Max-plus automata
or
how to link automata theory with max-plus algebra

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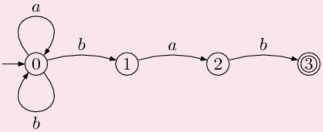
University of Warsaw

based on joint works with T.Colcombet,
P.Guillon, G.Merlet and F.Zuleger

Manchester, 23/11/2016

Two points of view...

Machine
Automata



Decidability ?
- Simplification-Minimisation
- Comparison

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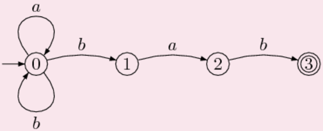

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Optimisation
Matrices over
$(\mathbb{N} \cup \{-\infty\}, \max, +)$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \quad \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$$

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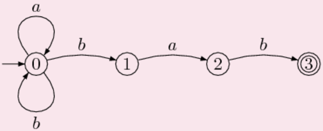

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$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

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Max-plus Automata

A very simple machine: Automata

alphabet $A = \{a, b\}$
set of words A^*: finite sequences of a and b

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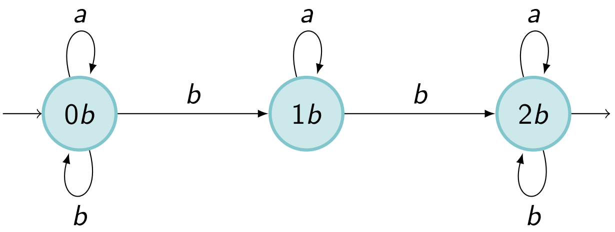

A word is *accepted* by the automaton if there is a path labelled by the word from an initial state to a final state.

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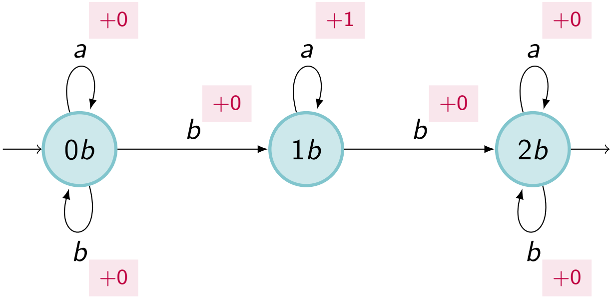

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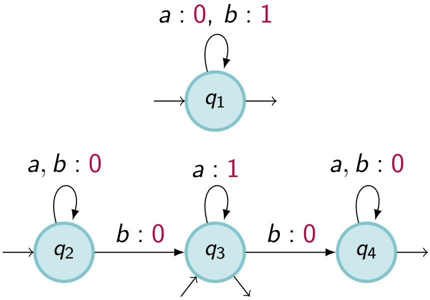
Max-plus automata: Machine point of view



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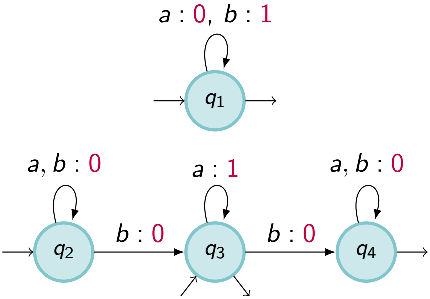

Syntax :

Non deterministic finite automaton for which each transition is labelled by a non negative integer (weight).

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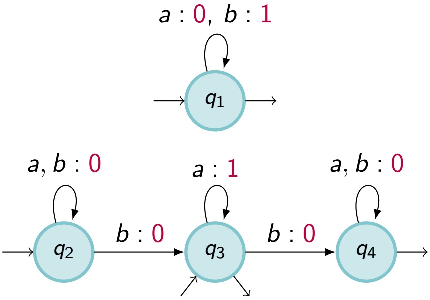

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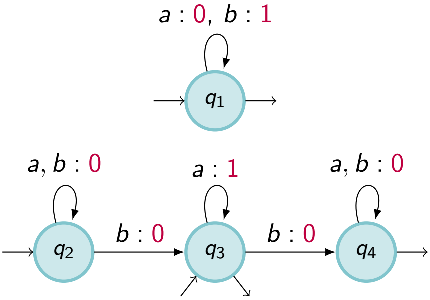
Weight of a run = sum of the weights of the transitions.

