Approximate comparison of functions computed by distance automata*

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Abstract

Distance automata are automata weighted over the semiring $(\mathbb{N} \cup \{\infty\}, \min, +)$ (the tropical semiring). Such automata compute functions from words to $\mathbb{N} \cup \{\infty\}$. It is known from Krob that the problems of deciding '$f \leqslant g$' or '$f = g$' for f and g computed by distance automata is an undecidable problem. The main contribution of this paper is to show that an approximation of this problem is decidable.

We present an algorithm which, given $\varepsilon > 0$ and two functions f, g computed by distance automata, answers "yes" if $f \leqslant (1-\varepsilon)g$, "no" if $f \not\leqslant g$, and may answer "yes" or "no" in all other cases. This result highly refines previously known decidability results of the same type.

The core argument behind this quasi-decision procedure is an algorithm which is able to provide an approximated finite presentation of the closure under products of sets of matrices over the tropical semiring.

We also establish another theorem, of affine domination, stating that previously known decision procedures for cost-automata have an improved precision when used over distance automata.

1 Introduction

One way to see language theory, and in particular the theory of regular languages, is as a toolbox of constructions and decision procedures allowing high level handling of languages. These high level operations can then be used as black-boxes in various decision procedures, such as in verification.

Since the early times of automata theory, the need for the effective handling of functions rather than sets (as languages) was already apparent. Schützenberger proposed already in the sixties models of finite state machines used for computing functions. These are now known as weighted automata [9] and are the subject of much attention from the research community. In general, weighted automata are non-deterministic automata, weighted over some semiring $(S, \oplus, \otimes)$. The value computed by such an automaton over a given word is then the sum (for $\oplus$) over every run over this word of the product (for $\otimes$) of the weights along the run.

Several instances of this model are very relevant for modelling the behaviour of systems, and henceforth attract much attention. This is in particular the case of probabilistic

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automata (over the semiring $(\mathbb{R}^+, +, \times)$ with some additional stochastic assumption enforcing weights to remain in $[0, 1]$), and distance automata which are automata weighted over the semiring $(\mathbb{N} \cup \{\infty\}, \min, +)$. In such an automaton, each transition is labelled with a non-negative integer (usually 0 or 1), and the weight of a word is the minimum over all possible paths of the sum of the weights. These automata naturally capture some optimisation problems since computing the value amounts to find the path of minimal weight.

The subject of this paper is to develop algorithmic tools for distance automata, and more precisely to develop the question of comparing distance automata. We know from the beginning that exact comparison is beyond reach.

Theorem 1 (Krob [6]). *The problem to determine, given two functions f, g computed by distance automata, if $f = g$ is undecidable. The problem $f \leqslant g$ is also undecidable, even if g is deterministic.*

Despite this, some positive results exist but for a comparison relation less precise than inequality, namely *domination*. Given two functions $\mathbb{A}^* \to \mathbb{N} \cup \{\infty\}$, *$f$ is dominated* by g (and we note $f \preccurlyeq g$) if there is a function $\alpha : \mathbb{N} \to \mathbb{N}$, extended with $\alpha(\infty) = \infty$, such that

$$f \leqslant \alpha \circ g \ .$$

Moreover, if α is a polynomial, we say that *f is polynomially dominated by g*. The following theorem shows the good properties of the domination relation.

Theorem 2 ([2] extending results and techniques from [4, 8, 11, 5, 1]). *The domination of functions computed by distance automata is decidable. Furthermore, if a function dominates another, then it polynomially dominates it*[1]*.*

The motivation of this work is to improve Theorem 2 and to answer the following question:

> Is it possible to decide "approximations" of the inequality of functions computed by distance automata that are finer than domination ?

We answer positively this question in two ways. We first show:

Theorem 3 (affine domination). *Given two functions f and g computed by distance automata, if f is dominated by g then f is affinely dominated by g, i.e., $f \leqslant \alpha \circ g$ for some polynomial α of degree 1.*

A consequence of this theorem is that the decision procedure provided by Theorem 2 in fact decides the affine domination, which is finer than the polynomial domination[2].

Our second, and main contribution is an even more accurate decision-like procedure. We say that an algorithm, given two functions f and g and some real $\varepsilon > 0$, *ε-approximates the inequality* if:

- if $f \leqslant (1 - \varepsilon)g$, the output is "yes",
- if $f \not\leqslant g$, the output is "no",

[1] Technically, polynomial domination is not stated in [2] , but can be derived directly from the proofs which explicitly compute the function α using operations preserving polynomials.

[2] Theorem 2 holds for more general classes of automata, cost automata, for which affine domination does not hold. Affine domination is specific to distance automata.

- otherwise the output can be either "yes" or "no".

Hence, if such an algorithm answers "yes", one has a guaranty that $f \leqslant g$. Conversely if f is ε-inferior to g (meaning $f \leqslant (1-\varepsilon)g$), one is sure that the algorithm answers "yes". Our second and main result reads as follows:

Theorem 4 (approximate comparison)**.** *There is an EXPSPACE algorithm which ε-approximates the inequality of functions computed by distance automata.*

This result is in fact a consequence of a theorem – called the core theorem below – stating that it is possible, given a set of matrices X in the tropical semiring, to approximate (in a suitable way) the set

$$\left\{ \frac{1}{k}(M_1 \otimes \cdots \otimes M_k) \;:\; M_1, \ldots, M_k \in X \right\} ,$$

where $\otimes$ denotes the product of matrices. More precisely, the core theorem states that it is possible to approximate the upper envelope of the set of pairs

$$\{(M_1 \otimes \cdots \otimes M_k, k) \;:\; M_1, \ldots, M_k \in X\}$$

for a suitable notion of approximation. This core theorem, Theorem 5, requires several definition to be introduced before hand. This is the subject of the first section of this paper.

Organization of the paper.

In Section 2 we present some classical definitions and formally state our core theorem. Section 3 is devoted to the proof of the core theorem. Section 4 applies the core theorem for answering our original motivation, and shows the decidability of the approximate comparison between distance automata. We prove on the way our result of affine domination, Theorem 3. Section 5 concludes the paper.

2 Description of the core theorem

In this first section, we introduce the basic definitions, and define sufficient material for stating our core theorem 5. Its proof is the subject of Section 3 and its application to the comparison of distance automata is the subject of Section 4.

We first introduce some classical algebraic definitions in Section 2.1, and finally state our core theorem in Section 2.2.

2.1 Standard definitions

A *semigroup* $(S, \cdot)$ is a set S equipped with an associative binary operation "$\cdot$". If the product has furthermore a neutral element, it is called a *monoid*. The monoid is said *commutative* when $\cdot$ is commutative. An *idempotent* in a monoid is an element e such that $e \cdot e = e$. Given a subset A of a semigroup, $\langle A \rangle$ denotes the closure of A under product, *i.e.,* the least sub-semigroup that contains A. Given two subsets X, Y of a semigroup, $X \cdot Y$ denotes the set $\{a \cdot b \;:\; a \in X,\; b \in Y\}$.

A *semiring* is a set S equipped with two binary operations $\oplus$ and $\otimes$ such that $(S, \oplus)$ is a commutative monoid of neutral element 0, $(S, \otimes)$ is a monoid of neutral element 1, 0 is absorbing for $\otimes$ (i.e., $x \otimes 0 = 0 \otimes x = 0$) and $\otimes$ distributes over $\oplus$. We will consider three semirings: $(\mathbb{R}^+ \cup \{\infty\}, \min, +)$, denoted $\overline{\mathbb{R}^+}$, its restriction to $\mathbb{N} \cup \{\infty\}$, denoted $\overline{\mathbb{N}}$, and its restriction to $\{0, \infty\}$ denoted $\mathbb{B}$. The semiring $\mathbb{B}$ is called the *Boolean semiring*, since if we identify 0 with "true" and ∞ with "false", then $\oplus$ is the disjunction and $\otimes$ the conjunction. Remark that in the three cases, the "0" is ∞, and the "1" is 0.

Let S be one of the above semirings. The set of matrices with m rows and n columns over S is denoted $\mathcal{M}_{m,n}(S)$. For $M \in \mathcal{M}_{m,n}(S)$, we denote by $\phi(M)$ the matrix over $\mathbb{B}$ in which all entries of M different from ∞ are changed into 0. We define the multiplication $A \otimes B$ of two matrices A, B (provided the number n of columns of A equals the number of rows of B) as usual by:

$$(A \otimes B)_{i,j} = \bigoplus_{0<k\leqslant n} (A_{i,k} \otimes B_{k,j}) = \min_{0<k\leqslant n} (A_{i,k} + B_{k,j}) \ .$$

For a positive integer k, we also use the notation $M^k = \underbrace{M \otimes \cdots \otimes M}_{k \text{ times}}$.

For $\lambda \in S$, we denote by λA the matrix such that $(\lambda A)_{i,j} = \lambda A_{i,j}$ for all i, j, with the convention that $\lambda\infty = \infty$ (the standard product is used here, not the one of the semiring). Note in particular that $\phi(M) = 0M$. We also denote by $B + \lambda$ the matrix such that $(B + \lambda)_{i,j} = B_{i,j} + \lambda$ for all i, j. Finally, we write $A \leqslant B$ if $A_{i,j} \leqslant B_{i,j}$ for all i, j.

