Approximate comparison of distance automata

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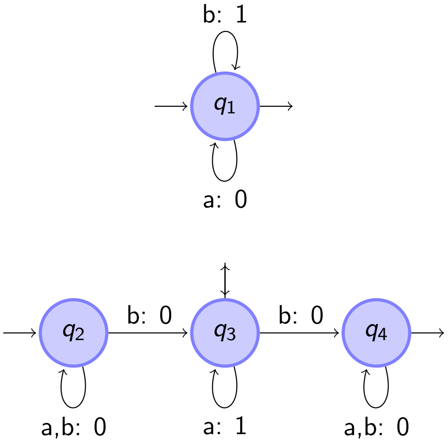
Distance automata

Distance automaton: Non deterministic finite automaton for which each transition is also labelled by a non-negative integer called the weight of the transition.

$$(\mathbb{A}, Q, I, T, E) \text{ with } E \subseteq (Q \times \mathbb{A} \times \mathbb{N} \times Q))$$



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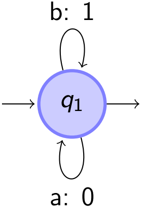
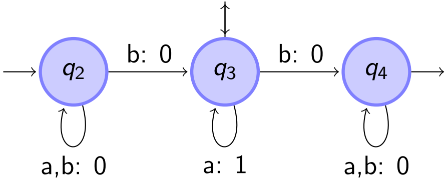


<u>Weight of a run</u>:
sum of the weights of the transitions



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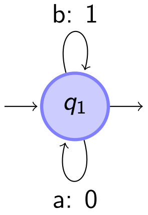
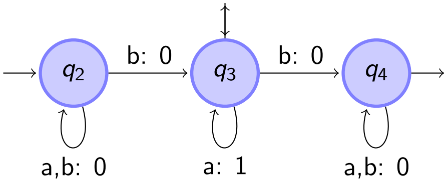


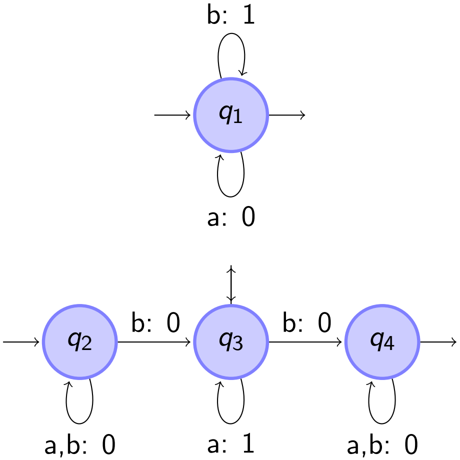
Weight of a run:
sum of the weights of the transitions

Computed function:

$$\begin{array}{rcl} A^* & \to & \mathbb{N} \cup \{+\infty\} \\ w & \mapsto & \text{minimum of the weights of} \\ & & \text{the runs labelled by } w \text{ going} \\ & & \text{from an initial state} \\ & & \text{to a final state} \\ & & (+\infty \text{ if no such run}) \end{array}$$

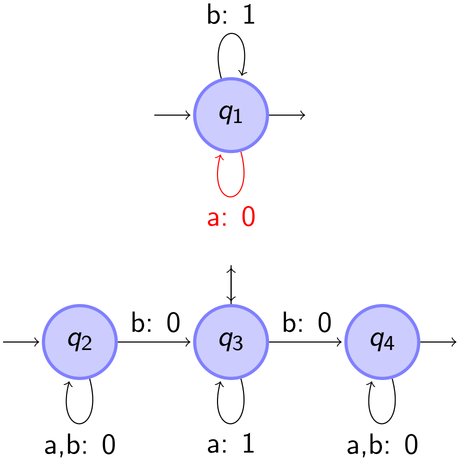


Distance automata



Example: $a^m b a^n b a^p$

Distance automata



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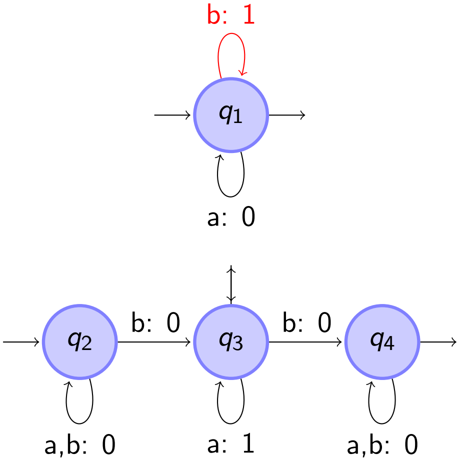
weight of run (1):