$$\begin{array}{lcl} A^+ & \to & \mathbb{N} \cup \{-\infty\} \\ w & \mapsto & \text{Maximum of the weights of the accepting runs labelled by } w \ (-\infty \text{ if no such run}) \end{array}$$

Max-plus automata: Machine point of view



$a^{n_0} b a^{n_1} b \cdots b a^{n_k}$

$\mapsto \max(n_0, n_1, \ldots, n_k, k)$

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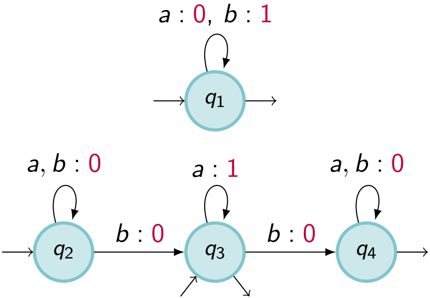
Matrix representation



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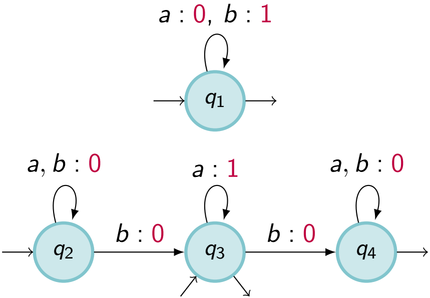

$$\begin{array}{c} \\ q_1 \\ q_2 \\ q_3 \\ q_4 \end{array} \begin{array}{c} \begin{array}{cccc} q_1 & q_2 & q_3 & q_4 \end{array} \\ \begin{pmatrix} 0 & \infty & \infty & \infty \\ \infty & 0 & \infty & \infty \\ \infty & \infty & 1 & \infty \\ \infty & \infty & \infty & 0 \end{pmatrix} \end{array} = \mu(a)$$

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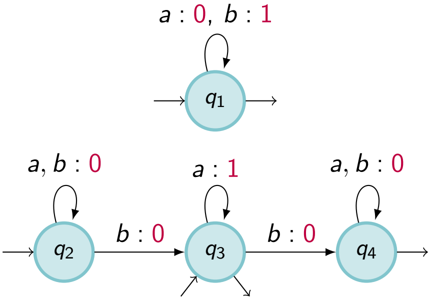

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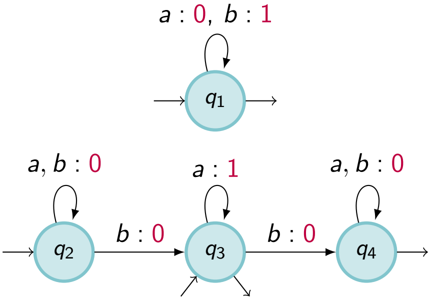
$$

$$
\begin{pmatrix} 1 & \infty & \infty & \infty \\ \infty & 0 & 0 & \infty \\ \infty & \infty & \infty & 0 \\ \infty & \infty & \infty & 0 \end{pmatrix} = \mu(b)
$$

Matrix representation



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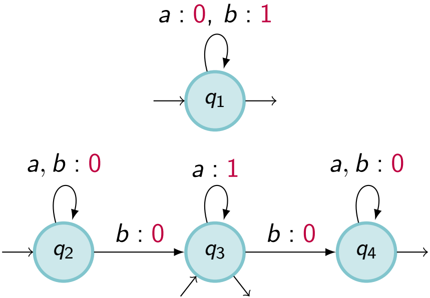

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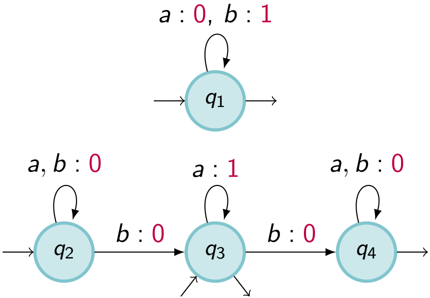

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$\mu(w)_{i,j} =$ max of the weights of the runs from i to j labelled by w

$$f(w) = I\mu(w)F$$

Correspondences...

