Size-Change Abstraction and Max-Plus Automata

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Outline

1 - Max-plus automata: definition and example

2 - Asymptotic behaviour of max-plus automata

3 - Application to the size-change abstraction

Max-Plus Automata

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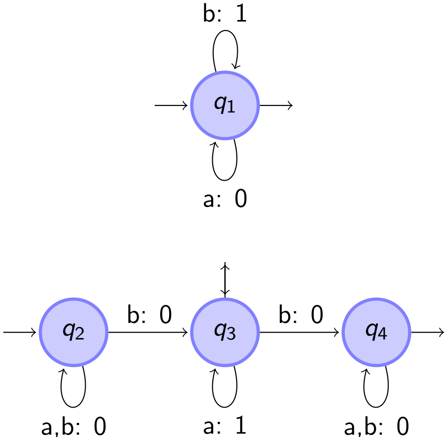
Max-plus automaton: Non deterministic finite automaton for which each transition is also labelled by a non-negative integer called the weight of the transition.

$$(\mathbb{A}, Q, I, T, E) \text{ with } E \subseteq (Q \times \mathbb{A} \times \mathbb{N} \times Q))$$



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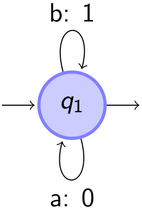
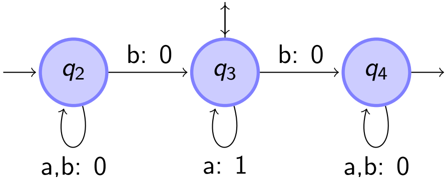


Weight of a run:
sum of the weights of the transitions



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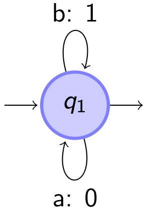
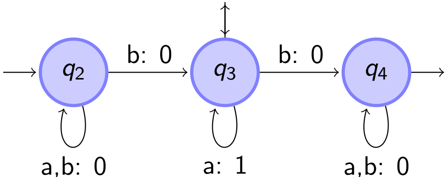


Weight of a run:
sum of the weights of the transitions

Computed function:

$$\begin{array}{ccl} \mathbb{A}^* & \to & \mathbb{N} \cup \{-\infty\} \\ w & \mapsto & \text{maximum of the weights of} \\ & & \text{the runs labelled by } w \text{ going} \\ & & \text{from an initial state} \\ & & \text{to a final state} \\ & & (-\infty \text{ if no such run}) \end{array}$$



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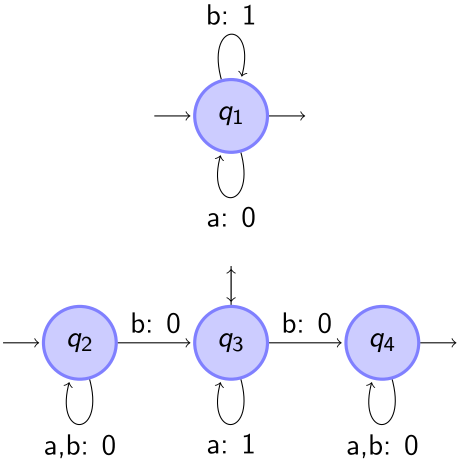
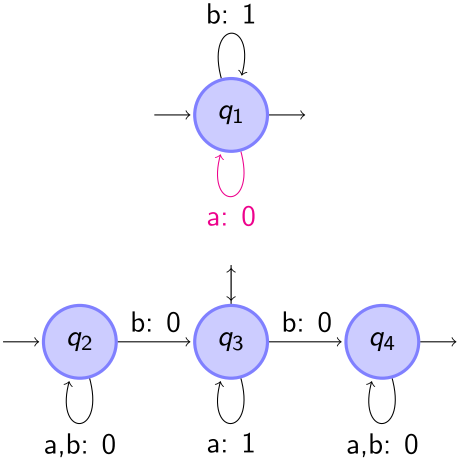


