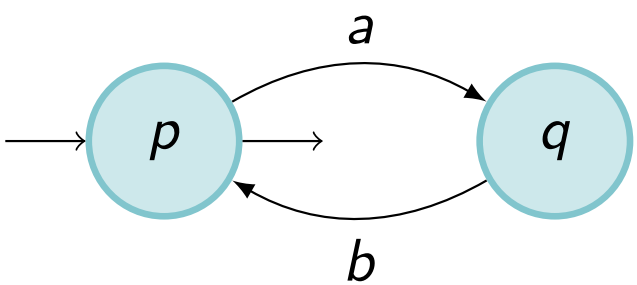

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What about our original question?...

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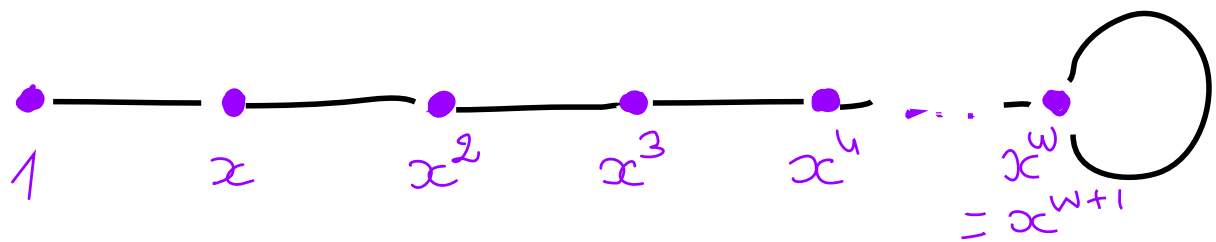
A monoid M is aperiodic if for all $x \in M$, $x^{\omega+1} = x^{\omega}$.



$\rightarrow (ab)^*$ is star-free.

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Identities ?

A monoid M satisfies a word-identity $u = v$ with $u, v \in A^*$, if for all morphisms $\varphi : A^* \to M$, $\varphi(u) = \varphi(v)$.

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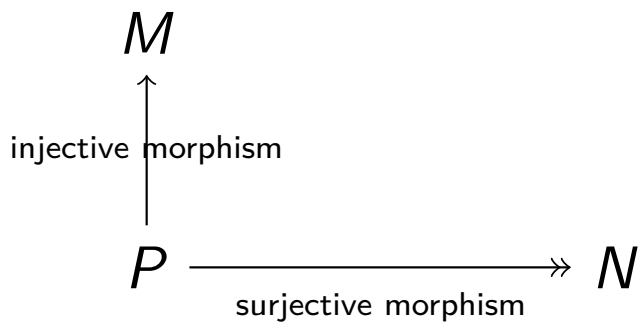
- direct product

