About the description of functions computed by max-plus automata

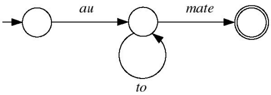
Laure Daviaud

University of Warsaw

based on joint works with T.Colcombet, P.Guillon, G.Merlet and F.Zuleger

PGMO Days 2016

Two points of view...



$+$

Optimisation

Semiring $(\mathbb{N} \cup \{-\infty\}, \max, +)$

$=$

Max-plus automata

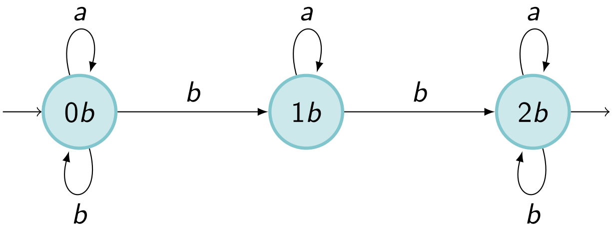
- automata representation
- matrix representation

Machine point of view: max-plus automata

alphabet $A = \{a, b\}$

words A^*: finite sequences of a and b

A word is *accepted* by the automaton if there is a path labelled by the word from an initial state to a final state.



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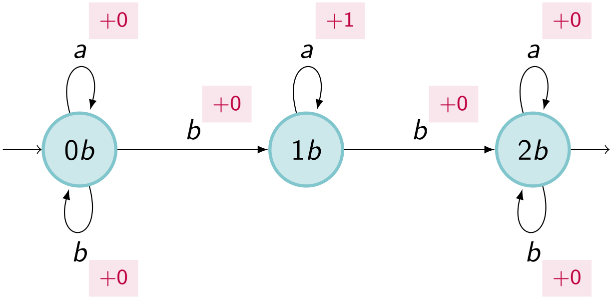

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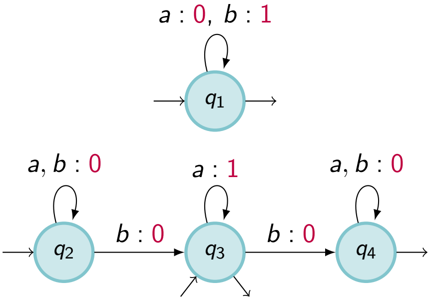

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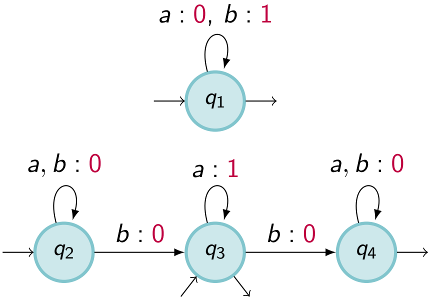

Syntax :

Non deterministic finite automaton for which each transition is labelled by a non negative integer (weight).

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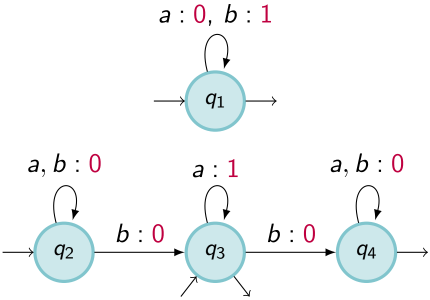

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Semantic :

Weight of a run = sum of the weights of the transitions.

$$\begin{array}{lll} \mathbb{A}^+ & \to & \mathbb{N} \cup \{-\infty\} \\ w & \mapsto & \text{Maximum of the weights of accepting runs labelled by } w \\ & & (-\infty \text{ if no such run}) \end{array}$$

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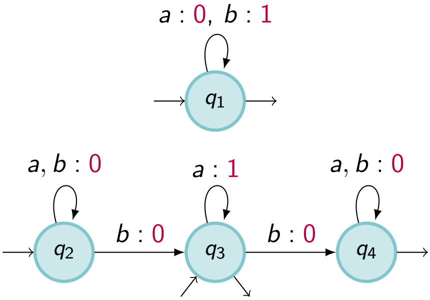

$a^{n_0} b a^{n_1} b \cdots b a^{n_k}$

$\mapsto \max(n_0, n_1, \ldots, n_k, k)$

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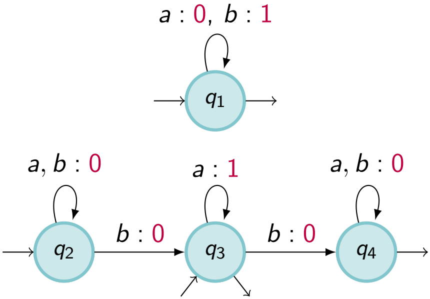
Or matrices...