Distance automata



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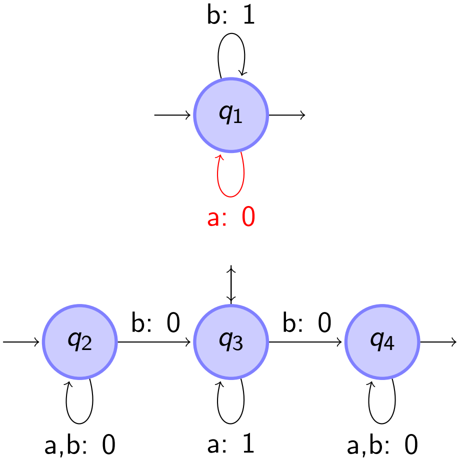

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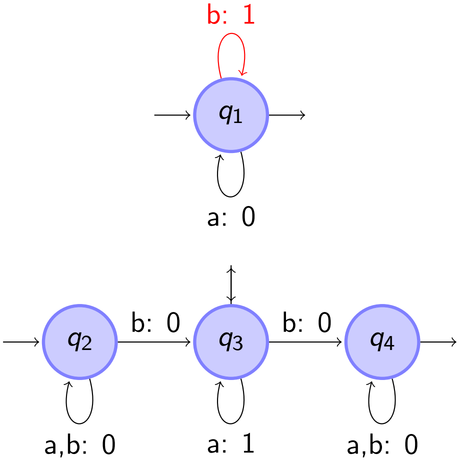

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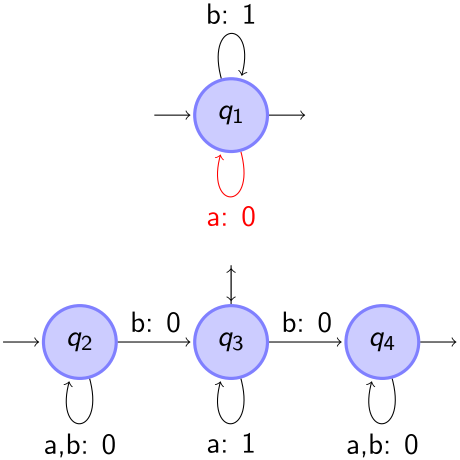
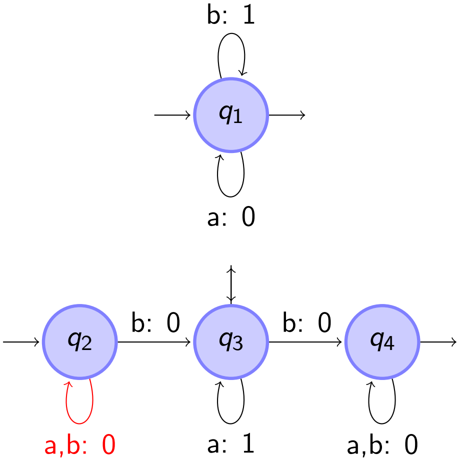


Example: $a^m b a^n b a^p$

weight of run (1): 2



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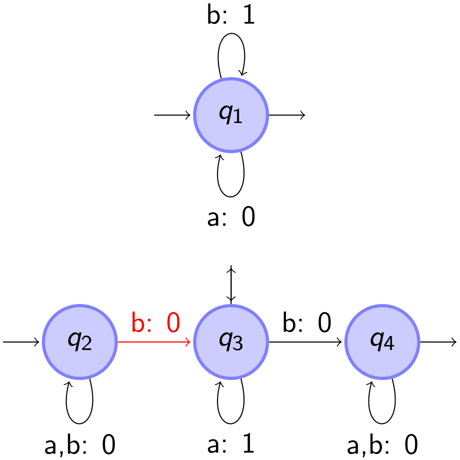
weight of run (1): 2
weight of run (2):

