
The Shortest Identities for Max-Plus Automata with Two States

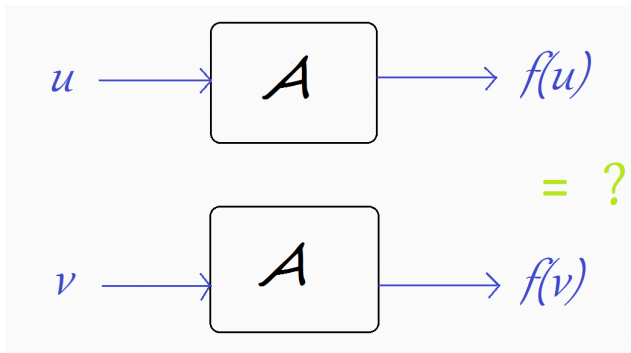
Laure Daviaud
University of Warsaw

With Marianne Johnson, University of Manchester

MFCS 2017

A natural and fundamental question:

Which pairs of inputs can be distinguished
by a given computational model?



Given a class of computational models:

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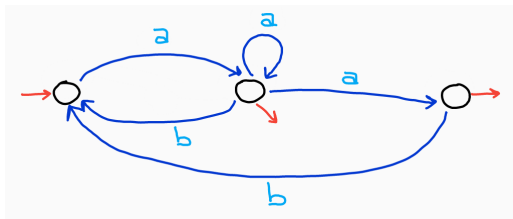
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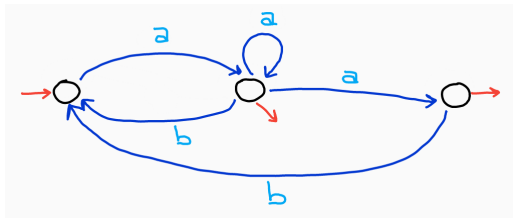
→ Is there an instance which can distinguish all the inputs?

→ What is the minimal size of an instance distinguishing two given inputs?

Automata

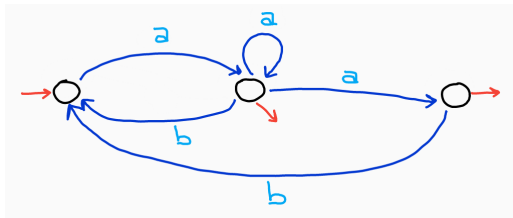


Automata



→ Are all the pairs of distinct inputs distinguishable by an instance of the class? **YES**

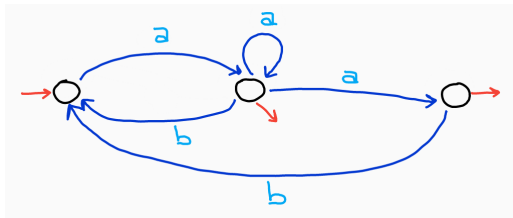
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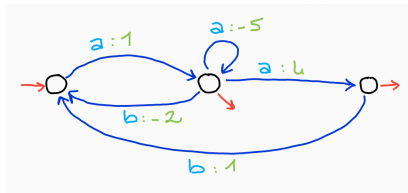
→ Are all the pairs of distinct inputs distinguishable by an instance of the class? **YES**

→ Is there an instance which can distinguish all the inputs? **NO**

→ What is the minimal size of an instance distinguishing two given inputs? **yields to the profinite theory**

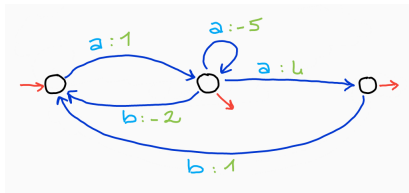
Weighted automata [Schützenberger]

Quantitative extensions of automata



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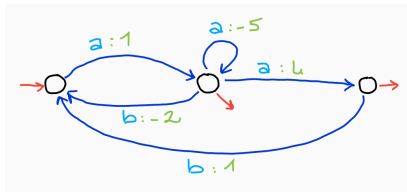
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Semiring (S, \oplus, \otimes) : transitions are weighted by elements of S

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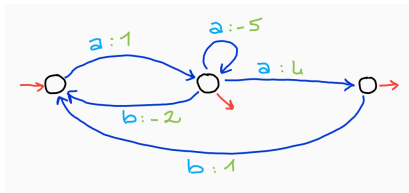
Paths: \otimes

Non-determinism: \oplus



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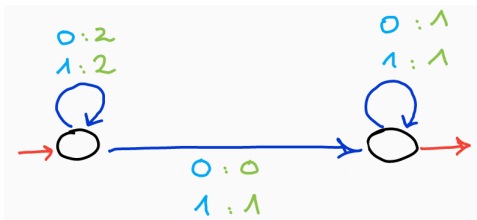
Non-determinism: \oplus



$$[\mathcal{A}] : w \mapsto \bigoplus_{\rho \text{ accepting path labelled by } w} (\rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_{|w|})$$