2.2 Weighted matrices, approximation, finitely presented sets, and the core theorem

In this section we state our core approximation result, Theorem 5. This theorem states that given a set of weighted matrices, it is possible to compute a finite presentation of its closure under product up to some approximation. Hence we have to introduce weighted matrices, the approximation, and what are finite presentations before disclosing the statement. This requires some specific definitions that we present beforehand. We fix now a positive integer n, and all matrices implicitly belong to $\mathcal{M}_{n,n}(\overline{\mathbb{R}^+})$.

As already mentioned in the introduction, our goal is to approximate a set of pairs (M, ℓ) where M is a matrix and ℓ is a positive integer. This is done using weighted matrices. A *weighted matrix* is an ordered pair (M, ℓ) where $M \in \mathcal{M}_{n,n}(\overline{\mathbb{R}^+})$ and $\ell \in \mathbb{N}$ is non-null. The positive integer ℓ is called the *weight* of the weighted matrix. The set of weighted matrices is denoted by $\mathcal{W}_{n,n}$. Weighted matrices have a semigroup structure $(\mathcal{W}_{n,n}, \otimes)$, where $(M, \ell) \otimes (M', \ell')$ stands for $(M \otimes M', \ell + \ell')$. Given X, Y subsets of $\mathcal{W}_{n,n}$, one denotes by $X \otimes Y$ the set $\{M \otimes N \ : \ M \in X, \ N \in Y\}$, and by $\langle X \rangle$ the closure under $\otimes$ of X. With this terminology, our goal is, given a finite set of weighted matrices X, to approximate $\langle X \rangle$.

Along the proof, it will be important to control the maximal coefficient of weighted matrices related to their weight. Formally, $\mathcal{W}^a_{n,n} \subseteq \mathcal{W}_{n,n}$ denotes these weighted matrices (M, ℓ) such that the largest coefficient of M is at most $a\ell$. It is easy to check that $(\mathcal{W}^a_{n,n}, \otimes)$ is a subsemigroup of $(\mathcal{W}_{n,n}, \otimes)$.

We describe now the notion of approximation that we use. Given some $\varepsilon > 0$ and two weighted matrices (M, ℓ) and (M', ℓ'), we write

$$(M, \ell) \preccurlyeq_\varepsilon (M', \ell') \qquad \text{if} \qquad \ell \geqslant \ell', \quad \phi(M) = \phi(M') \text{ and } M \leqslant M' + \varepsilon\ell \ .$$

Remark 1. *Note that $(M, \ell) \preccurlyeq_\varepsilon (M', \ell')$ implies $\frac{1}{\ell}M \leqslant \frac{1}{\ell'}M' + \varepsilon$, and this is the intention behind this definition, i.e., being able to consider weighted matrices up to a multiplicative error of ε. In fact $(M, \ell) \preccurlyeq_\varepsilon (M', \ell')$ is a more restrictive definition than simply $\frac{1}{\ell}M \leqslant$*

$\frac{1}{\ell'}M' + \varepsilon$. *This is necessary since we want this notion to be robust with respect to the operations used later on in the proof. This is particular the case for Lemma 1 below.*

This definition extends to sets of weighted matrices as follows. Given two sets X, X' of weighted matrices, $X \preccurlyeq_\varepsilon X'$ if for all $(M,\ell) \in X$, there exists $(M',\ell') \in X'$ such that $(M,\ell) \preccurlyeq_\varepsilon (M',\ell')$. We also define $X \approx_\varepsilon X'$ to hold if both $X \preccurlyeq_\varepsilon X'$ and $X' \preccurlyeq_\varepsilon X$ (and we write that X is ε-equivalent to X').

The following lemma establishes some simple, yet essential, properties of the $\preccurlyeq_\varepsilon$ relations (as a consequence, the same properties hold for $\approx_\varepsilon$).

Lemma 1. *Given $X, X', Y, Y', Z \subseteq \mathcal{W}_{n,n}$ and $\varepsilon, \eta > 0$,*

- *if $X \preccurlyeq_\varepsilon Y$ and $Y \preccurlyeq_\eta Z$ then $X \preccurlyeq_{\varepsilon+\eta} Z$,*
- *if $X \preccurlyeq_\varepsilon X'$ and $Y \preccurlyeq_\varepsilon Y'$ then $X \otimes Y \preccurlyeq_\varepsilon X' \otimes Y'$,*
- *if $X \preccurlyeq_\varepsilon X'$ then $\langle X\rangle \preccurlyeq_\varepsilon \langle X'\rangle$.*

Proof. First item. If $(M,\ell) \preccurlyeq_\varepsilon (M',\ell') \preccurlyeq_\eta (M'',\ell'')$, then one gets $\ell \geqslant \ell' \geqslant \ell''$, $\phi(M) = \phi(M') = \phi(M'')$ and $M \leqslant M' + \varepsilon\ell \leqslant M'' + \eta\ell' + \varepsilon\ell \leqslant M'' + (\varepsilon+\eta)\ell$. This easily extends to sets of weighted matrices.
Second item. Assume that both $(M,\ell) \preccurlyeq_\varepsilon (M',\ell')$ and $(N,t) \preccurlyeq_\varepsilon (N',t')$. Then, $\ell + \ell' \geqslant t + t'$, $\phi(M \otimes N) = \phi(M' \otimes N')$ and $M \otimes N \leqslant (M' + \varepsilon\ell) \otimes (N' + \varepsilon t) \leqslant M' \otimes N' + \varepsilon(\ell + t)$. This naturally extends to sets of weighted matrices.
Third item. By induction, applying the second item. □

The last ingredient required is to describe how to represent (infinite) sets of weighted matrices. Call a set of weighted matrices $W \subseteq \mathcal{W}_{n,n}$ *finitely presented* if it is a finite union of singleton sets, and of sets of the form $\{(kM, k) : k \geqslant \ell\}$ where $M \in \mathcal{M}_{n,n}(\overline{\mathbb{R}^+})$ and ℓ is a positive integer. Our algorithm manipulates finitely presented sets of weighted matrices. Note that for each finitely presented set P, there is an a such that $P \subseteq \mathcal{W}^a_{n,n}$.

The core technical contribution of this paper can now be stated, as follows.

Theorem 5 (core theorem)**.** *Given a finitely presented set $X \subseteq \mathcal{W}_{n,n}$ and a real $\varepsilon > 0$, there exists effectively a finitely presented set* $\texttt{closure}(\varepsilon, X) \subseteq \mathcal{W}_{n,n}$ *such that:*

$$\texttt{closure}(\varepsilon, X) \approx_\varepsilon \langle X\rangle \ .$$

The proof of this result will be the subject of Section 3. The application of this theorem to the comparison of distance automata is presented in Section 4. The two sections are independent.

3 Proof of the core theorem

In this section we prove our core theorem, Theorem 5. It is the combination of several arguments. The first one is the use of the factorisation forest theorem of Simon, and is the subject of Section 3.1. Thanks to this theorem, the proof is reduced to establishing two lemmas, namely Lemmas 2 and 3. Lemma 2 is the subject of Section 3.2, and Lemma 3 the subject of Section 3.3.

3.1 The main induction: the factorisation forest theorem of Simon

The factorisation forest theorem of Simon [10] is a powerful combinatorial tool for understanding the structure of finite semigroups. We will not describe the original statement of this theorem, in terms of trees of factorisations, but rather a direct consequence of it which is central in our proof.

Theorem 6 (equivalent to the factorisation forest theorem [10][3])**.** *Given a semigroup morphism ϕ from $(S, \otimes)$ (possibly infinite) to a finite semigroup $(T, \cdot)$, and some $X \subseteq S$, set $X_0 = X$ and for all $k \geqslant 0$ define*

$$X_{k+1} = X_k \cup X_k \otimes X_k \cup \bigcup_{e \cdot e = e \in T} \langle X_k \cap \phi^{-1}(e) \rangle \ ,$$

then $X_N = \langle X \rangle$ for $N = 3|T| - 1$.

This proposition teaches us that, for computing the closure under product in the semigroup S, it is sufficient to know how to compute (a) the union of sets, (b) the product of sets, (c) the intersection of a set with the inverse image of an idempotent under ϕ, and (d) the closure under product of sets of elements that all have the same idempotent image under ϕ. Of course, this proposition is interesting when the semigroup T is cleverly chosen.

In our case, we are going to use the above proposition with $(S, \otimes) = (\mathcal{W}_{n,n}, \otimes)$, $(T, \cdot) = (\mathcal{M}_{n,n}(\mathbb{B}), \otimes)$, and ϕ the morphism which maps (M, ℓ) to $\phi(M)$. Our algorithm will compute, given a finitely presented set of weighted matrices X, an approximation of $\langle X \rangle$ following the same inductive construction as in the factorisation forest theorem. This is justified by the two following lemmas, that are proved in Sections 3.2 and 3.3. respectively

Lemma 2. *For all $\varepsilon > 0$ and all finitely presented sets $X, Y \subseteq \mathcal{W}_{n,n}$ there exists effectively a finitely presented set $\texttt{product}(\varepsilon, X, Y) \subseteq \mathcal{W}_{n,n}$ such that*

$$\texttt{product}(\varepsilon, X, Y) \approx_\varepsilon X \otimes Y \ .$$

Let X be a set of weighted matrices, we set

$$\phi(X) = \{\phi(M) \mid (M, \ell) \in X\} \ .$$

Lemma 3. *For all $\varepsilon > 0$ and all finitely presented sets $X \subseteq \mathcal{W}_{n,n}$ such that $\phi(X) = \{E\}$ for some idempotent E, there exists effectively $\texttt{idempotent}(\varepsilon, X) \subseteq \mathcal{W}_{n,n}$ finitely presented such that*

$$\texttt{idempotent}(\varepsilon, X) \approx_\varepsilon \langle X \rangle \ .$$

Assuming that Lemmas 2 and 3 hold, it is easy to provide an algorithm for Theorem 5, i.e., an algorithm which, given finitely presented set $X \subseteq \mathcal{W}_{n,n}$ computes a finitely presented set $\texttt{closure}(\varepsilon, X) \subseteq \mathcal{W}_{n,n}$ such that $\texttt{closure}(X) \approx_\varepsilon \langle X \rangle$. This algorithm, in some sense, mimics the induction involved in the statement of the factorisation forest theorem, Theorem 6. The algorithm is presented in Figure 1.

