Comparison of max-plus automata and joint spectral radius of tropical matrices

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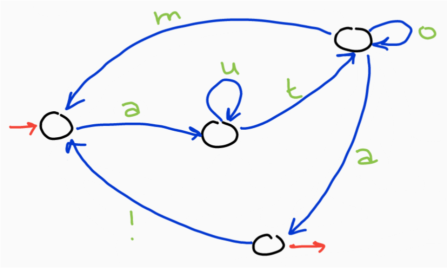
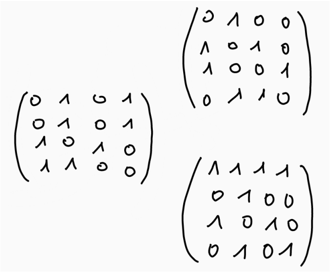
Two points of view

AUTOMATA

MATRICES





A quantitative extension of automata

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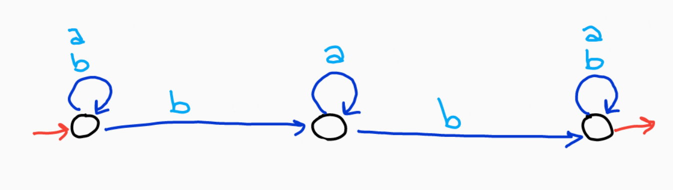
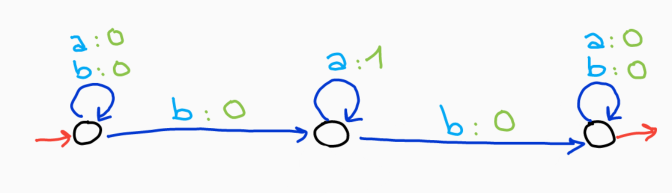
$\longrightarrow$ Check if a word contains two b's



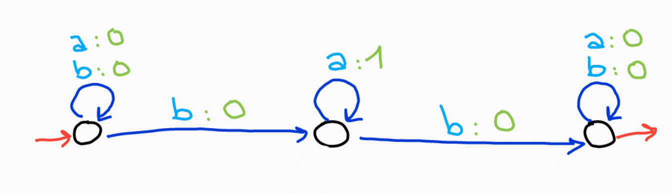
A quantitative extension of automata

$\longrightarrow$ Number of consecutive a's between two b's



A quantitative extension of automata

$\longrightarrow$ Maximal number of consecutive a's between two b's

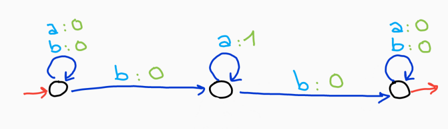


$w \mapsto$ MAXIMUM of the weights of the runs over w which are accepting

(weights of the runs: SUM of the weights of the transitions)

Max-plus automata: Worst-case analysis

$$[\![\mathcal{A}]\!] : A^+ \to \mathbb{N} \cup \{-\infty\}$$



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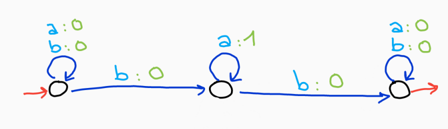
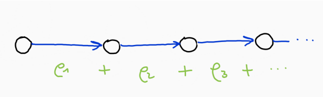
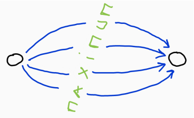


Run: Sum



Non-determinism: Maximum



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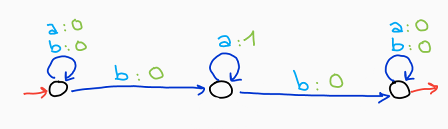

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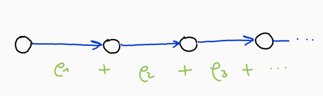

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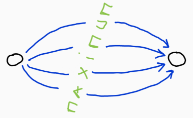

- Program analysis: complexity in worst case
- Game theory: gain optimisation

Some examples

$A = \{a, b\}$

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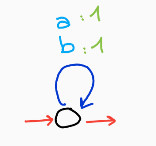

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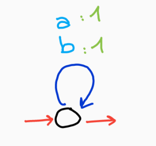

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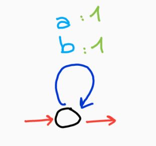

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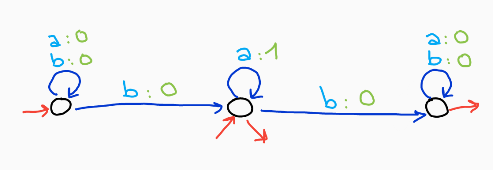

$w \mapsto$ length of the longest factor of consecutive a's

$$[\![\mathcal{A}]\!] = [\![\mathcal{B}]\!]?$$

The equivalence problem for max-plus automata is undecidable. [Krob 92]

Limits of undecidability?

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$\longrightarrow$ Relate several universal objects:
diophantine equations, two-counter machines

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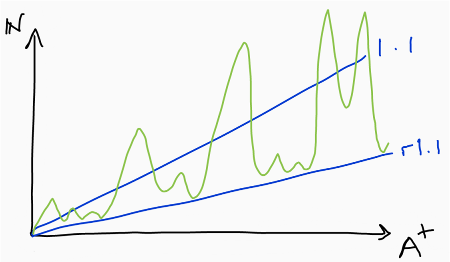

Joint spectral radius of tropical matrices

$\rightarrow$ *best previously known result: NP-hard [Blondel, Gaubert, Tsitsiklis]*

Theorem [D., Guillon, Merlet]

There is no algorithm that given a finite set of tropical matrices, computes its joint spectral radius.

But:

$\longrightarrow$ can be approximated - *from a result of [Colcombet,D.]*

$\longrightarrow$ can be computed in PSPACE when restricting to full matrices

Perspectives

Precise the limits of undecidability

- Equivalence problem for max-plus automata with 2 states (up to 546)? Link between the several universal objects?

- Computability-theoretic characterization of the joint spectral radius?