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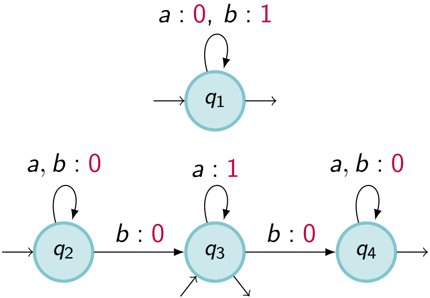

$$\begin{array}{c} \\ q_1 \\ q_2 \\ q_3 \\ q_4 \end{array} \begin{array}{c} \begin{array}{cccc} q_1 & q_2 & q_3 & q_4 \end{array} \\ \begin{pmatrix} 0 & \infty & \infty & \infty \\ \infty & 0 & \infty & \infty \\ \infty & \infty & 1 & \infty \\ \infty & \infty & \infty & 0 \end{pmatrix} \end{array} = \mu(a)$$

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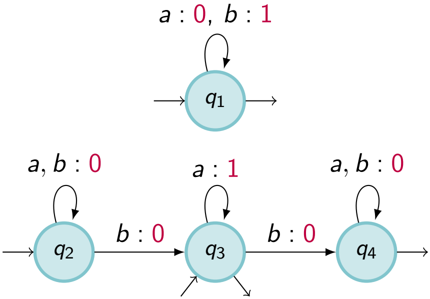

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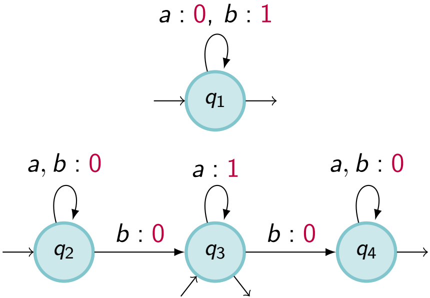

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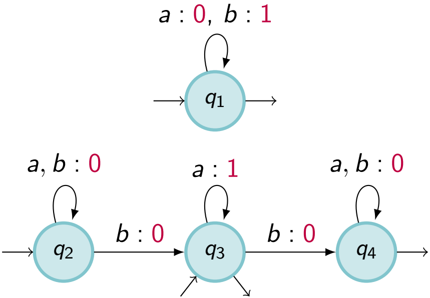

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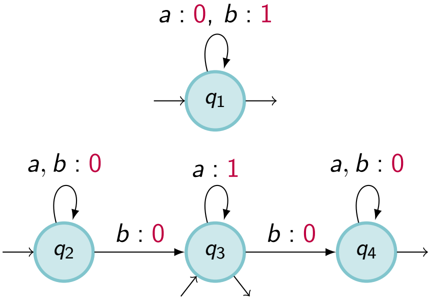

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$\mu(w)_{i,j} =$ max of the weights of the runs from i to j labelled by w

$$f(w) = I\mu(w)F$$

Correspondences...

Matrices

Finite set of ℓ matrices
$X = \{M_1, M_2, \ldots, M_\ell\}$
of dimension d

Automaton

Transitions over ℓ letters
$A = \{a_1, a_2, \ldots, a_\ell\}$
in an automaton $\mathcal{A}$ with d states

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$M_{i_1} M_{i_2} \cdots M_{i_k} \in \langle X \rangle$

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Automaton over the words
Behaviour over $a_{i_1} a_{i_2} \cdots a_{i_k}$

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Select the coefficient (i, j) of the matrix $M_{i_1} M_{i_2} \cdots M_{i_k}$	q_i initial, q_j final compute $[\![\mathcal{A}]\!](a_{i_1} a_{i_2} \cdots a_{i_k})$

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$\{\\|M\\|_\infty \mid M \in \langle X \rangle\}$	All states are both initial and final $\{[\![\mathcal{A}]\!](w) \mid w \in A^*\}$

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Joint spectral radius of a finite set of matrices:

$$\rho(X) = \inf_{k>0} \left\{ \frac{1}{k} \|M_{i_1} \cdots M_{i_k}\|_\infty \,\middle|\, M_{i_1}, \ldots, M_{i_k} \in X \right\}$$

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$\longrightarrow$ undecidable: Extension of the result of Krob to automata having all their states both initial and final [D., Guillon, Merlet]

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- Restrictions

- Approximations

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Input: A max-plus automaton $\mathcal{A}$ with at most *some number* states on a two-letter alphabet.

