Asymptotic behaviour of Max-Plus Automata

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- Max-plus automata: definition and example
- Asymptotic behaviour of a max-plus automaton
- Application to the computational time complexity of terminating size-change abstraction instances
- Ideas of the main proof

Max-Plus Automata

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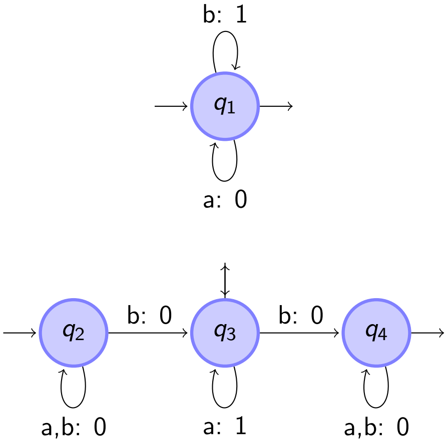
Max-plus automaton: Non deterministic finite automaton for which each transition is also labelled by a non-negative integer called the weight of the transition.

$$(\mathbb{A}, Q, I, T, E) \text{ with } E \subseteq (Q \times \mathbb{A} \times \mathbb{N} \times Q))$$



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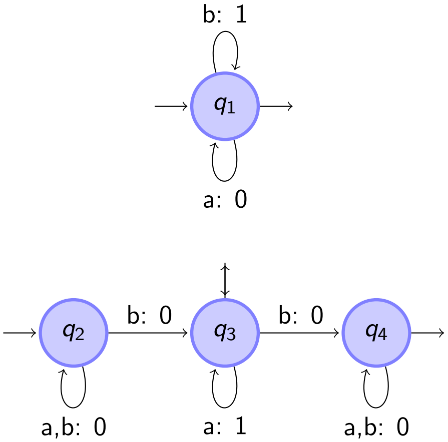

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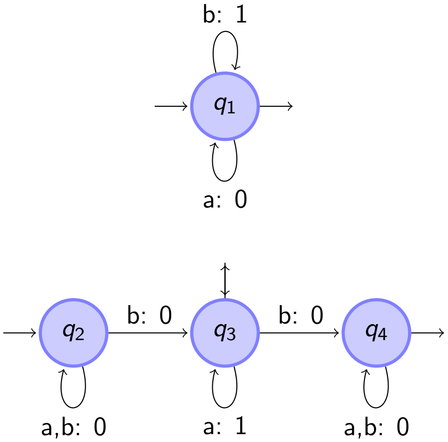
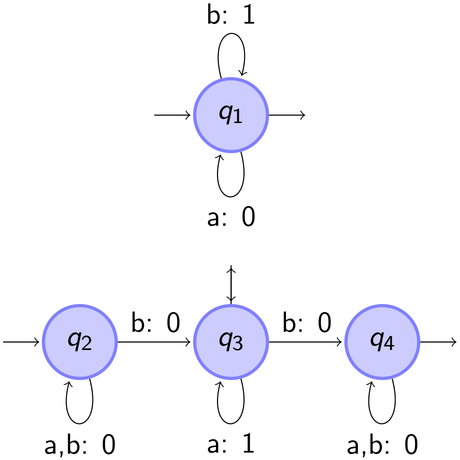


Weight of a run:
sum of the weights of the transitions

Computed function:

$$\begin{array}{lcl} \mathbb{A}^* & \to & \mathbb{N} \cup \{-\infty\} \\ w & \mapsto & \text{maximum of the weights of the runs labelled by } w \text{ going from an initial state to a final state } (-\infty \text{ if no such run}) \end{array}$$

Max-Plus Automata



Example: $a^m b a^n b a^p$

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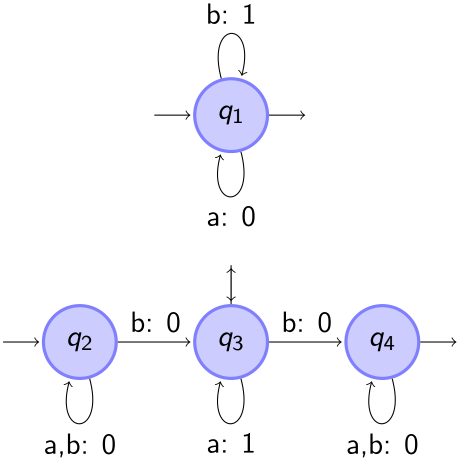