Example: $a^m b a^n b a^p$

Max-Plus Automata



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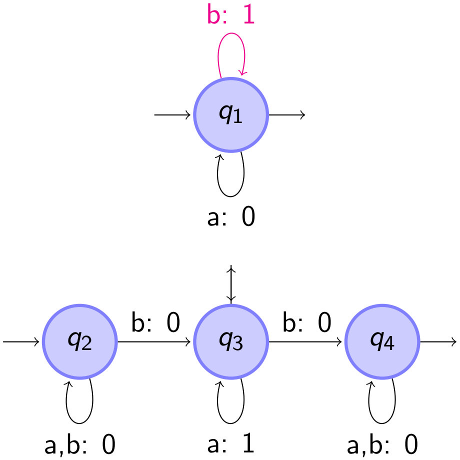
weight of run (1):

Max-Plus Automata



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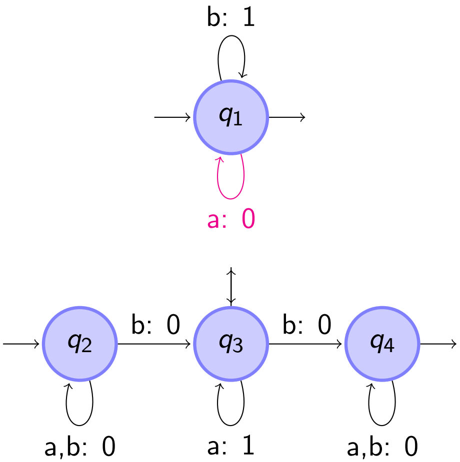

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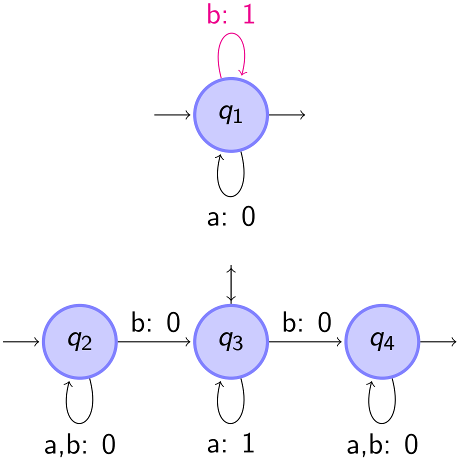

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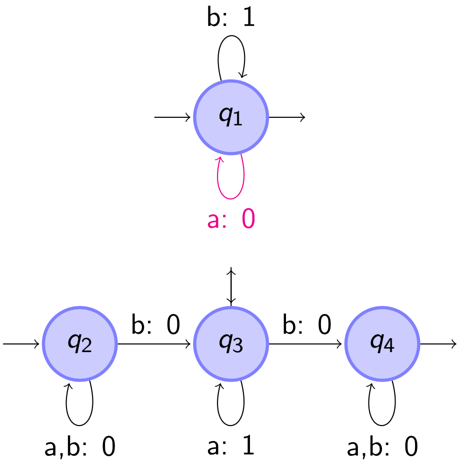
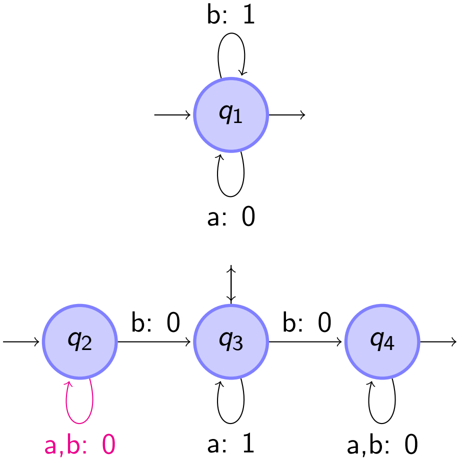


Example: $a^m b a^n b a^p$

weight of run (1): 2

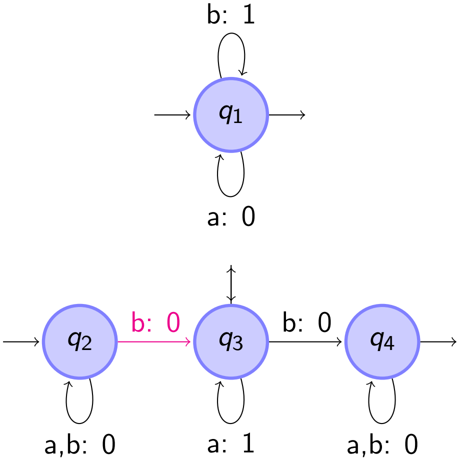
Max-Plus Automata



Example: $a^m b a^n b a^p$

weight of run (1): 2
weight of run (2):