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What about between 2 states and 553 states?

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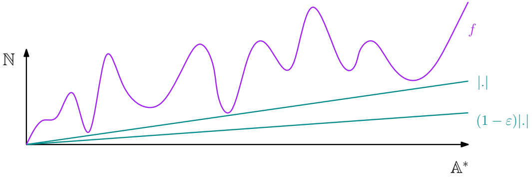

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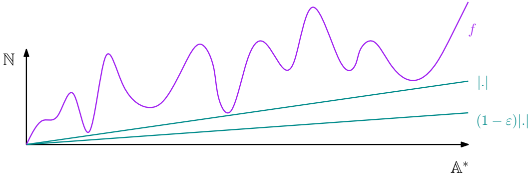

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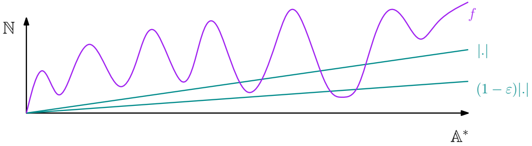

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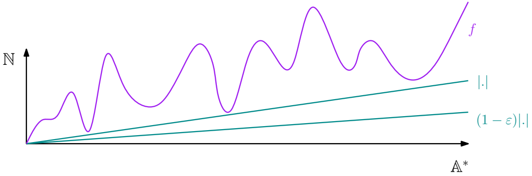

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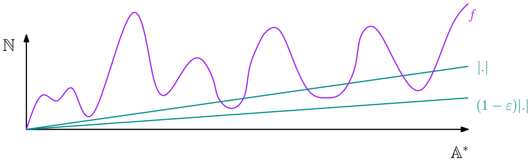
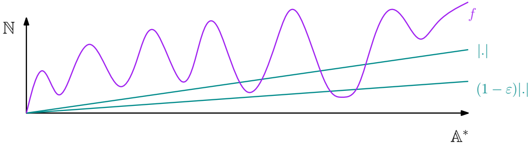


Yes or No



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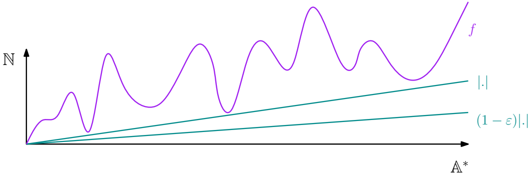

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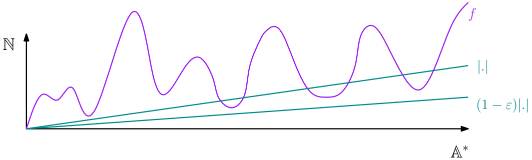

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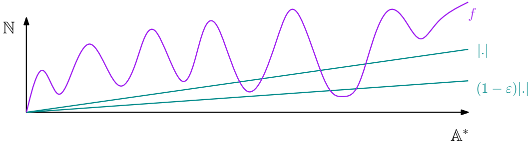
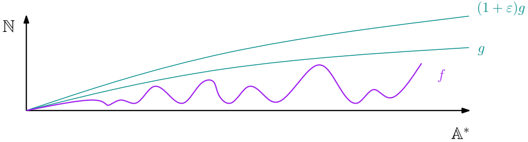


No

$\longrightarrow$ this also gives an approximation of the joint spectral radius.

Approximation [Colcombet,D.]