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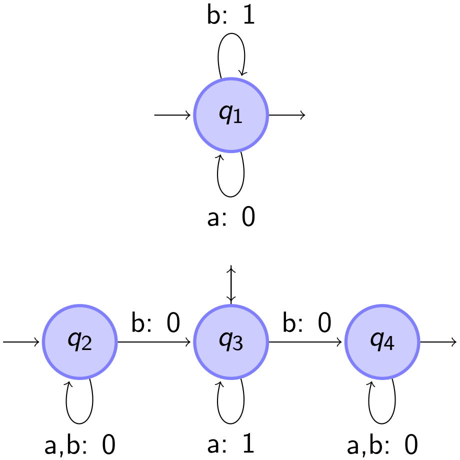
weight of run (1): 2

Max-Plus Automata



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weight of run (2): n

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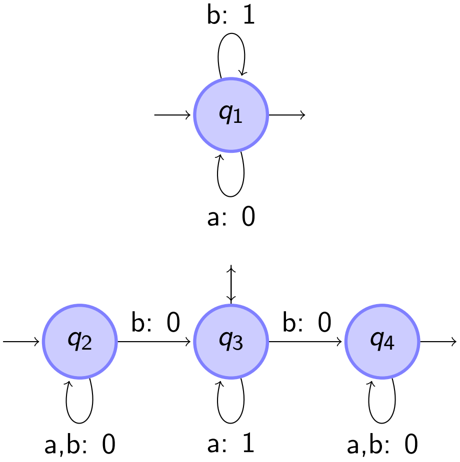

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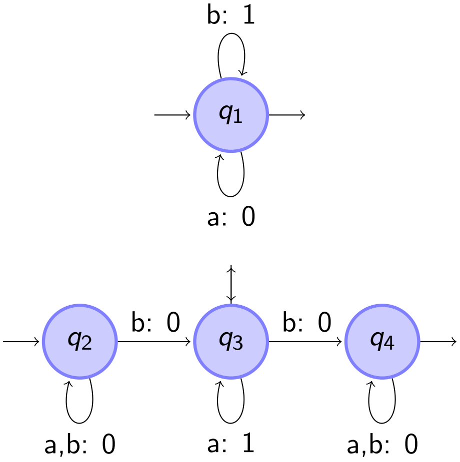

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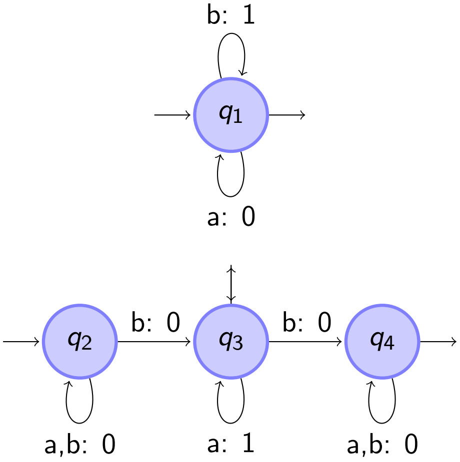

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$a^m b a^n b a^p \mapsto \max(2, m, n, p)$

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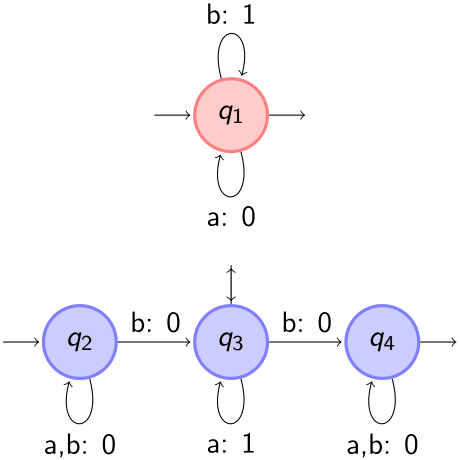