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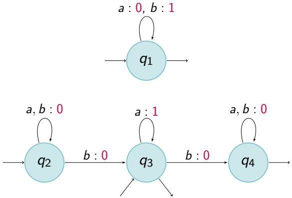

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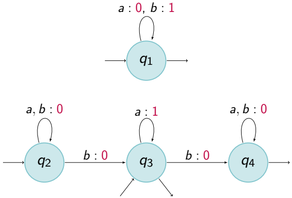

Finite set of ℓ matrices
$X = \{M_1, M_2, \ldots, M_\ell\}$
of dimension d

Transitions over ℓ letters
$A = \{a_1, a_2, \ldots, a_\ell\}$
in an automaton $\mathcal{A}$ with d states

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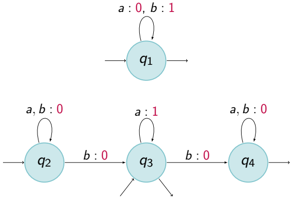

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Generated semigroup
$M_{i_1} M_{i_2} \cdots M_{i_k} \in \langle X \rangle$

Automaton over the words
Behaviour over $a_{i_1} a_{i_2} \cdots a_{i_k}$

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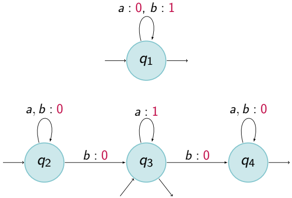

Finite set of ℓ matrices $X = \{M_1, M_2, \ldots, M_\ell\}$ of dimension d	Transitions over ℓ letters $A = \{a_1, a_2, \ldots, a_\ell\}$ in an automaton $\mathcal{A}$ with d states
Generated semigroup $M_{i_1} M_{i_2} \cdots M_{i_k} \in \langle X \rangle$	Automaton over the words Behaviour over $a_{i_1} a_{i_2} \cdots a_{i_k}$
???	Computed function $[\![\mathcal{A}]\!]$

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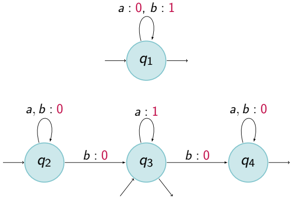

Finite set of ℓ matrices $X = \{M_1, M_2, \ldots, M_\ell\}$ of dimension d	Transitions over ℓ letters $A = \{a_1, a_2, \ldots, a_\ell\}$ in an automaton $\mathcal{A}$ with d states
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???	Computed function $[\![\mathcal{A}]\!]$
Select the coefficient (i, j) of the matrix $M_{i_1} M_{i_2} \cdots M_{i_k}$	q_i initial, q_j final compute $[\![\mathcal{A}]\!](a_{i_1} a_{i_2} \cdots a_{i_k})$

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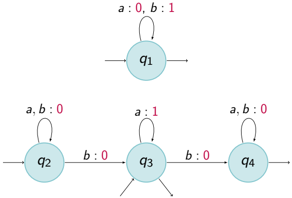

Finite set of ℓ matrices $X = \{M_1, M_2, \ldots, M_\ell\}$ of dimension d	Transitions over ℓ letters $A = \{a_1, a_2, \ldots, a_\ell\}$ in an automaton $\mathcal{A}$ with d states
Generated semigroup $M_{i_1} M_{i_2} \cdots M_{i_k} \in \langle X \rangle$	Automaton over the words Behaviour over $a_{i_1} a_{i_2} \cdots a_{i_k}$
???	Computed function $[\![\mathcal{A}]\!]$
Select the coefficient (i, j) of the matrix $M_{i_1} M_{i_2} \cdots M_{i_k}$	q_i initial, q_j final compute $[\![\mathcal{A}]\!](a_{i_1} a_{i_2} \cdots a_{i_k})$
$\{\|M\|_\infty \mid M \in \langle X \rangle\}$	All states are both initial and final $\{[\![\mathcal{A}]\!](w) \mid w \in A^*\}$

Weighted automata [Schützenberger '61]

$$\begin{array}{c} \\ q_1 \\ q_2 \\ q_3 \\ q_4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} q_1 & q_2 & q_3 & q_4 \end{array} \\ \begin{pmatrix} 0 & \infty & \infty & \infty \\ \infty & 0 & \infty & \infty \\ \infty & \infty & 1 & \infty \\ \infty & \infty & \infty & 0 \end{pmatrix} \end{array} = \mu(a) \quad \begin{pmatrix} 1 & \infty & \infty & \infty \\ \infty & 0 & 0 & \infty \\ \infty & \infty & \infty & 0 \\ \infty & \infty & \infty & 0 \end{pmatrix} = \mu(b)$$