Input: f, g computed by min-plus automata, $\varepsilon > 0$



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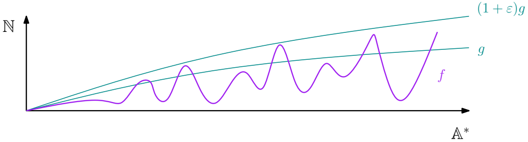

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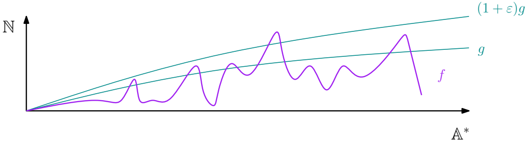
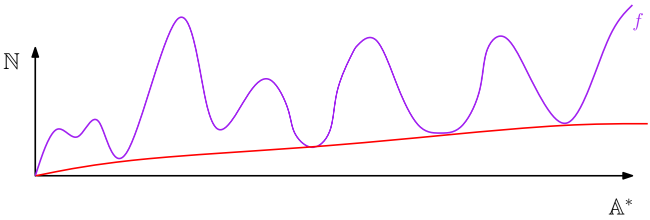


No

$\longrightarrow$ open for max-plus

Approximation [Colcombet,D.,Zuleger]



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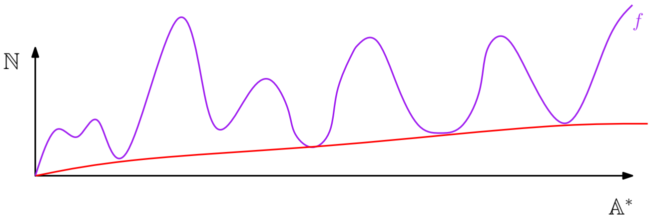

$$
\begin{array}{rccc}
f_{\inf}: & \mathbb{N} & \to & \mathbb{N} \cup \{-\infty\} \\
& n & \mapsto & \inf\limits_{|w| \geqslant n} f(w)
\end{array}
$$

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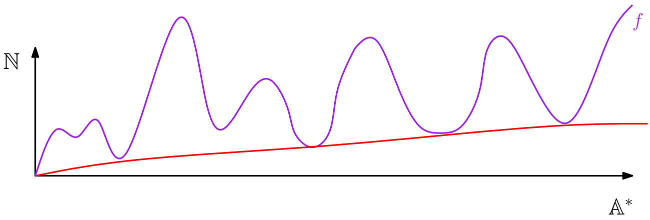

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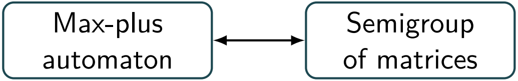
There is an algorithm that, given f computed by a max-plus automaton, computes a rational α (in $[0,1] \cup \{-\infty\}$) such that:

$$f_{\text{inf}}(n) = \Theta(n^{\alpha})$$

General ideas



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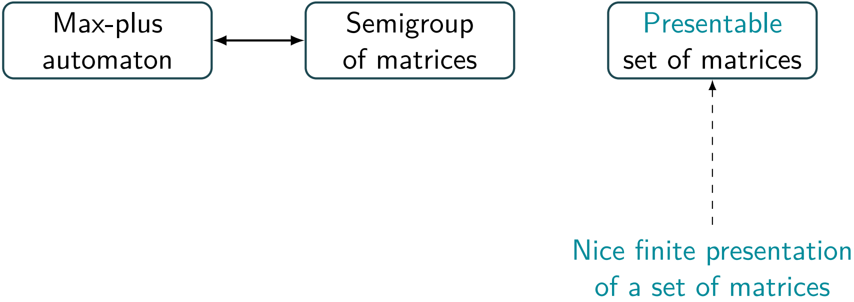

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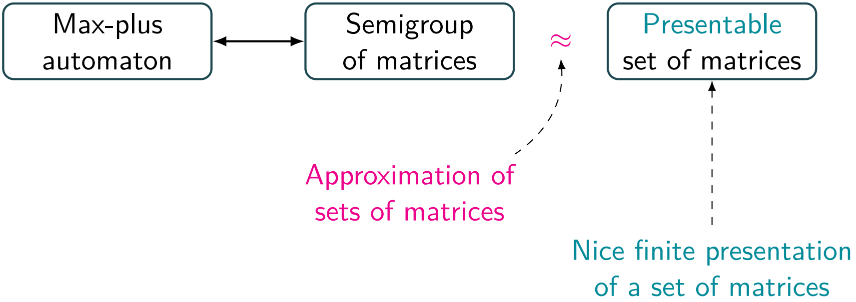

Forest factorisation theorem of Simon

Max-plus automaton	↔	Semigroup of matrices $< X >$	$\approx$	Presentable set of matrices

$M_1 \quad M_2 \quad M_3 \quad M_4 \quad M_5 \quad M_6 \quad M_7 \quad \cdots\cdots \quad M_k$