Distance automata



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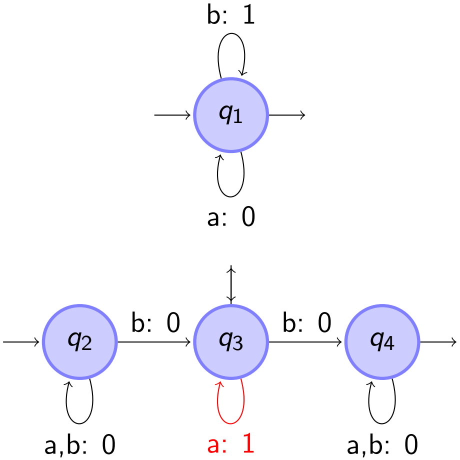

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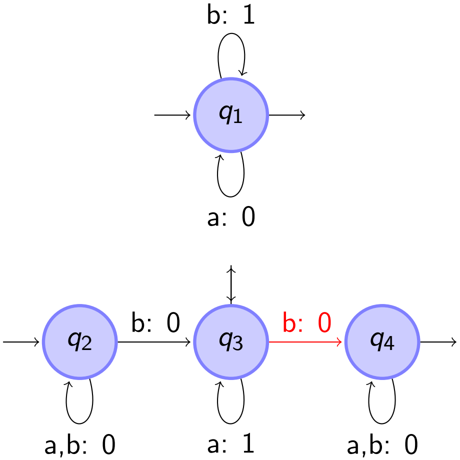

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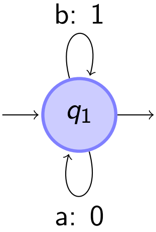
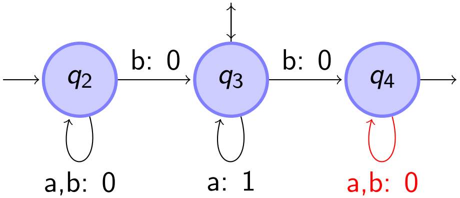


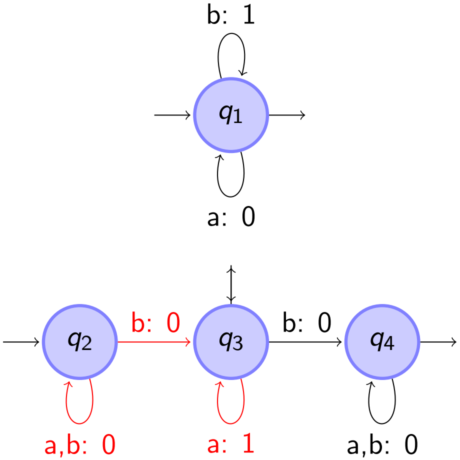


Example: $a^m b a^n b a^p$

weight of run (1): 2

weight of run (2): n

Distance automata



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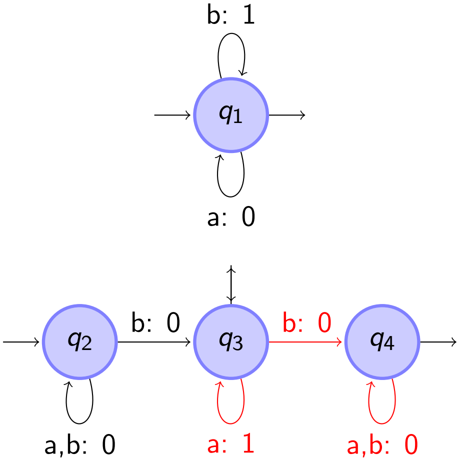

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weight of run (1): 2
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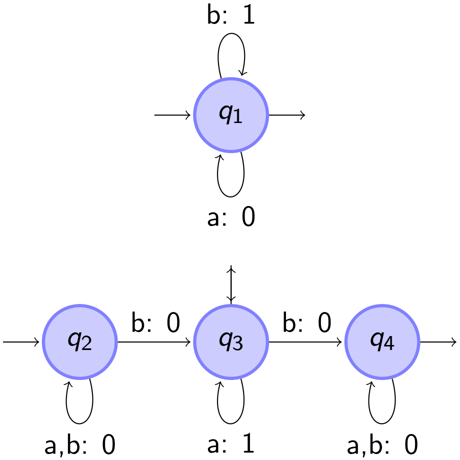
Distance automata