$$M_1 \times M_2 \times M_3 \times \cdots$$

- division



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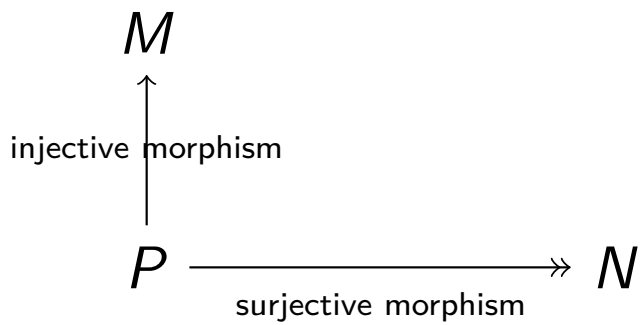
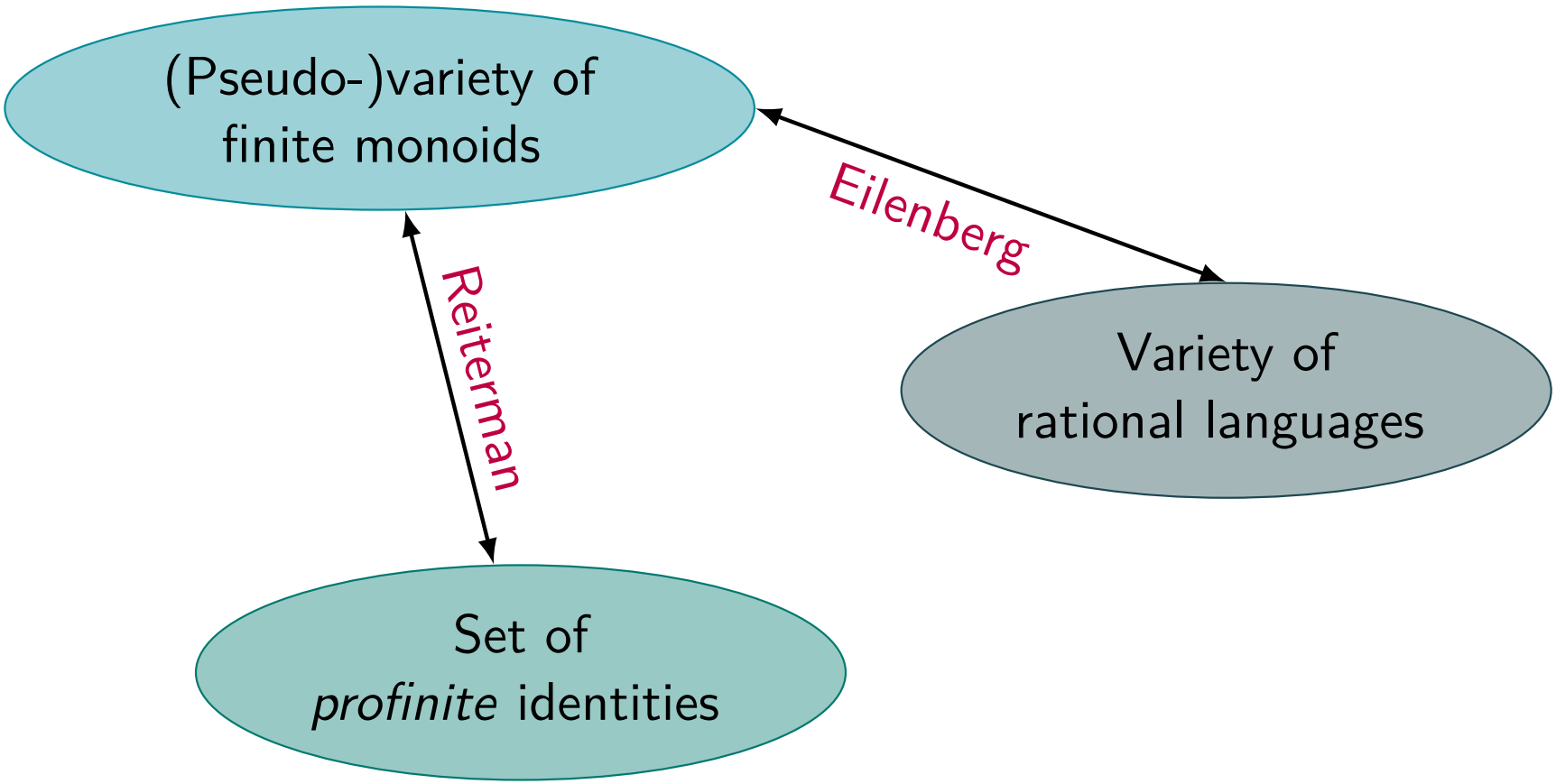


Birkhoff varieties of monoids are defined by a set of identities.

Example: the variety of commutative monoids: $xy = yx$

The variety theorem(s)



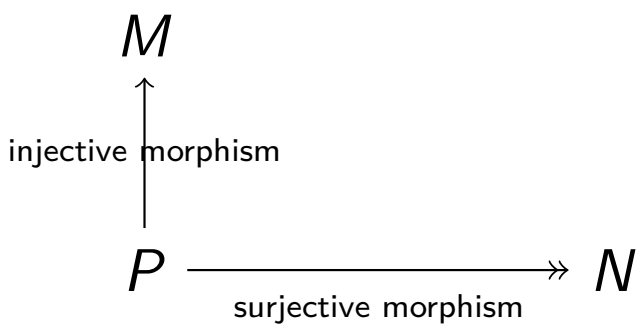
Varieties of finite monoids

A class of finite monoids forms a (pseudo-)variety if:

- it is closed under FINITE product

$$M_1 \times M_2 \times \cdots \times M_\ell$$

- it is closed under division



Varieties of languages

A variety of languages is a class of rational languages

$$\nu(A_1) \cup \nu(A_2) \cup \nu(A_3) \ldots$$

such that:

- for each alphabet A_i, $\nu(A_i)$ is a boolean algebra over A_i
  (closed under finite union, intersection, complement)
- for each alphabet A_i, $\nu(A_i)$ is closed under quotient:
  if $L \in \nu(A_i)$ and $u \in A_i$ then Lu^{-1} and $u^{-1}L \in \nu(A_i)$
- it is closed under inverse image: for each monoid morphism
  $\varphi : A_i^* \to A_j^*$, $L \in \nu(A_j)$ implies $\varphi^{-1}(L) \in \nu(A_i)$