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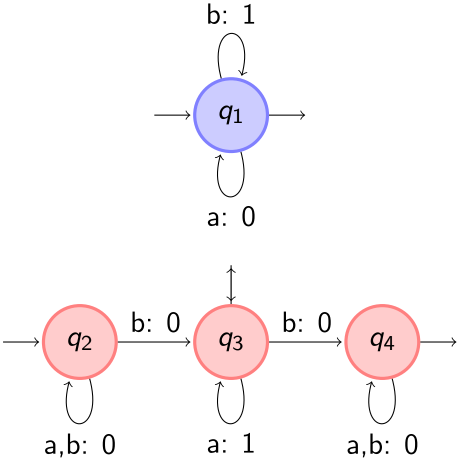

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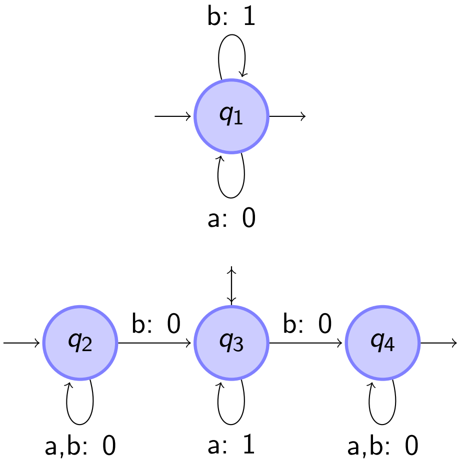

$$a^{n_0} b a^{n_1} b \cdots b a^{n_k} \mapsto \max(n_0, n_1, \ldots, n_k, k)$$

Asymptotic behaviour of a max-plus automaton

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Theorem [Krob, 92 (equivalent to min-plus automata)]

The following problems are undecidable:
Given f and g computed by max-plus automata,

- is $f \leqslant g$?
- is $f = g$?

$\rightsquigarrow$ Find other ways to look at the behaviour of functions computed by max-plus automata.

Asymptotic behaviour of a max-plus automaton

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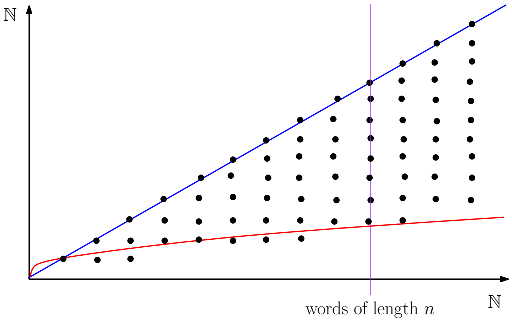

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Theorem

There exists effectively $\alpha \in (\mathbb{Q} \cap [0,1]) \cup \{-\infty\}$ such that $\overline{f}(n) = \Theta(n^\alpha)$.

- $\alpha = -\infty$: there is an infinite number of words of weight $-\infty$
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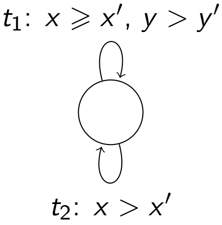
$\rightsquigarrow$ Length of the longest word having value at most n: $\Theta(n^{1/\alpha})$.

Application to the computational time complexity of terminating size-change abstraction instances

Size-Change Abstraction

Variables : x, y
Primed versions : x', y'



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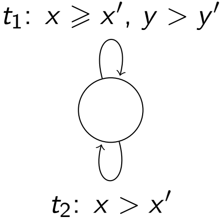

$$(5,5) \xrightarrow{t_1} (5,4) \xrightarrow{t_2} (4,8) \xrightarrow{t_1} (3,6) \xrightarrow{t_1} (3,2) \xrightarrow{t_2} (2,10)\ldots$$

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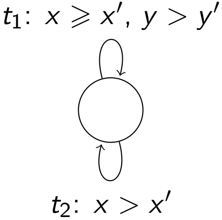

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- A finite number of variables that can take values in $\mathbb{N}$.
- Transitions: conjonction of a finite number of predicates of the form $x_i > x'_j$ or $x_i \geqslant x'_j$.
- A trace: sequence of transitions and valuations compatible with the transitions.

Size-Change Abstraction

Terminating SCA instance: no infinite trace.

Theorem [Lee, Jones, Ben-Amram]

It is decidable whether a given SCA instance is terminating.