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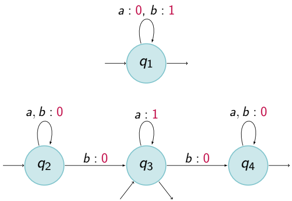

Matrices over the semiring
$(\mathbb{N} \cup \{-\infty\}, \max, +)$

Weight of a run:
sum of the weights of the transitions.

$$\begin{array}{lcl} A^+ & \to & \mathbb{N} \cup \{-\infty\} \\ w & \mapsto & \text{Maximum of the weights} \\ & & \text{of the accepting runs} \\ & & \text{labelled by } w \\ & & (-\infty \text{ if no such run}) \end{array}$$

Weighted automata [Schützenberger '61]

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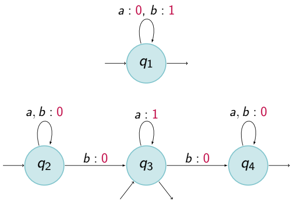

Matrices over the semiring
~~$(\mathbb{N} \cup \{-\infty\}, \max, +)$~~
$(S, \oplus, \otimes)$

Weight of a run:
~~sum~~ $\otimes$ of the weights of the transitions.

$A^+ \rightarrow$ ~~$\mathbb{N} \cup \{-\infty\}$~~ S

$w \mapsto$ $\oplus$ ~~Maximum~~ of the weights of the accepting runs labelled by w (~~$-\infty$~~ if no such run)
$0_{\oplus}$

Weighted automata [Schützenberger '61]

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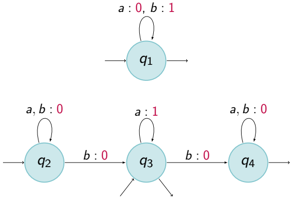

Matrices over the semiring
~~$(\mathbb{N} \cup \{-\infty\}, \max, +)$~~
$(S, \oplus, \otimes)$

Weight of a run:
~~sum~~ $\otimes$ of the weights of the transitions.

$A^+ \to$ ~~$\mathbb{N} \cup \{-\infty\}$~~ S
$w \mapsto$ $\oplus$ ~~Maximum~~ of the weights of the accepting runs labelled by w (~~$-\infty$~~ $0_\oplus$ if no such run)

For example: $(\mathbb{Z} \cup \{-\infty\}, \max, +)$, $(\mathbb{N} \cup \{+\infty\}, \min, +)$, $(\mathbb{R}, +, \times)$, *probabiliste...*

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$$\rho(X) = \inf_{k>0} \left\{ \frac{1}{k} \|M_{i_1} \cdots M_{i_k}\|_\infty \,\middle|\, M_{i_1}, \ldots, M_{i_k} \in X \right\}$$

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Given a max-plus automaton $\mathcal{A}$ with all states initial and final, does for all words w, $[\![\mathcal{A}]\!](w) \geqslant |w|$?

Undecidability

Comparison problem: Given a max-plus automaton $\mathcal{A}$, does for all words w, $[\![\mathcal{A}]\!](w) \geqslant |w|$?

Theorem [Krob, 92]

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$\longrightarrow$ *There is no algorithm taking as input a finite set of matrices and computing its joint spectral radius.*

What can we do?

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- Approximations

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  - $\rightarrow$ Weights (coefficients of the matrices)
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  - $\rightarrow$ Number of states (dimension of the matrices)
- Approximations

Restriction on the number of states

Input: A max-plus automaton $\mathcal{A}$ with at most *some number* states on a two-letter alphabet.

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What about between 2 states and 553 states?

Approximation [Colcombet,D.]

Input: f computed by a max-plus automaton, $\varepsilon > 0$

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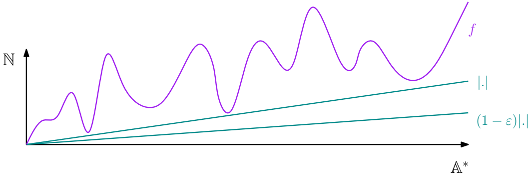
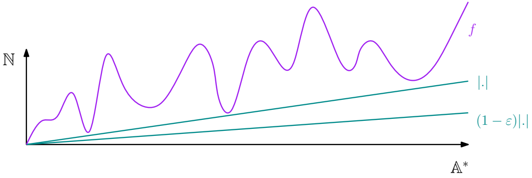


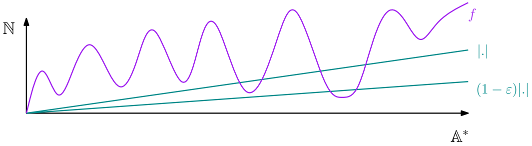
Yes

Approximation [Colcombet,D.]