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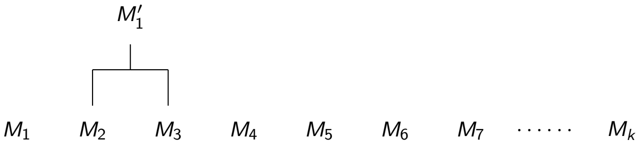

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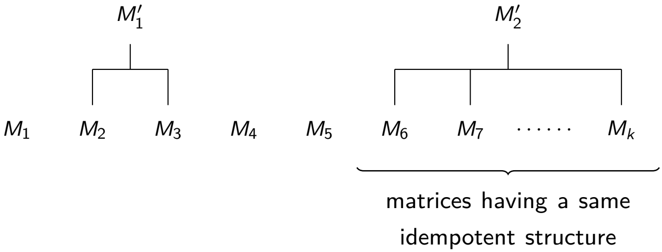

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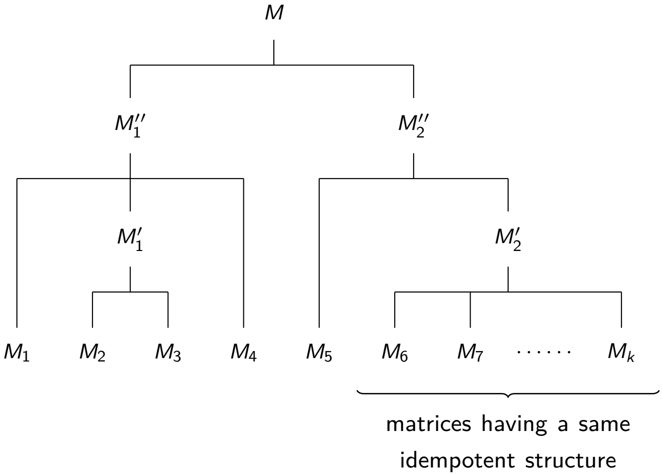

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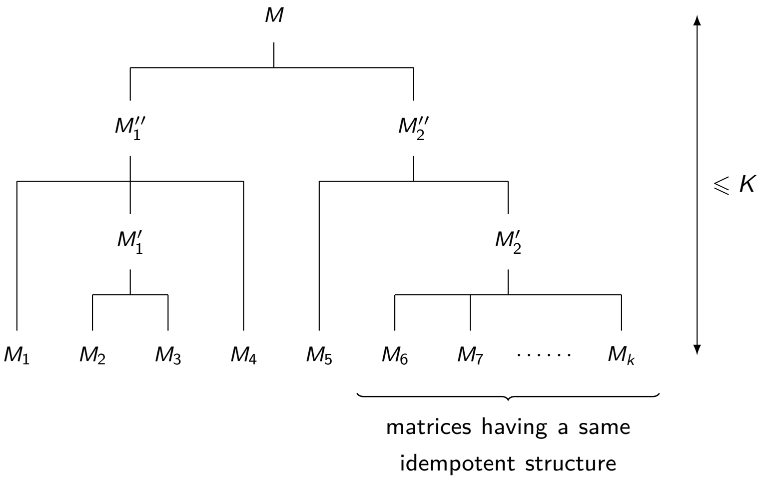

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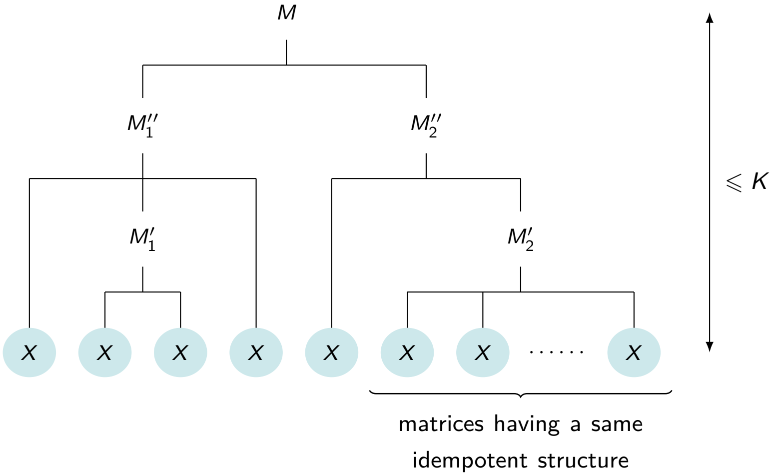