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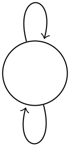

t_2: $x > x'$

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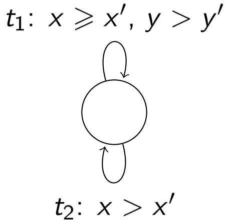

Restriction to $[0, n]$: What is the length of the longest trace ?

Size-Change Abstraction

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Given a terminating SCA instance, the longest trace is of order $\Theta(n^{\alpha})$, for some rational number α no smaller than 1.
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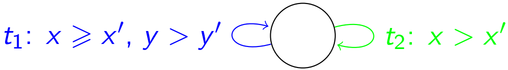

$$
\begin{array}{ccccccccc}
 & (n,n) & \xrightarrow{t_1} & (n,n-1) & \xrightarrow{t_1} & \cdots & \xrightarrow{t_1} & (n,0) \\
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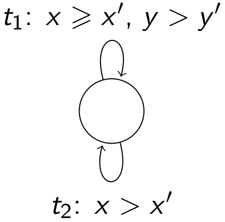
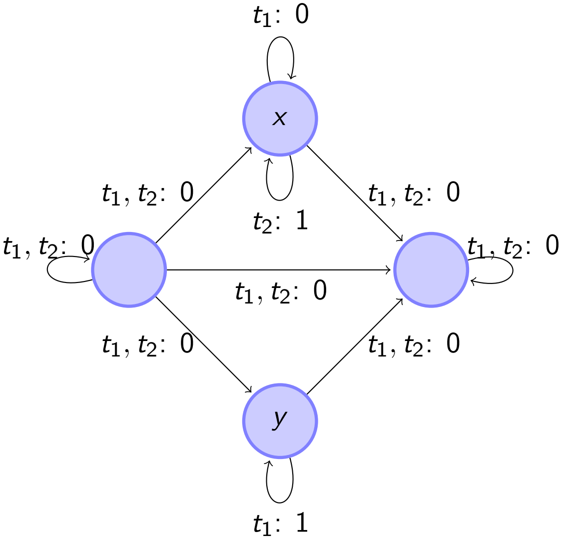
\end{array}
$$

Size-Change Abstraction

Longest trace
when variables belongs to $[0, n]$



Longest word
having value at most n



Ideas of the main proof

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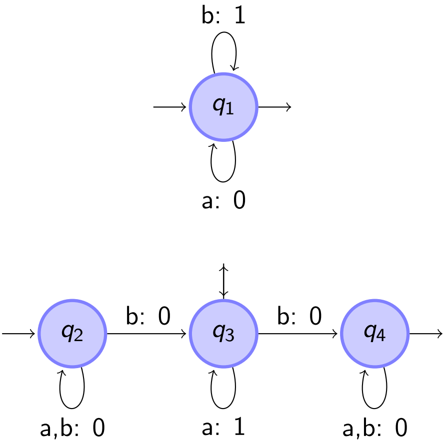
Max-Plus Automata: an algebraic view



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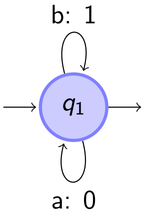

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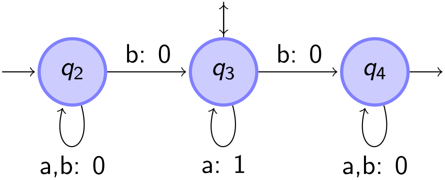

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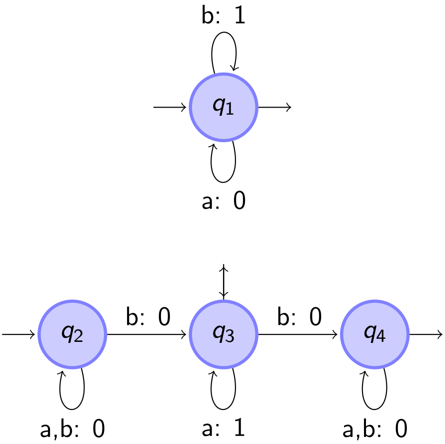