Input: *f* computed by a max-plus automaton, $\varepsilon > 0$



Yes



No

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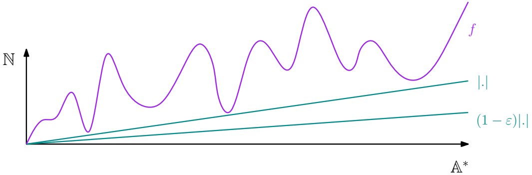
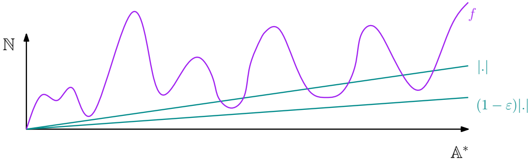
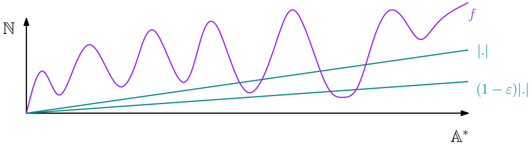


Yes



Yes or No



No

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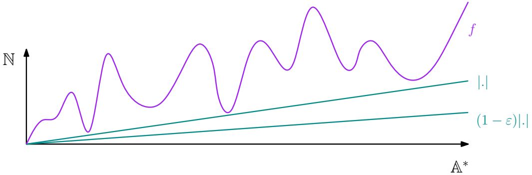
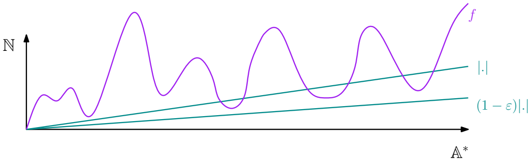


Yes



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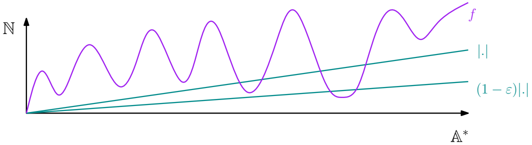
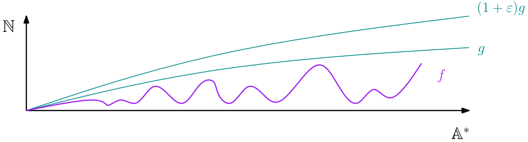


No

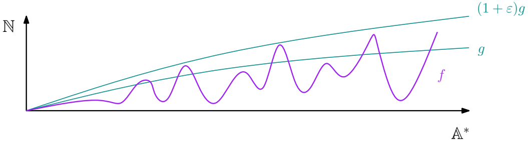
$\longrightarrow$ this also gives an approximation of the joint spectral radius.

Approximation [Colcombet,D.]

Input: f, g computed by min-plus automata, $\varepsilon > 0$



Yes



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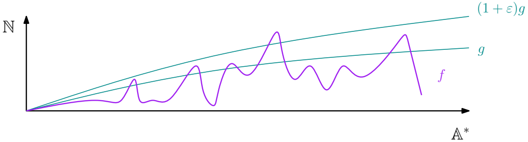
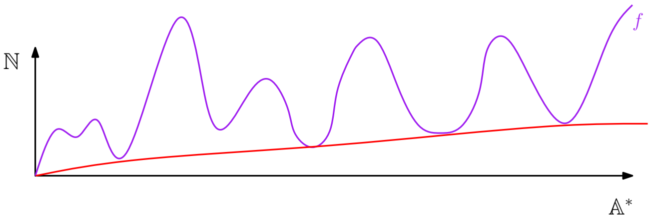


No

$\longrightarrow$ open for max-plus

Approximation [Colcombet,D.,Zuleger]



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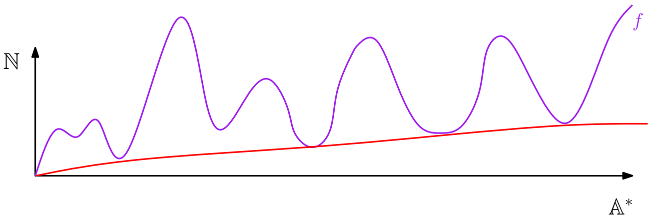

$$f_{\text{inf}} : \quad \mathbb{N} \rightarrow \mathbb{N} \cup \{-\infty\}$$
$$n \mapsto \inf_{|w| \geqslant n} f(w)$$