Max-Plus Automata



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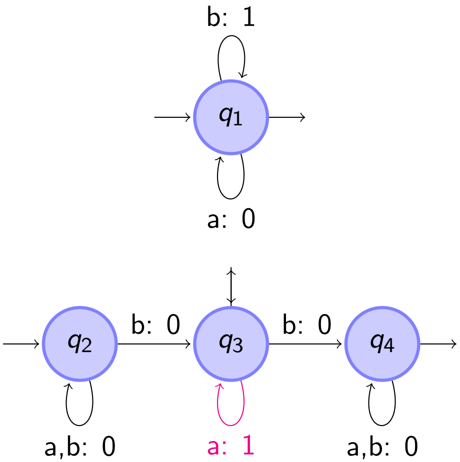

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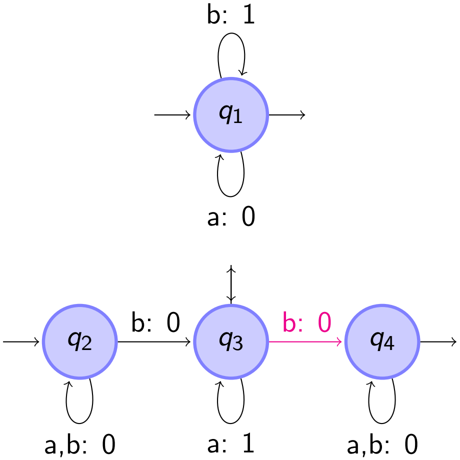

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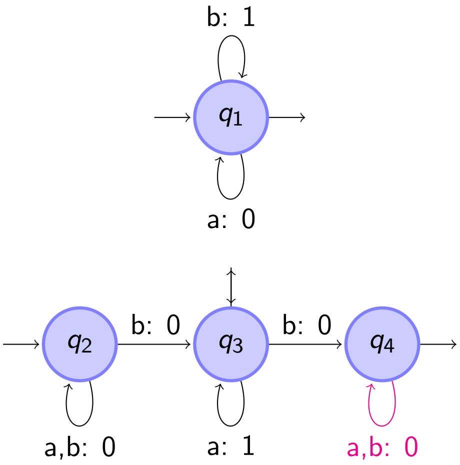
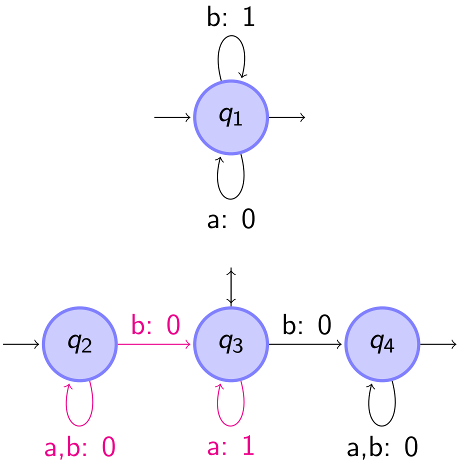


Example: $a^m b a^n b a^p$

weight of run (1): 2
weight of run (2): n

Max-Plus Automata



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weight of run (1): 2
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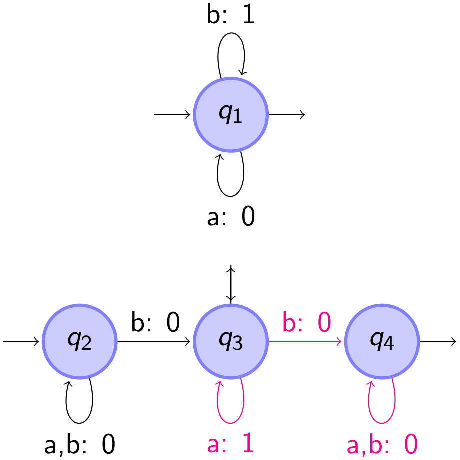