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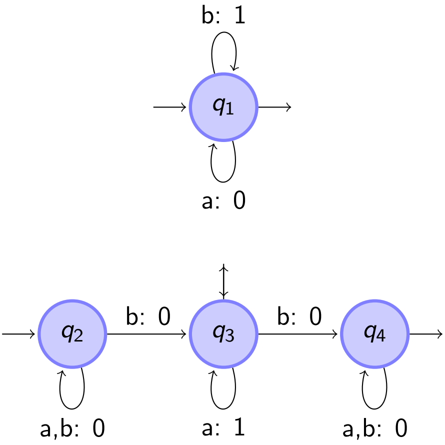

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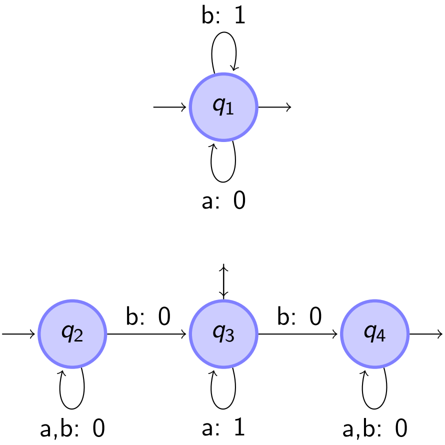

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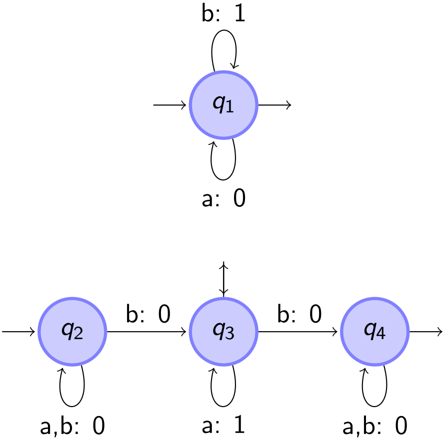

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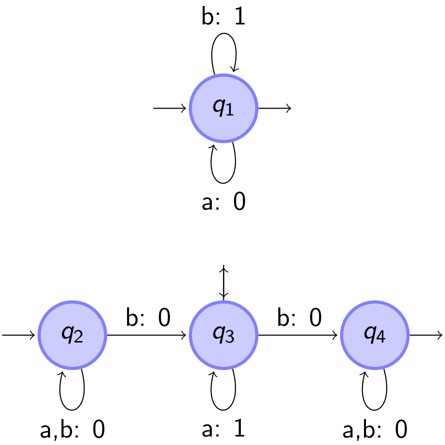
$$\mu(a_1 a_2 \cdots a_k) = \mu(a_1) \otimes \mu(a_2) \otimes \cdots \otimes \mu(a_k)$$

$\mu(w)_{i,j}$ is the maximal weight of runs going from i to j labelled by w.



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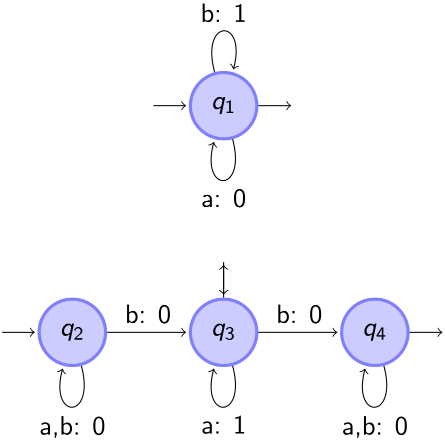

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$$f(w) = I \otimes \mu(w) \otimes F$$

Ideas for the proof

$f : \mathbb{A}^* \mapsto \mathbb{N} \cup \{-\infty\}$ computed by a max-plus automaton.

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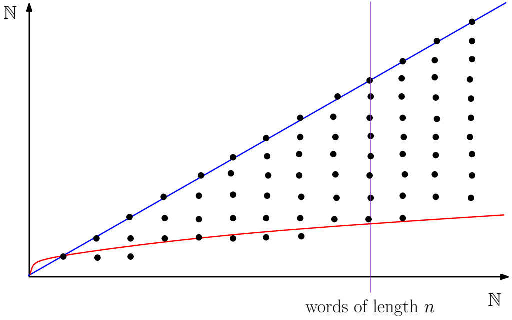
$\rightsquigarrow$ describe the asymptotic behaviour of infinite sequences of words.