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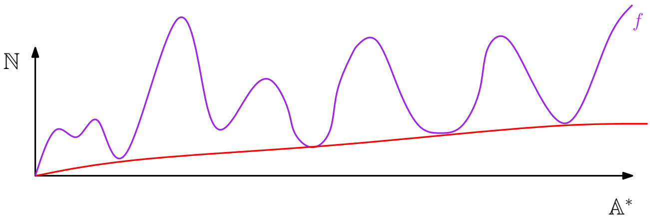
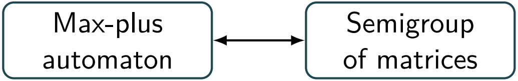


$$\begin{array}{rccc} f_{\mathsf{inf}}: & \mathbb{N} & \to & \mathbb{N} \cup \{-\infty\} \\ & n & \mapsto & \inf_{|w| \geqslant n} f(w) \end{array}$$

There is an algorithm that, given f computed by a max-plus automaton, computes a rational α (in $[0,1] \cup \{-\infty\}$) such that:

$$f_{\mathsf{inf}}(n) = \Theta(n^{\alpha})$$

General ideas



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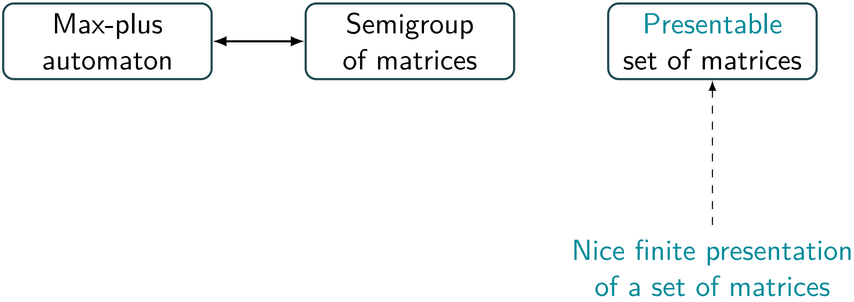

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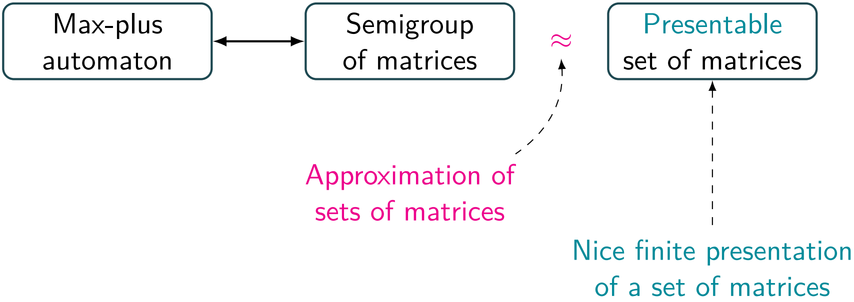

Forest factorisation theorem of Simon

Max-plus automaton ⟷ Semigroup of matrices $< X >$ $\approx$ Presentable set of matrices

M_1 M_2 M_3 M_4 M_5 M_6 M_7 $\cdots\cdots$ M_k

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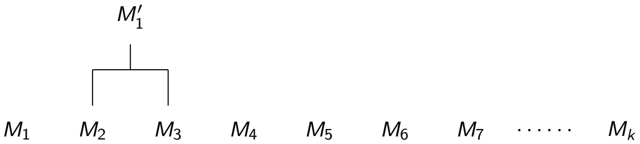

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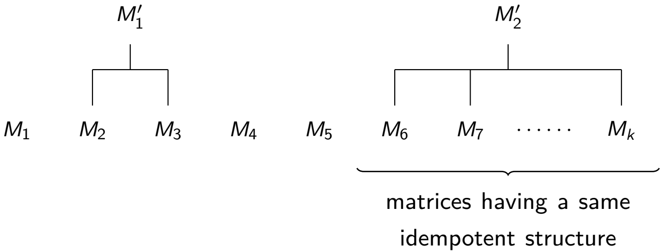