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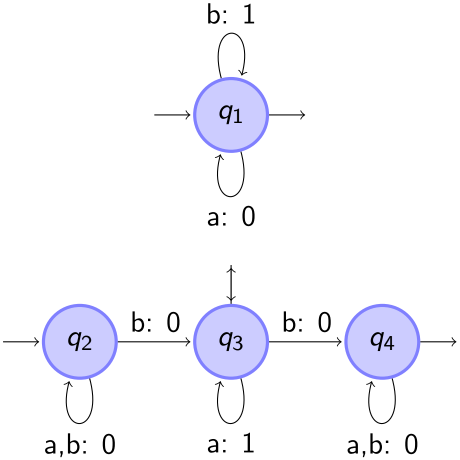

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$a^m b a^n b a^p \mapsto \max(2, m, n, p)$

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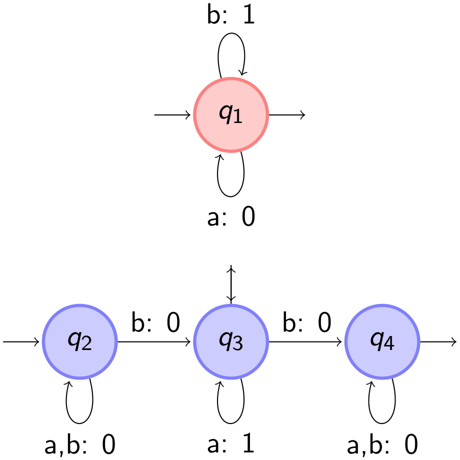

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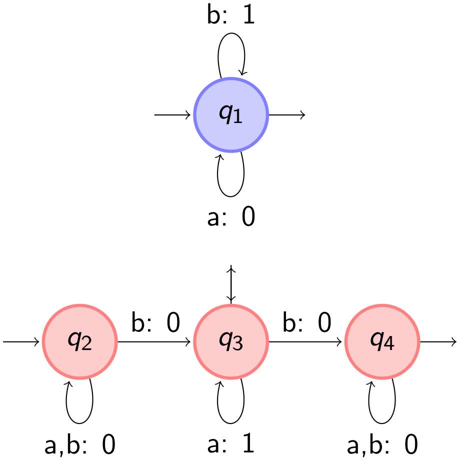

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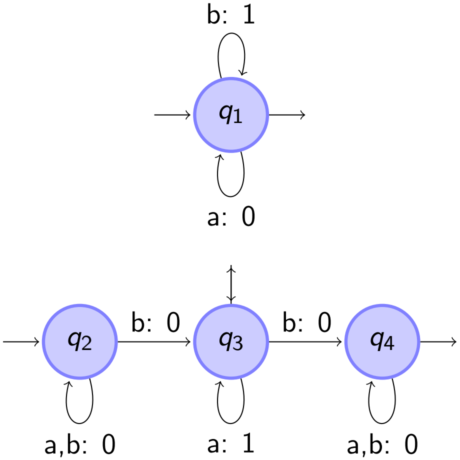

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$a^{n_0} b a^{n_1} b \cdots b a^{n_k} \mapsto \max(n_0, n_1, \ldots, n_k, k)$

Asymptotic behaviour of max-plus automata

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Theorem [Krob]

The following problems are undecidable:
Given f and g computed by max-plus automata,

- is $f \leqslant g$?
- is $f = g$?

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$$\begin{array}{rccl} f_{\text{min}} : & \mathbb{N} & \to & \mathbb{N} \cup \{-\infty\} \\ & n & \mapsto & \min\{f(w) \mid |w| \geqslant n\} \end{array}$$

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$f(w_n) = n$

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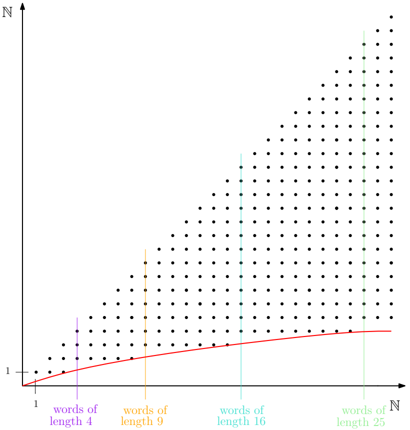

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Theorem

There exists effectively $\alpha \in (\mathbb{Q} \cap [0,1]) \cup \{-\infty\}$ such that $f_{\min}(n) = \Theta(n^\alpha)$.