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Describe the set $\{(\mu(w), |w|) \mid w \in \mathbb{A}^*\}$:

Ideas for the proof

$f : \mathbb{A}^* \mapsto \mathbb{N} \cup \{-\infty\}$ computed by a max-plus automaton.

$$\begin{array}{rccc} \overline{f}: & \mathbb{N} & \mapsto & \mathbb{N} \cup \{-\infty\} \\ & n & \rightarrow & \min\{f(w) \mid |w| = n\} \end{array}$$

Describe the set $\{(\mu(w), |w|) \mid w \in \mathbb{A}^*\}$:

- Represent the asymptotic behaviour of infinite sequences of words: presentable sets.

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Describe the set $\{(\mu(w), |w|) \mid w \in \mathbb{A}^*\}$:

- Represent the asymptotic behaviour of infinite sequences of words: presentable sets.
- Approximate the smallest pairs.

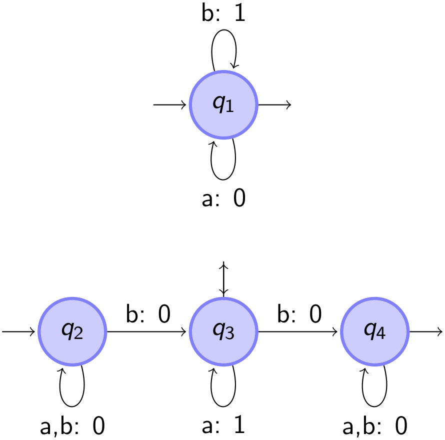
Ideas of the proof: presentable sets

Represent the asymptotic behaviour of an infinite sequence of words

$$\{(\begin{pmatrix} 0 & -\infty & n^{1/2} & n^{1/3} \\ 0 & 0 & n & 1 \\ n^{3/5} & -\infty & 1 & -\infty \\ -\infty & n^{2/3} & -\infty & 0 \end{pmatrix}, n) \mid n \in \mathbb{N}\}$$

Ideas of the proof: presentable sets

Represent the asymptotic behaviour of an infinite sequence of words



$$(a^n b)^n a^n$$

$$\begin{pmatrix} n^{1/2} & -\infty & -\infty & -\infty \\ -\infty & 0 & n^{1/2} & n^{1/2} \\ -\infty & -\infty & -\infty & n^{1/2} \\ -\infty & -\infty & -\infty & 0 \end{pmatrix}$$

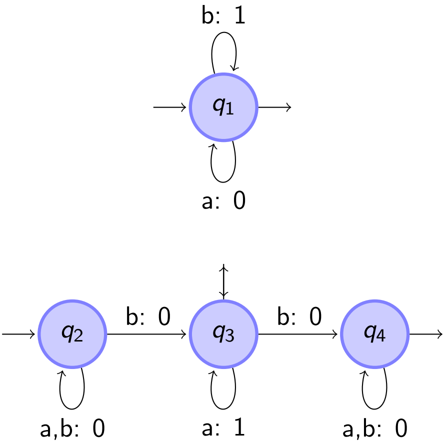
Ideas of the proof: presentable sets

Represent a convex polyhedra of exponents

$$\{(\begin{pmatrix} 0 & -\infty & n^{\lambda} & n^{\mu} \\ 0 & 0 & n & 1 \\ n^{\nu} & -\infty & 1 & -\infty \\ -\infty & n^{\eta} & -\infty & 0 \end{pmatrix}, n) \mid \left\{ \begin{array}{l} n \in \mathbb{N} \\ \lambda, \eta, \mu, \nu \in [0,1] \\ \mu + \eta \geqslant 1 \\ 5\lambda + 10\nu \geqslant 8 \end{array} \right. \}$$

Ideas of the proof: presentable sets

Represent a convex polyhedra of exponents



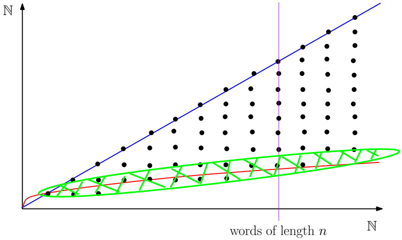
$$(a^*b)^*a^*$$

$$\begin{pmatrix} n^{1-\lambda} & -\infty & -\infty & -\infty \\ -\infty & 0 & 1 & n^{\lambda} \\ -\infty & -\infty & -\infty & n^{\lambda} \\ -\infty & -\infty & -\infty & 0 \end{pmatrix}$$