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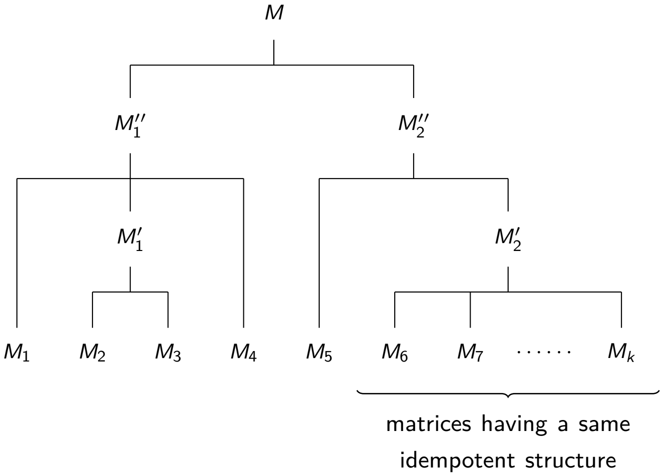

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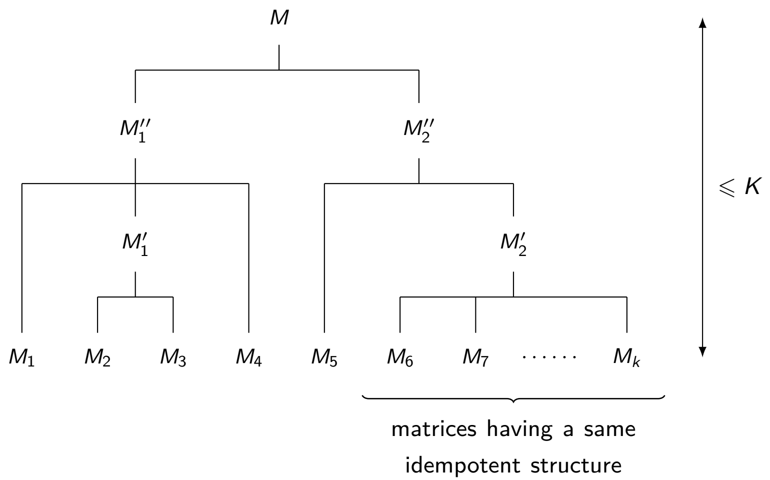

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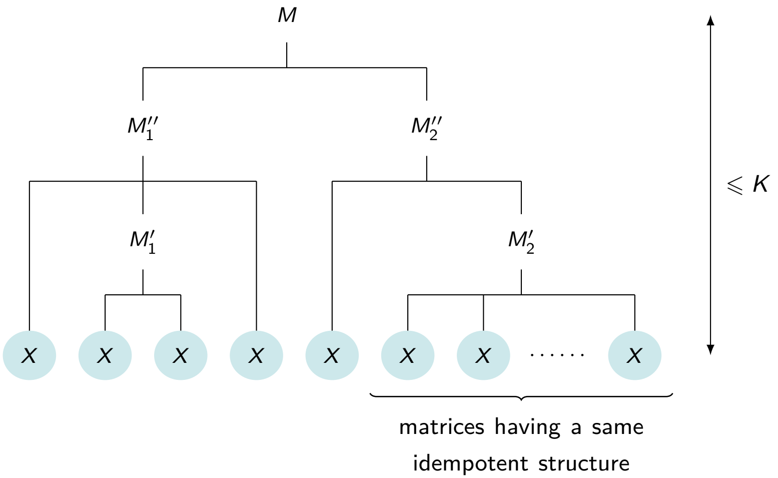

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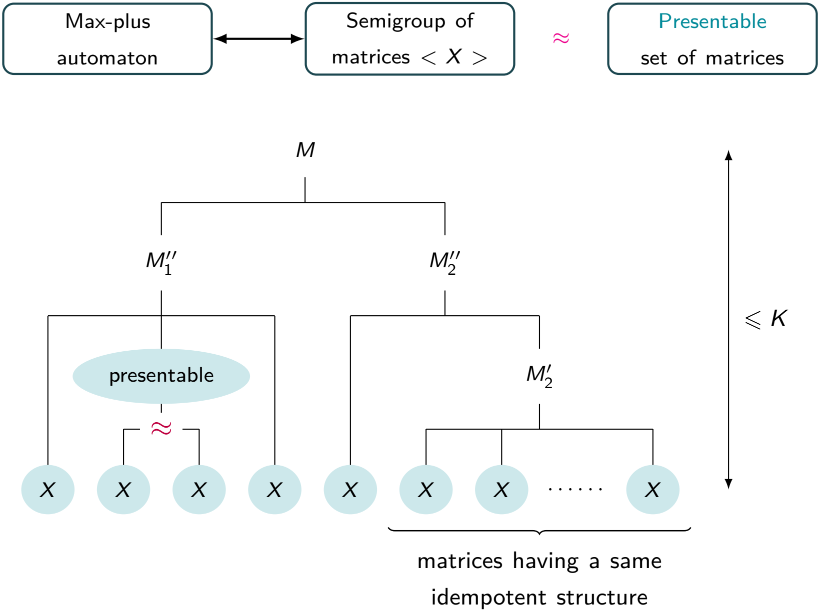