$\alpha = -\infty$: there is an infinite number of words of weight $-\infty$

$\alpha = 0$: there is an infinite sequence of words that is bounded

$\alpha = 1$: all infinite sequences of words are equivalent to the length

Asymptotic behaviour of a max-plus automaton

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$\rightsquigarrow$ Length of the longest word having value at most n: $\Theta(n^{1/\alpha})$.

Size-Change Abstraction

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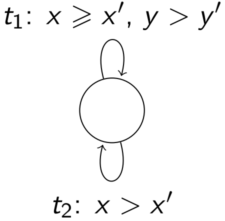
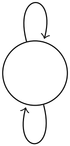


Variables: x and y

Size-Change Abstraction

t_1: $x \geqslant x'$, $y > y'$



t_2: $x > x'$

Variables: x and y

$(5,5) \xrightarrow{t_1} (5,4) \xrightarrow{t_2} (4,8) \xrightarrow{t_1} (3,6) \xrightarrow{t_1} (3,2) \xrightarrow{t_2} (2,10)\ldots$

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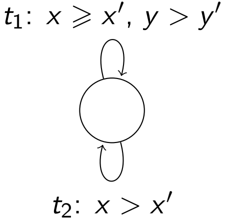

Variables: x and y

Terminating SCA instance:
no infinite trace.

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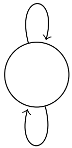

Variables: x and y

Terminating SCA instance: no infinite trace.

Theorem [Lee, Jones, Ben-Amram]

It is decidable whether a given SCA instance is terminating.

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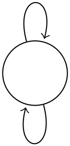

t_2: $x > x'$

Variables: x and y

Terminating SCA instance: no infinite trace.

Restriction to $[0, n]$:

Theorem

Given a terminating SCA instance, there is $\beta \geqslant 1$, rationnal, computable such that the longest trace is of order $\Theta(n^\beta)$.

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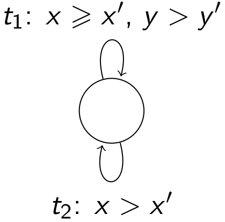
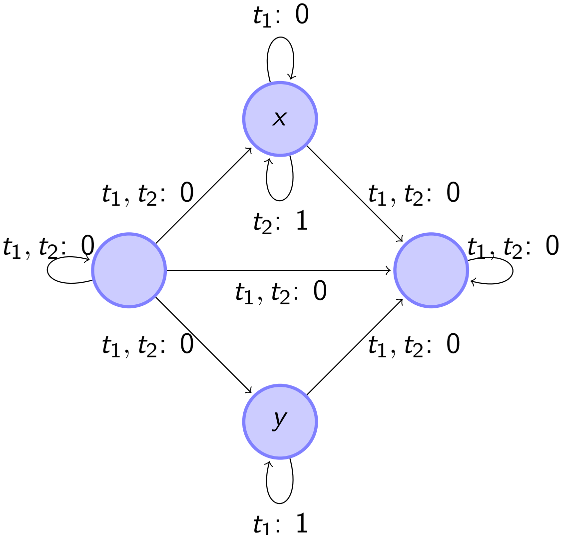
From Size-Change Abstraction to Max-Plus automaton

Longest trace
when variables belong to $[0, n]$

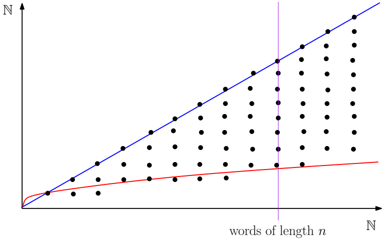


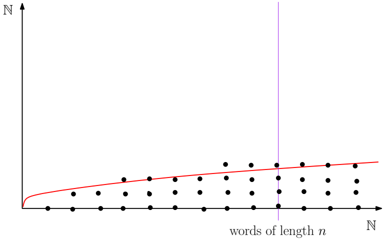
Longest word
having value at most n



Conclusion and further questions

- What about min-plus automata?





- Complexity ?