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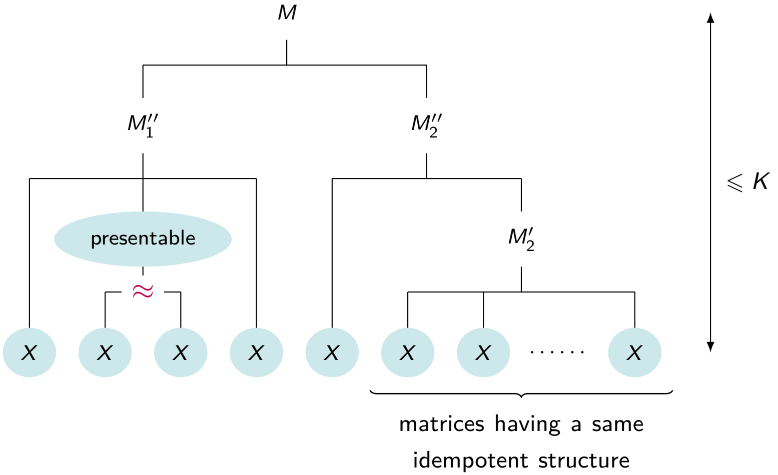

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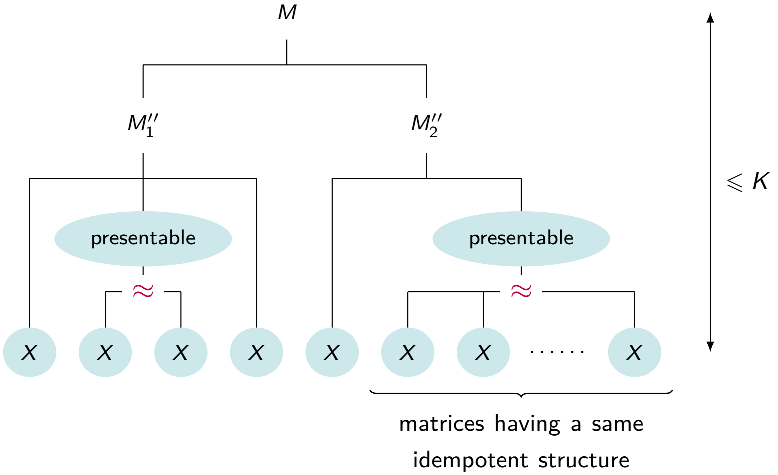

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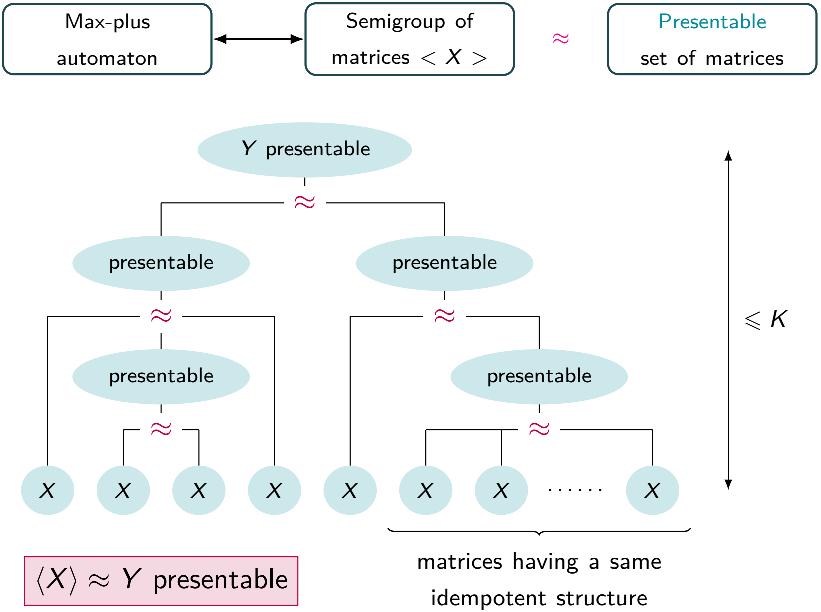

$\langle X \rangle \approx Y$ presentable