It is easy to prove that this construction is correct. This is done by induction on k, showing that $Y_k \approx_{\varepsilon(k)} X_k$ for all $k = 0 \dots N$ where X_k is defined as in Theorem 6 (where $S = \mathcal{W}_{n,n}$, $T = \mathcal{M}_{n,n}(\mathbb{B})$ and the morphism is ϕ).

[3]Modern proofs of this theorem can be found in [7, 3], in particular with the exact bound of $N = 3|T| - 1$ (Simon's original proof only provides $N = 9|T|$).

Input: A finitely presented set $X \subseteq \mathcal{W}_{n,n}$ and $\varepsilon > 0$,
Set $Y_0 = X$, $N = 3(2^{n^2}) - 1$,
For $k = 0$ to $N - 1$,

$$\text{define} \qquad \varepsilon(k) = \frac{\varepsilon}{2^{N-k}} \ ,$$

$$\begin{aligned} \text{and} \qquad Y_{k+1} = & \quad Y_k \\ \cup & \quad \texttt{product}(\varepsilon(k), Y_k, Y_k) \\ \cup & \bigcup_{\substack{e\otimes e=e \\ e\in\mathcal{M}_{n,n}(\mathbb{B})}} \texttt{idempotent}(\varepsilon(k), Y_k \cap \phi^{-1}(e)) \ . \end{aligned}$$

Output: $\texttt{closure}(X) = Y_N$.

Figure 1: The algorithm of Theorem 5 for computing $\texttt{closure}(\varepsilon, X)$.

For $k = 0$, it follows from the definitions: $X_0 = X = Y_0$.

Let $k \geqslant 0$, suppose that $Y_k \approx_{\varepsilon(k)} X_k$, then by Lemma 2, Lemma 1 and the induction hypothesis,

$$\texttt{product}(\varepsilon(k), Y_k, Y_k) \approx_{\varepsilon(k)} Y_k \otimes Y_k \approx_{\varepsilon(k)} X_k \otimes X_k \ .$$

Finally, by Lemma 1, $\texttt{product}(\varepsilon(k), Y_k, Y_k) \approx_{2\varepsilon(k)} X_k \otimes X_k$. Similarly, by Lemma 3, for all idempotent e, $\texttt{idempotent}(\varepsilon(k), Y_k \cap \phi^{-1}(e)) \approx_{2\varepsilon(k)} \langle X_k \cap \phi^{-1}(e)\rangle$. Thus $Y_{k+1} \approx_{\varepsilon(k+1)} X_{k+1}$ and finally $Y_N \approx_\varepsilon X_N = \langle X \rangle$.

Hence, what remains to be done is to establish Lemmas 2 and 3.

3.2 Approximate products of finitely presented sets

In this part, we give a proof of Lemma 2 which describes how to approximate the products of two finitely presented sets of weighted matrices. We also provide an extension of it to products of any bounded length, namely Lemma 6.

The proof of Lemma 2 shows explicit examples of the approximation arguments that are later used in a more advanced way for proving Lemma 3.

We first establish Lemma 4 which states that it is possible to control the effect of slight changes of length in the choices of weighted matrices in finitely presented sets. More precisely, it is possible to bound the difference between two products $(\ell_1 M_1 \otimes \cdots \otimes \ell_p M_p, \ell)$ and $(\ell'_1 M_1 \otimes \ell'_2 M_2 \otimes \cdots \otimes \ell'_p M_p, \ell)$, provided that the coefficients ℓ_i and ℓ'_i are sufficiently close.

Lemma 4. *For all $\varepsilon > 0$, all reals $a \geqslant 0$ and all positive integers p, there exists $\eta > 0$ such that for all positive integers $\ell_1, \ell_2, \ldots, \ell_p$, $\ell'_1, \ell'_2, \ldots, \ell'_p$, ℓ with*

$$\ell_i \leqslant \ell'_i + \eta\ell \qquad \text{for all } i = 1 \ldots p,$$

and all matrices $M_1, M_2, \ldots, M_p$ over $\overline{\mathbb{R}^+}$ with entries no greater than a,

$$(\ell_1 M_1 \otimes \ell_2 M_2 \otimes \cdots \otimes \ell_p M_p, \ell) \preccurlyeq_\varepsilon (\ell'_1 M_1 \otimes \ell'_2 M_2 \otimes \cdots \otimes \ell'_p M_p, \ell) \ .$$

Proof. Set $\eta = \frac{\varepsilon}{pa}$, we have:

$$\begin{aligned} \ell_1 M_1 \otimes \cdots \otimes \ell_p M_p &\leqslant (\ell'_1 + \eta\ell) M_1 \otimes \cdots \otimes (\ell'_p + \eta\ell) M_p \\ &\leqslant (\ell'_1 M_1 + a\eta\ell) \otimes \cdots \otimes (\ell'_p M_p + a\eta\ell) \\ &\leqslant \ell'_1 M_1 \otimes \cdots \otimes \ell'_p M_p + (pa\eta)\ell \\ &\leqslant \ell'_1 M_1 \otimes \cdots \otimes \ell'_p M_p + \eta\ell \ . \end{aligned}$$

Hence $(\ell_1 M_1 \otimes \ell_2 M_2 \otimes \cdots \otimes \ell_p M_p, \ell) \preccurlyeq_\varepsilon (\ell'_1 M_1 \otimes \ell'_2 M_2 \otimes \cdots \otimes \ell'_p M_p, \ell)$. □ □

Remark that in the above statement,η depends on p. This means that the argument is only valid for products of a bounded number of matrices.

The next lemma, Lemma 5 describes how to approximate reals by rationals, in a way suitable for the following.

Lemma 5. *For all positive integers x, p and all reals $\eta > 0$, there is a positive integer k such that, for all reals $\lambda_1, \ldots, \lambda_p \in [0, 1]$ satisfying*

$$\sum_{i=1}^{p} \lambda_i = 1 \ ,$$

and all $\ell \geqslant k$, there are integers $y_1, \ldots, y_p \geqslant x$ such that

$$\sum_{i=1}^{p} y_i = \ell \quad \textit{and} \quad |y_i - \lambda_i \ell| \leqslant \eta\ell \ .$$

Proof. Let us fix k to be any positive integer such that

$$\eta k > px \qquad \text{and} \quad k \geqslant p(p+1)x \ .$$

Let now $\ell \geqslant k$ and $\lambda_1, \lambda_2, \ldots, \lambda_p \in [0, 1]$ such that $\sum_{i=1}^{p} \lambda_i = 1$.

Before constructing the y_i's, let us construct the z_i's as follows:

$$z_i = \max(\lceil \lambda_i \ell \rceil, x) \qquad \text{for all } i = 1 \ldots p.$$

Note first that in all cases $z_i \geqslant \lambda_i \ell$. Furthermore, $z_i < \lambda_i \ell + x$ and since $x < \frac{\eta k}{p} \leqslant \eta k \leqslant \eta\ell$, we obtain $|z_i - \lambda_i \ell| \leqslant \eta\ell$ for all $i = 1 \ldots p$. Note also that

$$\ell = \sum_{i=1}^{p} \lambda_i \ell \leqslant \sum_{i=1}^{p} z_i \leqslant \sum_{i=1}^{p} (\lambda_i \ell + x) = \ell + px \ . \tag{1}$$

Hence, the z_i's would be a perfect choice for the y_i's, but for the fact that their sum may be a bit larger than ℓ, at most of px. We will correct this by modifying the largest of the z_i's.

Let m be the index maximizing λ_m. Let us fix $y_i = z_i$ for all $i \neq m$, and $y_m = \ell - \sum_{i \neq m} z_i$. By definition $\sum_{i=1\ldots p} z_i = \ell$. According to (1), $y_m \in [z_m - px, z_m]$. Let us prove that it respects the conclusion of the lemma.

By the pigeon-hole principle, $\lambda_m \geqslant \frac{1}{p}$, hence $z_m \geqslant \lambda_m \ell \geqslant \lambda_m k \geqslant \frac{k}{p} \geqslant (p+1)x$, hence $y_m \geqslant z_m - px \geqslant x$. Furthermore, since $\lambda_m \ell - \eta\ell \leqslant z_m - px \leqslant y_m$, we have $|y_m - \lambda_m \ell| \leqslant \eta\ell$. □ □

We can now complete the proof of Lemma 2, that we restate for the sake of completeness.

