Computational Complexity

Class 5

Exercise 1. The parity function $P_n : \{0,1\}^n \to \{0,1\}$ is defined by:

$$P(x_0 \cdots x_{n-1}) = \left(\sum_{i=0}^{n-1} x_i\right) \mod 2$$

Construct a Boolean circuit computing P_3 and do the computation for the input 011.

Exercise 2. Consider a variant of Boolean circuits where the fan-in of the $\wedge$-gates and the $\vee$-gates is restricted to 2. Simulate any boolean circuit in this restricted model and estimate the change of parameters.

Exercise 3. Consider an extended definition of Boolean circuits by allowing negation gates with fan-in 1 (denoted $\neg$), which compute the negation of the values of their unique childs. Simulate any circuit in this new model by a usual Boolean circuit. Estimate the change of parameters, and explain how the game defined on a Boolean circuit should be extended in this version.

Exercise 4. Prove that any Boolean circuit can be simulated by a circuit of depth 2.

Exercise 5. Prove that, for all $\varepsilon < 1$, for sufficiently large n, there is a function $f : \{0,1\}^n \to \{0,1\}$ that cannot be computed by a circuit with 2^{n^ε} gates.

Exercise 6. Show that, for sufficiently large n, there is a function $f : \{0,1\}^n \to \{0,1\}$ that cannot be computed by a Boolean circuit with fan-in 2 (see exercise 2) with $\frac{2^n}{2n}$ gates.

Exercise 7. Let $A_n : \{0,1\}^{2n} \to \{0,1\}^{n+1}$ be the mapping which adds two binary sequences $x_0, \ldots, x_{n-1}$ and $x_n, \ldots, x_{2n-1}$. Construct a circuit with $O(n)$ gates with $n+1$ outputs computing A_n.

Exercise 8. Let $M_n : \{0,1\}^n \to \{0,1\}$ be the majority function mapping n input bits with 1 if and only if at least half of them have value 1. Construct a circuit with $O(n \log(n))$ gates computing M_n.

Exercise 9. Simulate any Boolean circuit with k gates with a circuit with $O(k)$ gates and using only majority gates (see exercise 8).

Exercise 10. Show that every regular language can be recognised by a sequence of circuits of polynomial size and depth $O(\log(n))$, even if the fan-in is required to be at most 2 for each gate.

Exercise 11. A language is star-free if it can be represented by a star-free regular expression: $R := a \mid \emptyset \mid \neg R \mid RR \mid R + R$. Show that a star-free regular language can be recognised by a sequence of circuits of polynomial size and constant depth.