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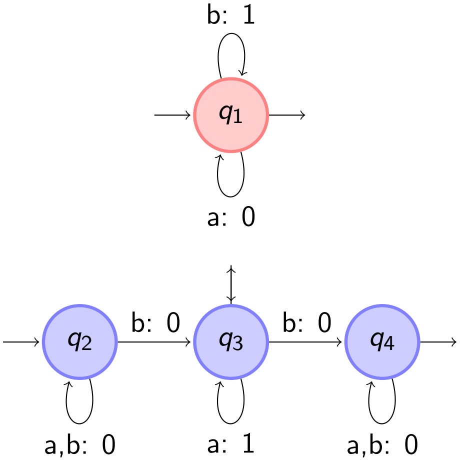

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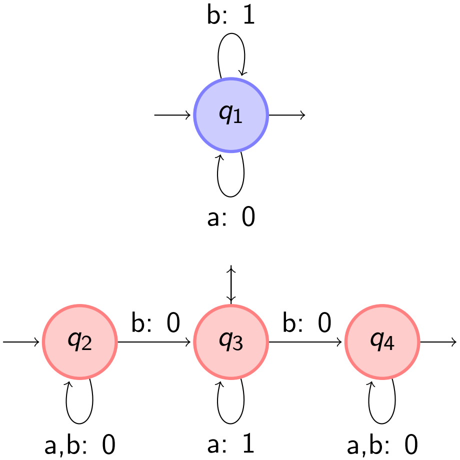

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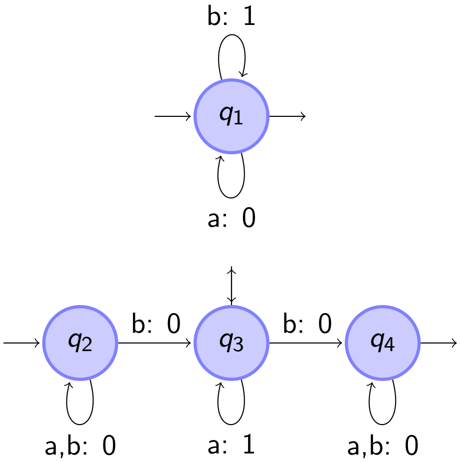
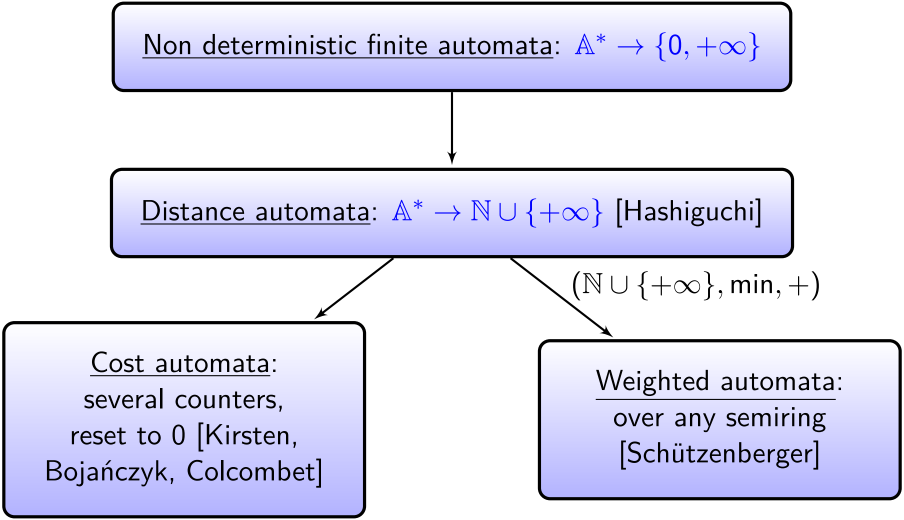


$$a^{n_0} b a^{n_1} b \cdots b a^{n_k} \mapsto \min(n_0, n_1, \ldots, n_k, k)$$

Context: different kinds of automata computing functions



How can we compare two functions given by distance automata?

Decision problems on comparison

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f, g computed by distance automata :

$$f \leqslant g \text{ if for all words } w,\ f(w) \leqslant g(w)$$

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Undecidable [Krob, 92]

Given f, g computed by distance automata, is $f \leqslant g$?

Decidable [Colcombet, 09]

Is there a polynomial P s.t $f \leqslant P \circ g$? (context of cost functions)

Generalisation of results by Hashiguchi, Leung and Simon

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Generalisation of results by Hashiguchi, Leung and Simon

Where does the border of decidability lie?