Lemma 2. *For all $\varepsilon > 0$ and all finitely presented sets $X, Y \subseteq \mathcal{W}_{n,n}$ there exists effectively a finitely presented set $\texttt{product}(\varepsilon, X, Y) \subseteq \mathcal{W}_{n,n}$ such that*

$$\texttt{product}(\varepsilon, X, Y) \approx_\varepsilon X \otimes Y \ .$$

Proof. Let X, Y be finitely presented sets of weighted matrices, and $\varepsilon > 0$. Since the finitely presented sets of weighted matrices are closed under union, it is sufficient to establish the result for the atomic blocks of the finite presentation. Namely, it is sufficient to consider the case $X = \{(M, \ell)\}$ or $X = \{(xM, x) \mid x \geqslant \ell\}$ together with $Y = \{(N, k)\}$ or $Y = \{(yN, y) \mid y \geqslant k\}$. This results in four possibilities, among which only three remain up to symmetry: (a) $X = \{(M, \ell)\}$ and $Y = \{(N, k)\}$, (b) $X = \{(M, \ell)\}$ and $Y = \{(yN, y) \mid y \geqslant k\}$, and finally (c) $X = \{(xM, x) \mid x \geqslant \ell\}$ and $Y = \{(yN, y) \mid y \geqslant k\}$.

- Case $X = \{(M, \ell)\}$ and $Y = \{(N, k)\}$, then we can set

$$\texttt{product}(\varepsilon, X, Y) = \{(M \otimes N, \ell + k)\} \qquad (= X \otimes Y) \ .$$

- Case $X = \{(M, \ell)\}$ and $Y = \{(yN, y) \mid y \geqslant k\}$, then we set a to be the greatest coefficient of $\frac{1}{\ell}M$ and N. Let us apply now Lemma 4 with parameters ε, a and $p = 2$, and obtain some $\eta > 0$. Set z to be an integer such that $\eta z \geqslant \ell$. Then set $Z = Z_1 \cup Z_2$ where

$$Z_1 = \bigcup_{k \leqslant y < z} \{(M \otimes yN, \ell + y)\}$$
$$\text{and} \qquad Z_2 = \{(y(\phi(M) \otimes N), y) \mid y \geqslant \ell + z\} \ .$$

Note that this set Z is finitely presented (in particular because Z_1 is finite). We now prove that furthremore $X \otimes Y \approx_\varepsilon Z$. The idea is that Z_1 captures exactly the products of matrices in X and Y of length smaller than $z + \ell$ while Z_2 gives an approximation of longer products.

First direction. Consider a weighted matrix in $X \otimes Y$. It is of the form $(M, \ell) \otimes (yN, y)$ for some $y \geqslant k$. If $k \leqslant y < z$, then $(M, \ell) \otimes (yN, y) \in Z_1$.

Otherwise $\eta y \geqslant \eta z \geqslant \ell \geqslant 1$. We obtain:

$$\begin{aligned}
(M, \ell) \otimes (yN, y) &= (1M \otimes yN, \ell + y) \\
&\preccurlyeq_\varepsilon (0M \otimes (\ell + y)N, \ell + y) \qquad \text{(by Lemma 4 with } 1 \leqslant \eta(\ell + y)) \\
&= ((\ell + y)(\phi(M) \otimes N), \ell + y) \\
&\in Z_2 \ .
\end{aligned}$$

Overall $X \otimes Y \preccurlyeq_\varepsilon Z$.

Opposite direction. We have to prove that $Z \preccurlyeq_\varepsilon X \otimes Y$. Since $Z_1 \subseteq X \otimes Y$ it is sufficient to prove $Z_2 \preccurlyeq_\varepsilon X \otimes Y$. Let us consider a weighted matrix in Z_2, it is of the form $(\phi(M) \otimes yN, y)$ for some $y \geqslant \ell + z$. We have $\eta y \geqslant \eta z \geqslant \ell$ and by Lemma 4,

$$\begin{aligned}
(\phi(M) \otimes yN, y) &= (0M) \otimes yN, y) \\
&\preccurlyeq_\varepsilon (1M \otimes (y - \ell)N, y) \qquad \text{(by Lemma 4)} \\
&= (M, \ell) \otimes ((y - \ell)N, y) \\
&\in X \otimes Y \ .
\end{aligned}$$

Overall $Z \preccurlyeq_\varepsilon X \otimes Y$.

- Case $X = \{(xM, x) \mid x \geqslant \ell\}$ and $Y = \{(yN, y) \mid y \geqslant k\}$. Let a be the greatest coefficient of M and N. Let us apply Lemma 4 with parameters ε, a, and $p = 2$, and obtain a corresponding η. We now use Lemma 5 with parameter $x = \max(k, \ell)$, $p = 2$ and η, and obtain an integer z as a result. Define now $Z = Z_1 \cup Z_2$ where,

$$Z_1 = \{(xM \otimes yN, x + y) \mid \ell \leqslant x < z,\ k \leqslant y < z\}$$

$$\text{and} \qquad Z_2 = \bigcup_{\lambda \in ([0,1] \cap \eta\mathbb{N})} \{(t(\lambda M \otimes (1 - \lambda)N), t) \mid t \geqslant z\}\ .$$

The set Z_1 is finite, and merely lists all weighted matrices of weight less than z in $X \otimes Y$. The set Z_2 takes barycenters of M and N, and produces corresponding weighted matrices for all possible weights greater or equal to z. To make Z_2 finitely presented, instead of taking all barycentres $\lambda M \otimes (1-\lambda)N)$ for $\lambda \in [0, 1]$, we discretize λ by having it ranging in $[0, 1] \cap \eta\mathbb{N}$. We note first that such a set Z is finitely presented. Let us prove now that $X \otimes Y \approx_\varepsilon Z$. There are two directions.

First direction. Let us consider a weighted matrix in $X \otimes Y$. It is of the form $(xM, x) \otimes (yN, y)$ for some $x \geqslant \ell$ and some $y \geqslant k$. If $x < z$ and $y < z$, then $(xM, x) \otimes (yN, y) \in Z_1$ by definition.

Otherwise, $x \geqslant z$ or $y \geqslant z$. The weighted matrix can then be rewritten as

$$(xM, x) \otimes (yN, y) = \left((x + y)\left(\frac{x}{x + y}M \otimes \frac{y}{x + y}N\right), x + y\right)\ .$$

Futhermore, $x + y \geqslant z$. Let us now choose $\lambda \in ([0, 1] \cap \eta\mathbb{N})$ such that $\left|\frac{x}{x+y} - \lambda\right| \leqslant \eta$. We also immediately have $\left|\frac{y}{x+y} - (1 - \lambda)\right| \leqslant \eta$. Hence by Lemma 4,

$$\begin{aligned}(xM, x) \otimes (yN, y) &\preccurlyeq_\varepsilon ((x + y)(\lambda M \otimes (1 - \lambda)N), x + y)\\ &\in Z_2\ .\end{aligned}$$

Overall $X \otimes Y \preccurlyeq_\varepsilon Z$.

Second direction. Conversely, let us first note that $Z_1 \subseteq X \otimes Y$. Hence, what remains to be proved is $Z_2 \preccurlyeq_\varepsilon X \otimes Y$. Let us consider a weighted matrix in Z_2. It is of the form $(t(\lambda M \otimes (1 - \lambda)N), t)$ with $t \geqslant z$ and $\lambda \in [0, 1]$. By Lemma 5, there are $x \geqslant \max(k, \ell)$ and $y \geqslant \max(k, \ell)$ such that $x + y = t$, $|x - \lambda t| \leqslant \eta t$ and $|y - (1 - \lambda)t| \leqslant \eta t$. By Lemma 4, we get:

$$\begin{aligned}(t(\lambda M \otimes (1 - \lambda)N), t) &\preccurlyeq_\varepsilon \left((x + y)\left(\frac{x}{x + y}M \otimes \frac{y}{x + y}N\right), x + y\right)\\ &\in X \otimes Y\ .\end{aligned}$$

Overall $Z \preccurlyeq_\varepsilon X \otimes Y$.

□ □

We have just proved Lemma 2 that gives an approximation of the product of two finitely presented sets of weighted matrices. This lemma will be used also in the proof of the more difficult Lemma 3. We will in fact use a slight generalisation of the result to a product of a bounded number of weighted matrices.

Lemma 6 (generalisation of Lemma 2)**.** *For all $\varepsilon > 0$, and all finitely presented sets $X_1, \ldots, X_p \subseteq \mathcal{W}_{n,n}$, there is a computable and finitely presented set Z such that:*

$$Z \approx_\varepsilon X_1 \otimes \cdots \otimes X_p \ .$$

Proof. This is true for $p = 2$ (Lemma 2). Suppose this is true for an integer $p \geqslant 2$, then $X_1 \otimes \cdots \otimes X_{p+1} \approx_{\frac{\varepsilon}{2}} \texttt{product}(\frac{\varepsilon}{2}, X_1, X_2) \otimes \cdots \otimes X_{p+1}$ by Lemma 1 and Lemma 2. Then by induction hypothesis, there is a computable and finitely presented set Z such that $\texttt{product}(\frac{\varepsilon}{2}, X_1, X_2) \otimes \cdots \otimes X_{p+1} \approx_{\frac{\varepsilon}{2}} Z$. Finally by Lemma 1, we obtain $X_1 \otimes \cdots \otimes X_{p+1} \approx_\varepsilon Z$. □ □

3.3 Approximate closure under products of finitely presented sets having the same idempotent projection

We shall prove now Lemma 3, which is the most difficult part in the proof. Let us fix an idempotent $E \in \mathcal{M}_{n,n}(\mathbb{B})$ and some finitely presented set of weighted matrices $X \subseteq \phi^{-1}(E)$. Our goal is to construct, given some $\varepsilon > 0$ a finitely presented set $\texttt{idempotent}(\varepsilon, X)$ such that $\texttt{idempotent}(\varepsilon, X) \approx_\varepsilon \langle X \rangle$. In the rest of this section, all weighted matrices belong to $\phi^{-1}(E)$.