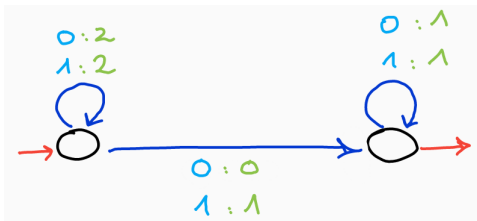
An example in $(\mathbb{R}, +, \times)$

$$A = \{0, 1\}$$



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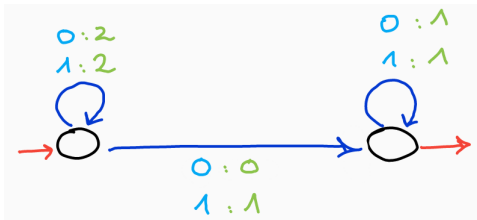
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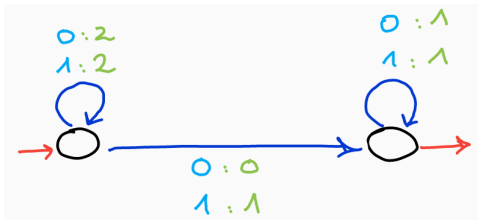


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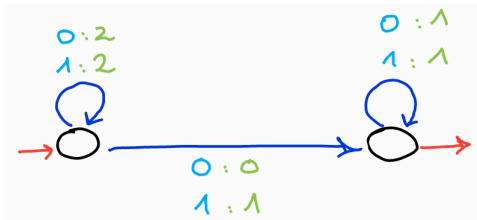
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→ Are all the pairs of distinct inputs distinguishable by an instance of the class? **YES**

→ Is there an instance which can distinguish all the inputs? **YES**

→ What is the minimal size of an instance distinguishing two given inputs? **1 or 2 states...**

Max-plus automata

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→ What is the minimal size of an instance distinguishing two given inputs? **???**

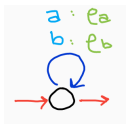
Given a positive integer n ,
are there $u \neq v$ such that
for all max-plus automata \mathcal{A} with at most n states:

$$[[\mathcal{A}]](u) = [[\mathcal{A}]](v) \quad ?$$

Some results

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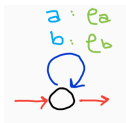
→ With one state



$$W \mapsto \rho_a |W|_a + \rho_b |W|_b$$

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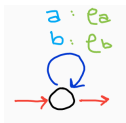
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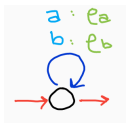
Existence of an identity (length 20) [Izhakian, Margolis]

→ With three states

Existence of an identity (length 1795308) [Shitov]

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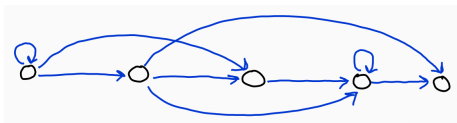
Existence of an identity (length 20) [Izhakian, Margolis]

→ With three states

Existence of an identity (length 1795308) [Shitov]

→ Triangular

Existence of identities for all n [Izhakian]



Max-plus automata with two states

Theorem [D.,Johnson]

There are two identities of minimal length which hold in the class of max-plus automata with two states:

$$a^2 b^3 a^3 babab^3 a^2 = a^2 b^3 ababa^3 b^3 a^2$$

and

$$ab^3 a^4 baba^2 b^3 a = ab^3 a^2 baba^4 b^3 a$$

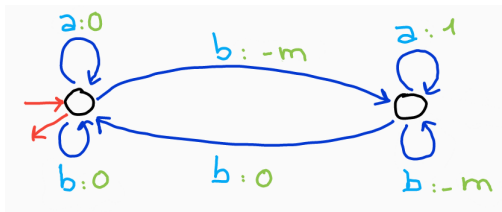
→ counter-example to a conjecture of Izhakian

An interesting list of criteria

→ list of criteria to eliminate all the shorter identities

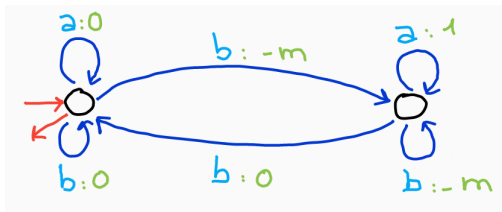
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- block-permutation
- triangular
- weights restricted to $\{0, 1\}$

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Characterize all the identities holding for the class of max-plus automata with at most n states, for all n ...

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- still quite far: Does there exist an identity for all n ?
- again quite far: Characterize the set of all the identities for $n = 2$
- Is there a strict subset of max-plus automata containing all their computational power?