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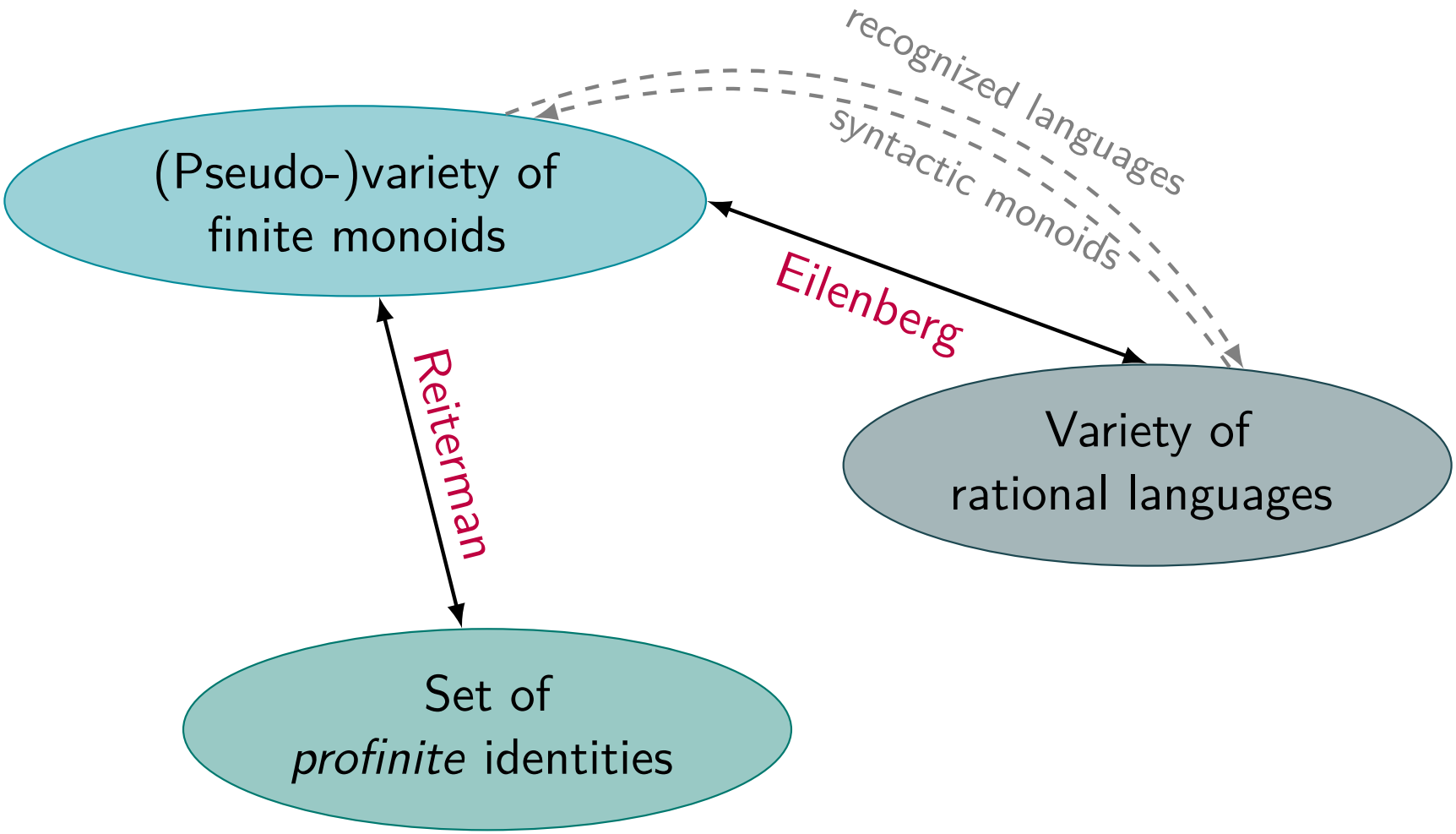

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Example 3: $u \in A^*$, $n \in \mathbb{N}$ - separate $u^{n!}$ and $u^{(n+1)!}$?

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$x \in M$ then $x^{|M|!} = x^{(|M|+1)!} =$ the idempotent power of x in M

$\implies \varphi(u)^{|M|!} = \varphi(u)^{(|M|+1)!}$

$u^{n!}$ and $u^{(n+1)!}$ cannot be separated by a monoid of size less than n

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d is an ultrametric distance:

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The words $u^{n!}$ and $u^{(n+1)!}$ are closer and closer...

Profinite monoids

Definition

Profinite monoid $\widehat{A^*}$:
completion of A^* with respect to the distance d.

- Monoid if u and v sequences of words, $(u.v)_n = u_n v_n$
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Idempotent power of $u \in A^*$

$$u^\omega = \lim_{n \to \infty} u^{n!}$$

Profinite identities

Profinite identity: $u = v$ with $u, v \in \widehat{A^*}$.