The proof is divided in four parts. We first describe the general structure of the proof in Section 3.3.1, stating the key intermediate lemmas, and using them for establishing Lemma 3. The subsequent sections, namely Sections 3.3.1, 3.3.2 and 3.3.3, are then devoted to the proofs of these intermediate lemmas.

3.3.1 The key intermediate lemmas and the proof of Lemma 3

Our goal is to approximate $\langle X \rangle$ for $X \subseteq \phi^{-1}(E)$. The fact that all matrices are sent to the same idempotent is a big help in the sense that the structure of matrices is now fixed. Nevertheless, it is far from being sufficient, and still in a product the coefficients of the entries of the matrices may vary a lot. To overcome this problem, we introduce the central notion of uniform (weighted) matrices.

A matrix M such that $\phi(M) = E$ is *uniform* if

$$E \otimes M \otimes E = M \ .$$

A weighted matrix is *uniform* if its matrix part is uniform. Note that for all M such that $\phi(M) = E$, then $E \otimes M \otimes E$ is a uniform matrix.

We will see below several properties of uniform matrices. What is interesting for us is that (a) the closure of a presentable set of uniform weighted matrices is approximable (Lemma 7 below). Another important point is that we are able to disclose a notion of 'small product', and that it is possible to approximate the set of small uniform products of weighted matrices from X. Finally, an extraction argument (c) states that in any sufficiently long product it is possible to extract products of uniform small products. The combination of these three points yields the proof of Lemma 3.

In this section describing the general structure of the proof we make all the points (a), (b) and (c) precise, and then conclude the proof of Lemma 3. We postpone to the following subsection the precise proofs involved in points (a), (b) and (c), that happen to use fairly distinct arguments.

The point (a) above is the easiest to state.

Lemma 7. *For all $\varepsilon > 0$ and all finitely presented sets of uniform matrices $X \subseteq \phi^{-1}(E)$, there exists effectively a finitely presented set Z such that:*

$$Z \approx_\varepsilon \langle X \rangle \ .$$

The proof of this statement is the subject of Section 3.4.

For describing point (b) we have to provide the notion of a 'small product'. The results concerning small products are developed in Section 3.3.2. Such products are parameterized by some $\eta > 0$ and some integer p. Essentially, a small product is a product in which in the total weight ℓ, a weight at least equal to $(1-\eta)\ell$ has been contributed by a small number of weighted matrices, namely at most p of them. Formally, let p be some positive integer and $\eta > 0$. Define $\langle X \rangle_{p,\eta}$ to be the set of weighted matrices

$$(M, \ell) = (M_1, \ell_1) \otimes \cdots \otimes (M_k, \ell_k)$$

where each (M_i, ℓ_i) belongs to X, and there exists $1 \leqslant i_1 < \cdots < i_s \leqslant k$ with $s \leqslant p$ such that

$$\sum_{j=1}^{s} \ell_{i_j} \geqslant (1-\eta)\ell \ .$$

The idea behind $\langle X \rangle_{p,\eta}$ is that it is an under approximation of $\langle X \rangle$, that it contains all products of weighted matrices from X up to length p, and that even better, it is robust to the use of (possibly many) matrices of small weights inserted everywhere. The following lemma states that small products can be effectively approximated (note the precise alternation of quantifiers, that is necessary for the rest of the proof to go through).

Lemma 8. *For all $\varepsilon > 0$ and all $a \geqslant 0$, there exists effectively $\eta > 0$ such that for all finitely presented $X \subseteq \mathcal{W}^a_{n,n} \cap \phi^{-1}(E)$ and all $p \geqslant 1$, there exists a finitely presented set Y such that*

$$\langle X \rangle_{p,\eta} \preccurlyeq_\varepsilon Y \preccurlyeq_\varepsilon \langle X \rangle \ .$$

The proof of this theorem is developed in Section 3.3.2.

We have now to combine the notion of small product with uniformity. For this, we define $\langle X \rangle^u_{p,\eta}$ exactly as $\langle X \rangle_{p,\eta}$, but for the fact that the indices $i_1, \ldots, i_s$ have to satisfy $1 < i_1 < \cdots < i_s < k$, i.e., that i_1 cannot be 1 and i_s cannot be k, which means that the first and last matrices of the product have to have a small weight. It happens that matrices in $\langle X \rangle^u_{p,\eta}$ are almost uniform in the sense that they are ε-close to a uniform matrix since they are products (of weighted matrices sent to the same idempotent) such that the first and last term account for a sufficiently small percentage of the weight. The way we use this remark is by having an adapted version of Lemma 8.

Lemma 9. *For all $\varepsilon > 0$, all finitely presented $X \subseteq \phi^{-1}(E)$ and all $p \geqslant 1$, there exists effectively $\eta > 0$ and a finitely presented set Y such that*

$$\langle X \rangle^u_{p,\eta} \preccurlyeq_\varepsilon Y \preccurlyeq_\varepsilon \langle X \rangle \ .$$

Furthermore, Y does only contain uniform weighted matrices.

At this point in the proof, we know how to approximate small products, their uniform variant, and we know how to approximate the closure of presentable sets of uniform weighted matrices. The key missing ingredient is to prove that, by combining these results together, we capture all possible products constructed upon some set of matrices (included in $\phi^{-1}(E)$). This is the subject of the following extraction lemma.

Lemma 10. *For all $X \subseteq \phi^{-1}(E)$ and all $\eta > 0$ there is an integer p such that:*

$$\langle X\rangle = \langle X\rangle_{p,\eta} \cup \langle X\rangle_{p,\eta} \otimes \langle \langle X\rangle^u_{p,\eta}\rangle \otimes \langle X\rangle_{p,\eta} \ .$$

The proof of this result is the subject of Section 3.3.3.

The combination of the above lemmas yields a direct proof of Lemma 3, that we recall now.

Lemma 3. *For all $\varepsilon > 0$ and all finitely presented sets $X \subseteq \mathcal{W}_{n,n}$ such that $\phi(X) = \{E\}$ for some idempotent E, there exists effectively* $\mathtt{idempotent}(\varepsilon, X) \subseteq \mathcal{W}_{n,n}$ *finitely presented such that*

$$\mathtt{idempotent}(\varepsilon, X) \approx_\varepsilon \langle X\rangle \ .$$

Proof. Let $\varepsilon > 0$. By Lemmas 8 and 9 we obtain some $\eta > 0$ (we take the minimum of the values of η produced by these two lemmas). Then, using Lemma 10 (with this value of η), we obtain an integer p such that

$$\langle X\rangle = \langle X\rangle_{p,\eta} \cup \langle X\rangle_{p,\eta} \otimes \langle \langle X\rangle^u_{p,\eta}\rangle \otimes \langle X\rangle_{p,\eta} \ .$$

Relying then on Lemmas 8 and 9 (with this value of p), there are effectively finitely presented sets T and V (V consisting of uniform matrices) such that

$$\langle X\rangle_{p,\eta} \preccurlyeq_{\frac{\varepsilon}{4}} T \preccurlyeq_{\frac{\varepsilon}{4}} \langle X\rangle \qquad \text{and} \qquad \langle X\rangle^u_{p,\eta} \preccurlyeq_{\frac{\varepsilon}{4}} V \preccurlyeq_{\frac{\varepsilon}{4}} \langle X\rangle \ .$$

Then, using Lemma 1, we obtain

$$\langle X\rangle \preccurlyeq_{\frac{\varepsilon}{4}} T \cup T \otimes \langle V\rangle \otimes T \preccurlyeq_{\frac{\varepsilon}{4}} \langle X\rangle \cup \langle X\rangle \otimes \langle\langle X\rangle\rangle \otimes \langle X\rangle = \langle X\rangle \ ,$$

and hence $\langle X\rangle \approx_{\frac{\varepsilon}{4}} T \cup T \otimes \langle V\rangle \otimes T$.

Moreover, since all the weighted matrices in V are uniform, by Lemma 7, there is effectively a finitely presented set Y, such that $\langle V\rangle \approx_{\frac{\varepsilon}{4}} Y$. Now using Lemma 1, we get

$$\langle X\rangle \approx_{\frac{\varepsilon}{2}} T \cup T \otimes Y \otimes T \ .$$

Finally, using Lemma 6 and the closure of finitely presented sets under union, there exists effectively a finitely presented set Z such that $T \cup T \otimes Y \otimes T \approx_{\frac{\varepsilon}{2}} Z$. We conclude using once more Lemma 1 that $\langle X\rangle \approx_\varepsilon Z$. □ □

3.3.2 Computing with small products

Let us recall and prove our first lemma concerning small products.

Lemma 8. *For all $\varepsilon > 0$ and all $a \geqslant 0$, there exists effectively $\eta > 0$ such that for all finitely presented $X \subseteq \mathcal{W}^a_{n,n} \cap \phi^{-1}(E)$ and all $p \geqslant 1$, there exists a finitely presented set Y such that*

$$\langle X\rangle_{p,\eta} \preccurlyeq_\varepsilon Y \preccurlyeq_\varepsilon \langle X\rangle \ .$$

Proof. Given $\varepsilon > 0$ and $a \geqslant 0$, let $\eta = \frac{\varepsilon}{2a}$.