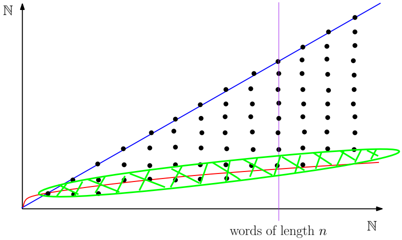
for all $\lambda \in [0,1]$

$$\begin{pmatrix} 1 & -\infty & -\infty & -\infty \\ -\infty & 0 & n & 1 \\ -\infty & -\infty & -\infty & 1 \\ -\infty & -\infty & -\infty & 0 \end{pmatrix}$$

Ideas of the proof: approximation



Ideas of the proof: approximation

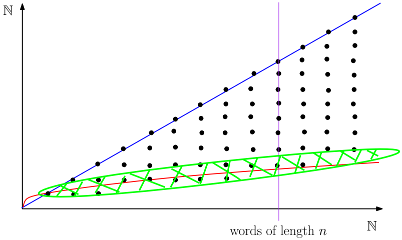


$(M, \ell) \preccurlyeq_a (N, k)$ if $M \leqslant aN$

$k \leqslant a\ell$

$\widetilde{M} = \widetilde{N}$

Ideas of the proof: approximation



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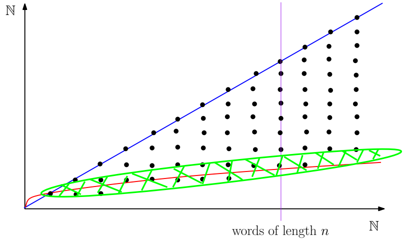
$\widetilde{M} = \widetilde{N}$

$X \preccurlyeq_a Y$ if for all $(N, k) \in Y$, there is $(M, \ell) \in X$ such that $(M, \ell) \preccurlyeq_a (N, k)$

Ideas of the proof: approximation



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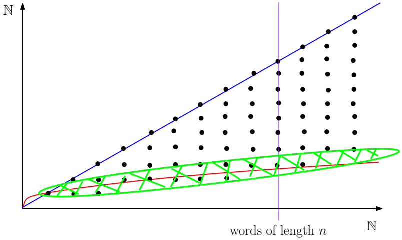
Ideas of the proof: approximation



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$X \approx_a Y$ if $X \preccurlyeq_a Y$ and $Y \preccurlyeq_a X$

$X \approx Y$ if there is a such that $X \approx_a Y$

Structure of the proof: forest factorization theorem

Approximate by presentable sets:

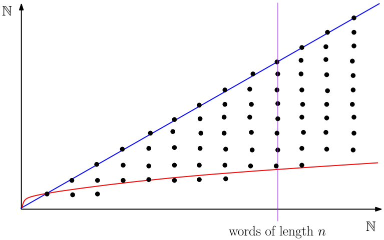
- the "product" of two presentable sets
- the "closure under product" of "idempotent" presentable sets

$\rightsquigarrow$ use of the forest factorization theorem of Simon:

- start with matrices corresponding to letters
- apply the two previous operations
- after a finite number of steps, we get a presentable set that approximates the set $\{(\mu(w), |w|) \mid w \in \mathbb{A}^*\}$

Conclusion and further questions

- What about min-plus automata?



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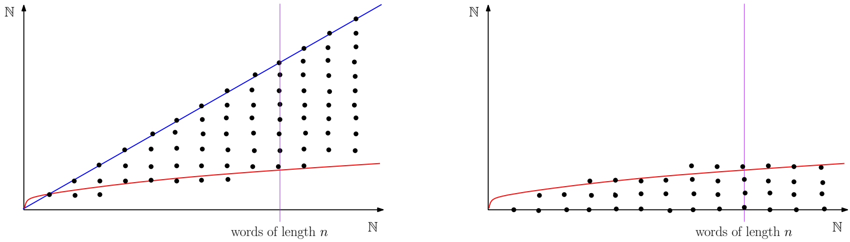

- Compute the multiplicative coefficient.
  (done for min-plus and $|.|$ up to an ε-approximation)