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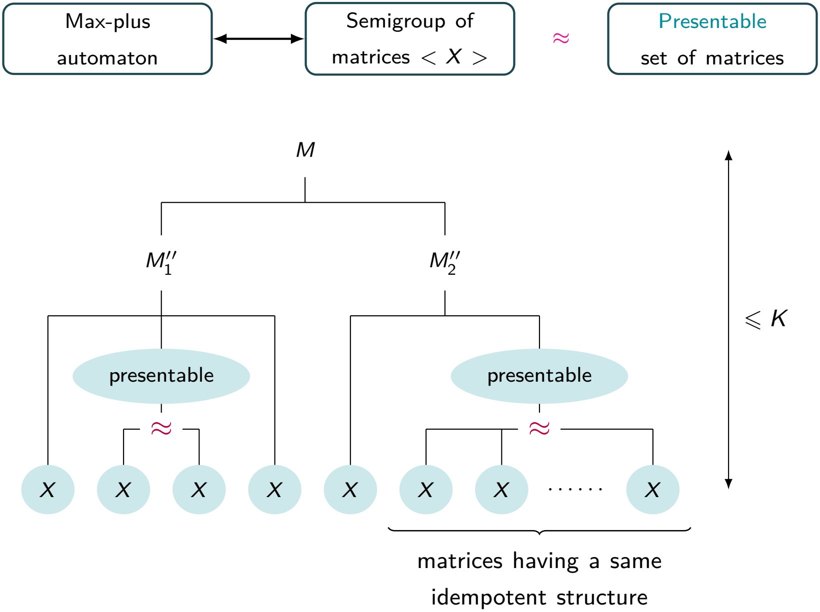

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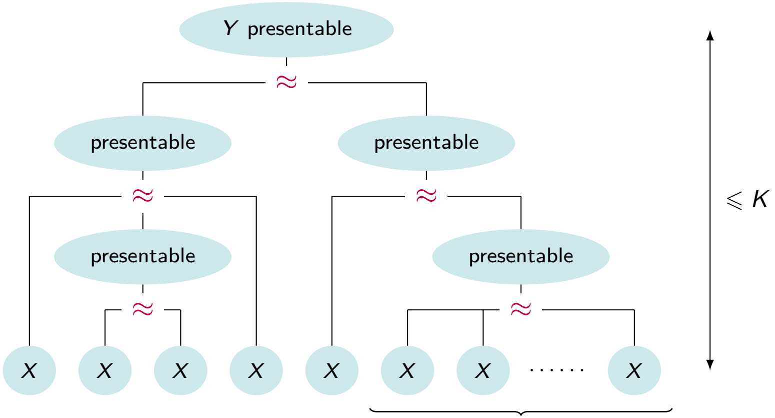

matrices having a same idempotent structure

$\langle X \rangle \approx Y$ presentable

Some open problems...: separation of words

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= Is there an identity on finitely generated semigroup of matrices of dimension d?
$d = 2$ [Izhakian, Margolis], $d = 3$ [Shitov], triangular [Izhakian]

Some open problems...: Determinisation

Problem:
Input: a max-plus automaton computing a function f
Output: "yes" if f is computable by a deterministic max-plus automaton, "no" otherwise

Is this problem decidable?

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Translation:

Input: a finite set of matrices Γ

Question: is there a finite set Γ' of matrices with at most one finite coefficient per row and μ a bijection $\Gamma \to \Gamma'$ such that for all $M_1, \ldots, M_k \in \Gamma$:

$$(M_1 \cdots M_k)_{1,2} = (\mu(M_1) \cdots \mu(M_k))_{1,2} \text{ ?}$$