Now, given a finitely presented set X and an integer p, define Z to be the results of products of at least one, and at most $2p+1$ weighted matrices from X, i.e.,

$$Z = \bigcup_{1 \leqslant r \leqslant 2p+1} \overbrace{X \otimes \cdots \otimes X}^{r \text{ times}} \ .$$

Two things are immediately clear from this definition. The first one is that $Z \subseteq \langle X \rangle$. The second is that, according to Lemma 6, there exists effectively a finitely presented set Y such that $Y \approx_{\frac{\varepsilon}{2}} Z$. We prove below that this Y fulfills the conclusion of the lemma. For this it is sufficient to prove that $\langle X \rangle_{p,\eta} \preccurlyeq_{\frac{\varepsilon}{2}} Z$. Indeed, then we would have

$$\langle X \rangle_{p,\eta} \preccurlyeq_{\frac{\varepsilon}{2}} Z \preccurlyeq_{\frac{\varepsilon}{2}} Y \preccurlyeq_{\frac{\varepsilon}{2}} Z \preccurlyeq_0 \langle X \rangle ,$$

which by Lemma 1 implies $\langle X \rangle_{p,\eta} \preccurlyeq_{\varepsilon} Y \preccurlyeq_{\varepsilon} \langle X \rangle$.

Hence, let us prove now that $\langle X \rangle_{p,\eta} \preccurlyeq_{\frac{\varepsilon}{2}} Z$. Let us consider a matrix (M, ℓ) from $\langle X \rangle_{p,\eta}$. Our goal is to turn it into a 'resemblant' matrix from Z. Let us consider the product that has produced the matrix (M, ℓ):

$$(M_1, \ell_1) \otimes \cdots \otimes (M_k, \ell_k) .$$

By hypothesis, there are $i_1 < \cdots < i_s$ with $1 \leqslant s \leqslant p$ such that $\ell_{i_1} + \cdots + \ell_{i_s} \geqslant (1 - \eta)\ell$. For ease of use, let us define $i_0 = 0$ and $i_{s+1} = k + 1$. The idea is to factorize this product as follows:

$$(M, \ell) = (N_0, n_0) \otimes (M_{i_1}, \ell_{i_1}) \otimes (N_1, n_1) \otimes \cdots \otimes (M_{i_s}, \ell_{i_s}) \otimes (N_s, n_s)$$

where for all $j = 0 \ldots s$,

$$(N_j, n_j) = (M_{i_{j-1}+1}, \ell_{i_{j-1}+1}) \otimes \cdots \otimes (M_{i_j - 1}, \ell_{i_j - 1}) .$$

Remark that the definition of (N_j, n_j) may involve an empty product. In this case, set $(N, n_j) = 1$ where 1 is a neutral element added to weighted matrices, i.e., corresponding to the virtual (virtual since weight 0 is not allowed) weighted matrix $(I_n, 0)$. Let us define now (N'_j, n'_j) to be 1 if $(N_j, n_j) = 1$, and to be (S, m) otherwise, where (S, m) is a matrix of minimal weight in X. Clearly now the matrix

$$(M', \ell') = (N'_0, n'_0) \otimes (M_{i_1}, \ell_{i_1}) \otimes (N'_1, n'_1) \otimes \cdots \otimes (M_{i_s}, \ell_{i_s}) \otimes (N'_s, n'_s)$$

belongs to Z. Let us show that $(M, \ell) \preccurlyeq_{\frac{\varepsilon}{2}} (M', \ell')$.

For this, let us note that for all $j = 0 \ldots s$ such that $N_j \neq 1$, all the non-infinity coefficients of N_j are at most equal to $a n_j$ (they can't be ∞ because N_j and N'_j are both sent by ϕ to E). Hence $N_j \leqslant N'_j + a n_j$. Since furthermore $\sum_{j=1}^{s} n_j \leqslant \eta \ell$, it follows that $M \leqslant M' + a\eta\ell = M' + \frac{\ell}{2}$. On the weight side, since for all $j = 0 \ldots s$, $n'_j \leqslant n_j$, we clearly have $\ell' \leqslant \ell$. Overall, $\langle X \rangle_{p,\eta} \preccurlyeq_{\frac{\varepsilon}{2}} Z$ as announced. □ □

Our next goal is to establish Lemma 9. As a preparation, let us prove Lemma 11 which states a sufficient condition for asserting that a weighted matrix is uniform (up to $\approx_{\varepsilon}$).

Lemma 11. *For all $\varepsilon > 0$ and all $a \geqslant 0$, there exists η such that given three weighted matrices $(M_1, \ell_1), (M_2, \ell_2), (M_3, \ell_3) \in \mathcal{W}^a_{n,n} \cap \phi^{-1}(E)$ and $\ell = \ell_1 + \ell_2 + \ell_3$, if $\ell_1 + \ell_3 \leqslant \eta\ell$, then*

$$(M_1, \ell_1) \otimes (M_2, \ell_2) \otimes (M_3, \ell_3) \approx_{\varepsilon} (E \otimes M_2 \otimes E, \ell) .$$

Proof. Clearly the two weighted matrices involved in the equivalence have the same weight. Also, since $E = \phi(M_1) \leqslant M_1$, and similarly for M_3, we have $E \otimes M_2 \otimes E \leqslant M_1 \otimes M_2 \otimes M_3$. For the converse inequality, let us choose η such that $a\eta \leqslant \varepsilon$. Clearly $M_1 \leqslant E + a\ell_1$ and $M_3 \leqslant E + a\ell_3$. Thus

$$\begin{aligned} M_1 \otimes M_2 \otimes M_3 &\leqslant E \otimes M_2 \otimes E + a(\ell_1 + \ell_3) \\ &\leqslant E \otimes M_2 \otimes E + a\eta\ell \leqslant E \otimes M_2 \otimes E + \varepsilon\ell . \end{aligned}$$

Overall $(M_1, \ell_1) \otimes (M_2, \ell_2) \otimes (M_3, \ell_3) \approx_{\varepsilon} (E \otimes M_2 \otimes E, \ell)$. □ □

We are now ready to prove Lemma 9, which we restate for the sake of completeness.

Lemma 9. *For all $\varepsilon > 0$, all finitely presented $X \subseteq \phi^{-1}(E)$ and all $p \geqslant 1$, there exists effectively $\eta > 0$ and a finitely presented set Y such that*

$$\langle X \rangle^u_{p,\eta} \preccurlyeq_\varepsilon Y \preccurlyeq_\varepsilon \langle X \rangle \ .$$

Furthermore, Y does only contain uniform weighted matrices.

Proof. Let $\varepsilon > 0$ and $a \geqslant 0$ be fixed. Using Lemma 8, we obtain η.

Now, given a finitely presented set $X \subseteq \mathcal{W}^a_{n,n} \cap \phi^{-1}(E)$. By Lemma 8, there exists effectively a finitely presented set Y such that

$$\langle X \rangle_{p,\eta} \preccurlyeq_\varepsilon Y \preccurlyeq_\varepsilon \langle X \rangle \ .$$

Let now m be a sufficiently large value (to be fixed below), define

$$Z = \{(E \otimes M \otimes E, \ell) \mid (M, \ell) \in Y,\ \ell \geqslant m\} \ .$$

It is clear that Z does only contain uniform matrices. It is also clear that Z is effectively finitely presented.

Let us prove that $\langle X \rangle^u_{p,\eta} \preccurlyeq_\varepsilon Z$. Consider a matrix in (M, ℓ) in $\langle X \rangle^u_{p,\eta}$, then

$$(M, \ell) = (M_1, \ell_1) \otimes (M_2, \ell_2) \otimes (M_3, \ell_3) \ ,$$

where $(M_1, \ell_1), (M_3, \ell_3) \in X$, $(M_2, \ell_2) \in \langle X \rangle_{p,\eta}$ and $\ell_1 + \ell_3 \leqslant \eta\ell$. According to Lemma 11, this implies $(M, \ell) \approx_\varepsilon (E \otimes M_2 \otimes E, \ell)$

Let us prove now that $Z \preccurlyeq_\varepsilon \langle X \rangle_{p,\eta}$.

TO FINISH cette preuve ne doit pas inspecter le codage de l'ensemble finement présenté

□ □

3.3.3 Finding uniform matrices in a long product: proof of Lemma 10

We shall now establish Lemma 10, which states that any product of weighted matrices in X can be decomposed according to a simple pattern. It states formally that given η, we can find p such that

$$\langle X \rangle = \langle X \rangle_{p,\eta} \cup \langle X \rangle_{p,\eta} \otimes \langle \langle X \rangle^u_{p,\eta} \rangle \otimes \langle X \rangle_{p,\eta} \ .$$

Technically, this result should be understood as a decomposition lemma (to some extent a Ramsey-like statement). It expresses that given a product of weighted matrices from X, either it belongs to $\langle X \rangle_{p,\eta}$, or it can be factorized as $(N_1, n_1) \otimes \cdots \otimes (N_m, n_m)$ such that $(N_1, n_1), (N_m, n_m) \in \langle X \rangle_{p,\eta}$ and $(N_2, n_2), \ldots, (N_{m-1}, n_{m-1}) \in \langle X \rangle^u_{p,\eta}$. We can dive even further into this statement, and note that the property for a product of weighted matrices to belong to $\langle X \rangle_{p,\eta}$ or $\langle X \rangle^u_{p,\eta}$ is a property that does only involve the weight of the weighted matrices, and not at all the content of the matrices themselves.