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unique way to extend φ by continuity

About $x^\omega = x^{\omega+1}$

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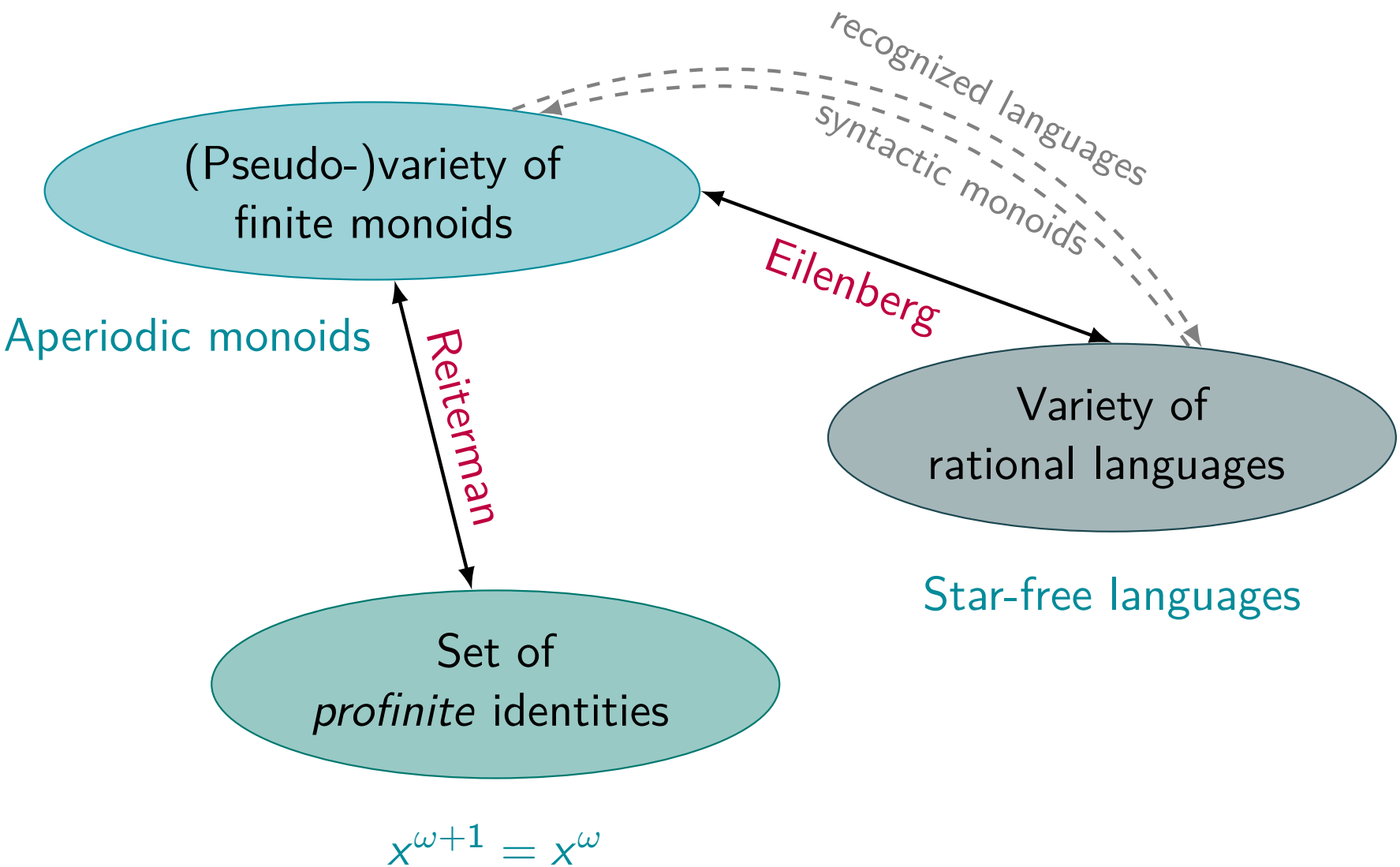

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Examples of rational languages of a given generalised star-height?

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$\longrightarrow$ OPEN : we do not even know if there exist a rational language with star-height at least 2.

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- One can decide if a given rational language is star-free.

- $(aa)^*$ is not star-free.

- *Generalised star-height*: minimal number of nested stars in a generalised expression ($\cup$, $\cdot$, ^{c}, *) representing a rational language.

Examples of rational languages of a given generalised star-height?

$\longrightarrow$ OPEN : we do not even know if there exist a rational language with star-height at least 2.

Mathematical Fondations of Automata Theory - `https://www.irif.fr/~jep/PDF/MPRI/MPRI.pdf`