Decision problems on comparison: a first contribution

Proposition

Given f, g computed by distance automata, the two assertions are equivalent:

1. There is a polynomial P s.t $f \leqslant P \circ g$.
2. There is an integer a s.t $f \leqslant ag + a$.

Theorem

Given f, g computed by distance automata, one can decide if there is an integer a s.t $f \leqslant ag + a$.

Our main contribution: the approximate comparison of distance automata

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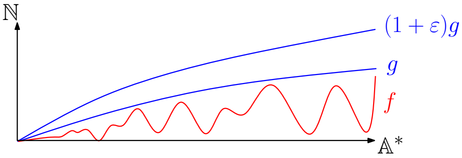
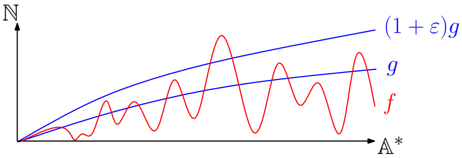
f,g computed by distance automata - Case $\varepsilon = 0$: undecidable result of Krob.



$\Longrightarrow$ YES



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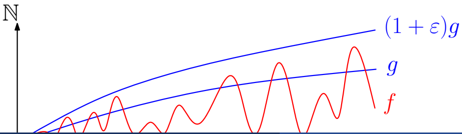

$\Longrightarrow$ YES or NO

Our main contribution

Theorem: There is an algorithm with the following behaviour:

Input:

- f, g computed by distance automata and $\varepsilon > 0$.

Output:

- *yes* if $f \leqslant g$,
- *no* if $f \nleqslant (1+\varepsilon)g$ (i.e if $\exists\ w$ such that $f(w) > (1+\varepsilon)g(w)$),
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Alternative *Output:*

- *yes* if $f \leqslant (1-\varepsilon)g$,
- *no* if $f \nleqslant g$ (i.e $\exists$ w such that $f(w) > g(w)$),
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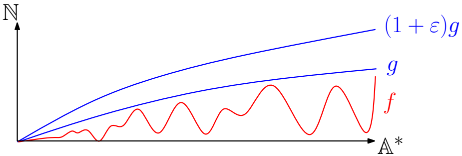
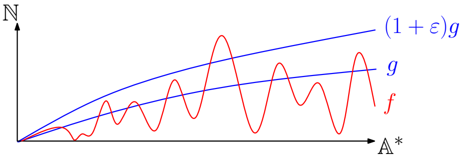
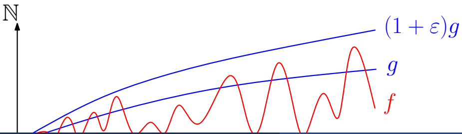
f,g computed by distance automata - Case $\varepsilon = 0$: undecidable result of Krob.



$\Longrightarrow$ YES



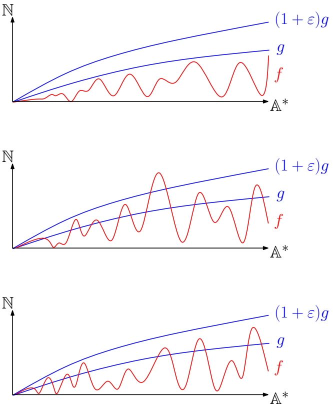
$\Longrightarrow$ NO



$\Longrightarrow$ YES or NO

Our main contribution: some ideas for the proof

f, g computed by distance automata



$\Longrightarrow$ YES

$\Longrightarrow$ NO

$\Longrightarrow$ YES or NO

- reduction to $g = |.|$
- use an algebraic view of distance automata with matrices in $(\mathbb{N} \cup \{+\infty\}, \min, +)$
- define an equivalence relation on matrices
- analyse with a small error long products of matrices that have the same structure
- use the forest factorization theorem of Simon to make the number of steps of the algorithm finite

Conclusion

Conclusion and further questions

- Given f, g computed by distance automata
  - is $f \leqslant g$? $\rightarrow$ undecidable [Krob]
  - is there a polynomial P s.t $f \leqslant P \circ g$? $\rightarrow$ decidable [Colcombet]
- Our contribution
  - is there an integer a s.t $f \leqslant ag + a$? $\rightarrow$ decidable
  - algorithm of approximate comparison - EXPSPACE
    (the problem is PSPACE-hard)
- Our algorithm approximates the joint spectral radius in $(\mathbb{N} \cup \{+\infty\}, \min, +)$.
- Next step: capture other kinds of asymptotic behaviours.