This means that the problem can be restated simply as a simplified one that involve only a sequence of positive integers. For this, let us redefine the notions of smallness to our case: a sequence of numbers $\ell_1, \ldots, \ell_k$ of sum ℓ is *p, η-small* if there are $1 \leqslant i_1 < \cdots < i_r \leqslant k$ with $r \leqslant p$ such that $\sum_{j=1}^{r} \ell_{j_i} \geqslant (1-\eta)\ell$. It is *uniform p, η-small* if $1 < i_1 < \cdots < i_r < k$ in the above definition.

We can now restate our problem as follows: for all $\eta > 0$ we have to find an integer p such that given a sequence of positive integers,

$$\bar{\ell} = \ell_1, \ldots, \ell_k \ ,$$

either it is p, η-small, or it can be be factorized into subsequences as $\overline{\ell}^1, \ldots, \overline{\ell}^m$ such that $\overline{\ell}^1, \overline{\ell}^m$ are p, η-small, and $\overline{\ell}^2, \ldots, \overline{\ell}^{m-1}$ are uniform p, η-small.

Our first result in this direction is a criterion for proving that a product is p, η-small.

Lemma 12. *Let $\eta > 0$, there exists p such that for all sequences of positive integers $\bar{\ell} = \ell_1, \ldots, \ell_k$ such that for all $i = 1 \ldots k$,*

$$\frac{\ell_i}{\ell_1 + \cdots + \ell_i} \geqslant \eta \ , \qquad \left(\textit{or equivalently} \quad \ell_i \geqslant \frac{\ell_1 + \cdots + \ell_{i-1}}{1 - \eta} \right)$$

then $\bar{\ell}$ is p, η-small.

Proof. Given η, let us fix some $p \geqslant \frac{1}{\eta}$.

Consider now a sequence of positive integers $\bar{\ell} = \ell_1, \ldots, \ell_k$. Note first that if $k \leqslant p$, then the conclusion obviously holds. Otherwise, we split this sequence into $\bar{\ell} = \bar{\ell}^1, \bar{\ell}^2$ where $\bar{\ell}^2$ has length p. Let s_1 be the sum of the sequence $\bar{\ell}^1$ and s_2 be the sum of $\bar{\ell}_2$. From the hypothesis, we know that each integer in $\bar{\ell}^2$ is at least equal to $\frac{s_1}{1-\eta}$. Thus $s_2 \geqslant \frac{ps_1}{1-\eta}$, which means

$$s_1 \leqslant \frac{1-\eta}{p} s_2 \leqslant \eta s_2 \leqslant \eta(s_1 + s_2) \ .$$

Hence $\bar{\ell}$ is p, η-small. □ □

Let us continue to advance toward the factorisation we aim at. We want to extract uniform p, η-small sequences of positive integers. Uniformity is a two-sided notion, since it means requiring that both the first ant last integer in the sequence are 'small'. That is why, as an intermediate step, we consider the one-sided versions of uniform p, η-smallness. Formally, a sequence of positive integers $\ell_1, \ldots, \ell_k$ of sum ℓ is *right-uniform p, η-small* (*resp. left-uniform p, η-small*) if there exist $1 \leqslant i_1 < \cdots < i_m < k$ (*resp.* $1 < i_1 < \cdots < i_m \leqslant k$) for some $m \leqslant p$ such that $\sum_{j=1}^{m} \ell_{i_j} \leqslant \eta \ell$.

We are now ready to prove a one-sided variant of the Lemma 10 we aim at.

Lemma 13. *Let $\eta > 0$, there exists an integer p such that all sequences of positive integers $\bar{\ell}$ can be factorized as $\bar{\ell} = \bar{\ell}^1, \ldots, \bar{\ell}^k$ such that*

- *$\bar{\ell}^1, \ldots, \bar{\ell}^{k-1}$ are right-uniform p, η-small, and*
- *$\bar{\ell}^k$ is p, η-small.*

Proof. Let $\eta > 0$. Set p be obtained from Lemma 12 for the value $\frac{\eta}{2}$.

We use the following claim. Given any sequence $\bar{\ell}$ of positive integers, then

- either $\bar{\ell}$ is p, η-small, or
- it has a non-empty prefix which is right-uniform p, η-small.

Indeed, once this claim established, it is straightforward to prove the lemma by induction on the length of the sequence.

Let us now prove the above claim. Let $\bar{\ell}$ be a sequence of positive integers. Two cases may arise, either for all $k \geqslant 1$

$$\frac{\ell_k}{\ell_1 + \cdots + \ell_k} \geqslant \frac{\eta}{2} \ .$$

and then by Lemma 12, $\bar{\ell}$ is $p, \frac{\eta}{2}$-small and hence p, η-small,

Otherwise, let $\ell_1, \ldots, \ell_k$ be the shortest non-empty prefix such that

$$\frac{\ell_k}{\ell_1 + \cdots + \ell_k} < \frac{\eta}{2} \ .$$

Note first that, by minimality in its construction, the sequence $\ell_1, \ldots, \ell_{k-1}$ satisfies the hypothesis of Lemma 12 for the value $\frac{\eta}{2}$. Hence, it is $p, \frac{\eta}{2}$-small. Since furthermore $\ell_k \leqslant \frac{\eta}{2}(\ell_1 + \cdots + \ell_k)$, it follows that $\ell_1, \ldots, \ell_k$ is right-uniform p, η-small. □ □

Let us now extend the above result into a two-sided version.

Lemma 14. *Let $\eta > 0$, there exists an integer p such that all sequences of positive integers $\bar{\ell}$ can be factorized as $\bar{\ell}^1, \ldots, \bar{\ell}^k$ such that*

- *$\bar{\ell}^1$ and $\bar{\ell}^k$ are p, η-small, and*
- *$\bar{\ell}^2, \ldots, \bar{\ell}^{k-1}$ are uniform p, η-small.*

Proof. The principle is to compose Lemma 13 with itself, or more precisely, with its mirror variant. For this, we need to be able to compose several use of such lemmas. This is the subject of the following claim.

We first claim the following: Given $\bar{\ell}^1, \ldots, \bar{\ell}^k$ sequences of positive integers of respective sums $s_1, \ldots, s_k = \bar{s}$.

- If the sequences $\bar{\ell}^1, \ldots, \bar{\ell}^k$ are p, η-small, and $\bar{s}$ is p, η-small, then $\bar{\ell} = \bar{\ell}^1, \ldots, \bar{\ell}^k$ is $p^2, 2\eta$-small.
- If the sequences $\bar{\ell}^1, \ldots, \bar{\ell}^k$ are right-uniform p, η-small, and $\bar{s}$ is left-uniform p, η-small, then $\bar{\ell} = \bar{\ell}^1, \ldots, \bar{\ell}^k$ is uniform $p^2, 2\eta$-small.

For the first item, consider the s_i's that are involved in the fact that $\bar{s}$ is p, η-small. For each of them, consider the ℓ_j's that witness the fact $\bar{\ell}^i$ is p, η-small. Overall, this means extracting TO FINISH

Consider now a sequence $\bar{\ell}$. According to Lemma 13, it can be decomposed as $\bar{\ell} = \bar{\ell}^1, \ldots, \bar{\ell}^k$ where $\bar{\ell}^1, \ldots, \bar{\ell}^{k-1}$ are right-uniform p, η-small, and $\bar{\ell}^k$ is p, η-small. Let $s_1, \ldots, s_k$ be the respective sums of $\bar{\ell}^1, \ldots, \bar{\ell}^k$. We can apply Lemma 13, but this time a mirrored version, to $\bar{s}^1, \ldots, \bar{s}^n$, where $\bar{s}^1$ is p, η-uniform, and $\bar{s}^2, \ldots, \bar{s}^n$ are left-uniform p, η-small. Now, in each $\bar{s}^i$, let us replace each term s_j by the sequence $\bar{\ell}^j$ it comes from, and obtain $\bar{t}^i$. Then, $\bar{t}^1, \ldots, \bar{t}^n$ is again $\bar{\ell}$. Furthermore, by the above claim, $\bar{t}^1$ and $\bar{t}^n$ are $p^2, 2\eta$-small, and $\bar{t}^2, \ldots, \bar{t}^n$ are uniform $p^2, 2\eta$-small. TO FINISH (in particular, modify the above proof for directly dealing with the correct coefficients). □

Let us recall now Lemma 10.

Lemma 10. *For all $X \subseteq \phi^{-1}(E)$ and all $\eta > 0$ there is an integer p such that:*

$$\langle X \rangle = \langle X \rangle_{p,\eta} \cup \langle X \rangle_{p,\eta} \otimes \langle \langle X \rangle^u_{p,\eta} \rangle \otimes \langle X \rangle_{p,\eta} \ .$$

Proof. Given a product of weighted matrices resulting in a weighted matrix in $\langle X \rangle$, it is sufficient to apply Lemma 14 to the sequence of the weights of the weighted matrices. The resulting decomposition exactly matches the conclusion of the lemma. □ □

3.4 The closure of finitely presented sets of uniform matrices: proof of Lemma 7

The goal of this section is to prove that the closure of a finitely presented set of uniform matrices is effectively approximable.

Lemma 15. *A product of uniform matrices (with the same idempotent projection) is uniform.*

Proof. Let M and M' two uniform matrices of idempotent projection E, then:

$$E \otimes M \otimes M' \otimes E = E \otimes E \otimes M \otimes E \otimes E \otimes M' \otimes E \otimes E = M \otimes M'.$$

This generalizes to longer products by a straightforward induction. □ □

Now, let us explain what is the structure of a uniform matrix.

Given a uniform matrix M of idempotent projection E, let i and j be two indices. Let us define the relation $\leftrightarrow$ by $i \leftrightarrow j$ if $E_{i,j} = 0$ and $E_{j,i} = 0$, or if $i = j$. This relation $\leftrightarrow$ is an equivalence relation (since E is idempotent). Hence it provides a partition of the indices into equivalence classes.

If $i \leftrightarrow j$ then i and j play exactly the same role in the following sense: for all indices h, $M_{i,h} = M_{j,h}$ and $M_{h,i} = M_{h,j}$ (indeed, $M_{i,h} = (E \otimes M)_{i,h} \leqslant E_{i,j} + M_{j,h} = M_{j,h}$ and conversely). Moreover, it is then easy to compute the entry (i,i) of products of uniform matrices that have the same idempotent projection:

Lemma 16. *Given $M_1, \ldots M_m$ uniform matrices of idempotent projection E,*

$$(M_1 \otimes \ldots \otimes M_m)_{i,i} = (M_1)_{i,i} \otimes \ldots \otimes (M_m)_{i,i}$$

Proof. □ □

The idea behind the proof is to study what happens in a long product of uniform matrices between indices i and j. If $i \leftrightarrow j$ then, by using the previous lemma, we can see that the order of matrices in the product does not matter.

Let us have a look to what happens if i and j are not in the same class. A pair of indices (g,h) is said transient if $E_{g,h} = 0$ but $E_{h,g} = \infty$. Then in a product of matrices $M_1 \otimes \ldots \otimes M_m$, the computation of $(M_1 \otimes \ldots \otimes M_m)_{i,j}$ can only use once a coefficient $(M_k)_{g,h}$ if (g,h) is transient. Using this knowledge, it is possible to show that the 'worst situation' occurs for very simple patterns of repetition of the matrices in X. Since the relation $\approx_\eta$ just refers to the upper envelope, this worst situation is sufficient for us to conclude.

We can now establish the key result of this section, that we restate for the sake of completeness.

Lemma 7. *For all $\varepsilon > 0$ and all finitely presented sets of uniform matrices $X \subseteq \phi^{-1}(E)$, there exists effectively a finitely presented set Z such that:*

$$Z \approx_\varepsilon \langle X \rangle \ .$$

Proof. Let $\eta > 0$, we can write:

$$X = \bigcup_{1 \leqslant i \leqslant p} \{(p_i M_i, p_i)\} \cup \bigcup_{p+1 \leqslant i \leqslant m} \{(x M_i, x) \mid x \geqslant q_i\} \ .$$

As usual, we will approximate the set $\langle X\rangle$ by a union of two sets: the set of exact products up to some length, and the set of asymptotic matrices that will be the products 'barycenters' of matrices in X.

Let a be the greatest coefficient of the matrices M_i.

Let $\gamma > 0$ such that $\gamma \leqslant \frac{\eta}{2am^2}$.

Set r a positive integer such that $r \geqslant \frac{2n(a+1)}{\gamma}$, where n is the number of rows of the matrices.

Lemma 5 gives an integer z', such that for all $k \geqslant z'$, for all $0 \leqslant \lambda_i \leqslant 1$ with $\sum_{i=1}^m \lambda_i = 1$, for all $i \leqslant p$, there are integers $y_i \geqslant p_i$ and for all $p+1 \leqslant i \leqslant m$, $y_i \geqslant q_i$ such that $\sum_{j=1}^m y_j = k$ and $|y_i - \lambda_i k| \leqslant \gamma k$.

Let $z \geqslant 2p_1 \cdots p_p r \eta^{-1} z'$.

Set $Z = Z_1 \cup Z_2$ with:

$$Z_1 = \{(N_1, t_1) \otimes \cdots \otimes (N_k, t_k) \mid \forall i, (N_i, t_i) \in X, t_1 + \ldots + t_k < z\}$$
$$\text{and } Z_2 = \bigcup_{\substack{\{(\lambda_1,\ldots,\lambda_m)| \\ \lambda_1+\ldots+\lambda_m=1 \\ \lambda_1,\ldots,\lambda_{m-1}\in[0,1]\cap\gamma\mathbb{N}\}}} \{(x\left(\frac{1}{r}\left(\lambda_1 M_1 \otimes \cdots \otimes \lambda_m M_m\right)^r\right), x) \mid x \geqslant z\}\ .$$

The set Z is computable and finitely presented.

First, let us show that $Z \preccurlyeq_\eta \langle X\rangle$. This part only has to do with the fact that we can approximate a product $x\left(\frac{1}{r}\left(\lambda_1 M_1 \otimes \cdots \otimes \lambda_m M_m\right)^r\right)$ with non integer values λ_i, $1/r$ by a true product in $\langle X\rangle$.

We have $Z_1 \subseteq \langle X\rangle$. Let us prove that $Z_2 \preccurlyeq_\eta \langle X\rangle$.

$$A = (k\left(\frac{1}{r}\left(\lambda_1 M_1 \otimes \cdots \otimes \lambda_m M_m\right)^r\right), k) \in Z_2\ .$$

Let k' be an integer such that $p_1 \cdots p_p r k' \leqslant k < p_1 \cdots p_p r(k'+1)$. Since $k \geqslant z$, we have $k' \geqslant z'$. Then, for all $i \leqslant p$, there are integers $y_i \geqslant p_i$ and for all $p \leqslant i \leqslant m$, there are integers $y_i \geqslant q_i$ such that $\sum_{j=1}^m y_j = k'$ and $|y_i - \lambda_i k'| \leqslant \gamma k'$.

Then:

$$\begin{aligned}
&k\left(\frac{1}{r}\left(\lambda_1 M_1 \otimes \cdots \otimes \lambda_m M_m\right)^r\right) = \left(\frac{k\lambda_1}{r} M_1 \otimes \cdots \otimes \frac{k\lambda_m}{r} M_m\right)^r \\
&\leqslant (\frac{p_1 \cdots p_p r y_1 + p_1 \cdots p_p r \gamma k' + p_1 \cdots p_p r \lambda_1}{r} M_1 \\
&\otimes \cdots \otimes \\
&\frac{p_1 \cdots p_p r y_m + p_1 \cdots p_p r \gamma k' + p_1 \cdots p_p r \lambda_m}{r} M_m)^r \\
&\leqslant (p_1 \cdots p_p y_1 M_1 \otimes \cdots \otimes p_1 \cdots p_p y_m M_m)^r \\
&+ p_1 \cdots p_p a m r \gamma k' + p_1 \cdots p_p r \\
&\leqslant (p_1 \cdots p_p y_1 M_1 \otimes \cdots \otimes p_1 \cdots p_p y_m M_m)^r \\
&+ k\frac{\eta}{2} + k\frac{\eta}{2} \\
&\leqslant ((p_1 M_1)^{p_2 \cdots p_p y_1} \otimes \cdots \otimes M_m^{p_1 \cdots p_p y_m})^r + k\eta\ .
\end{aligned}$$

Then $A \preccurlyeq_\eta (((p_1 M_1)^{p_2 \cdots p_p y_1} \otimes \cdots \otimes M_m^{p_1 \cdots p_p y_m})^r, p_1 \cdots p_p k' r) \in \langle X\rangle$.

Now let us show that $\langle X\rangle \preccurlyeq_\eta Z$. This part of the proof deals with the uniform structures of matrices.

Let:

$$A = (\ell_1 M_{i_1}, \ell_1) \otimes \cdots \otimes (\ell_k M_{i_k}, \ell_k) \in \langle X \rangle.$$

Set $\ell = \sum_{i=1}^{k} \ell_i$, and $\lambda_i = \frac{\sum_{i_j = i} \ell_j}{\ell}$ (λ_i represents the proportion of M_i in the product). If $\ell < z$ then $A \in Z_1$. Otherwise, there is $(\lambda'_i)_{i=1,\ldots,m}$ such that for all $i = 1, \ldots, m-1$, $\lambda'_i \in \gamma\mathbb{N}$, $\sum_{i=1}^{m} \lambda'_i = 1$ and for all $i = 1, \ldots, m$, $|\lambda_i - \lambda'_i| \leqslant m\gamma$.

Set $B = (\ell \left(\frac{1}{r} \left(\lambda'_1 M_1 \otimes \cdots \otimes \lambda'_m M_m\right)^r\right), \ell)$. Then $B \in Z_2$. Let us prove that $A \preccurlyeq_\eta B$.

First,

$$\ell \left(\frac{1}{r} \left(\lambda_1 M_1 \otimes \cdots \otimes \lambda_m M_m\right)^r\right) \leqslant \ell \left(\frac{1}{r} \left(\lambda'_1 M_1 \otimes \cdots \otimes \lambda'_m M_m\right)^r\right) + m^2 \gamma a \ell$$

thus

$$\ell \left(\frac{1}{r} \left(\lambda_1 M_1 \otimes \cdots \otimes \lambda_m M_m\right)^r\right) \leqslant \ell \left(\frac{1}{r} \left(\lambda'_1 M_1 \otimes \cdots \otimes \lambda'_m M_m\right)^r\right) + \frac{\eta}{2} \ell.$$

Then we only need to prove that:

$$\ell_1 M_{i_1} \otimes \cdots \otimes \ell_k M_{i_k} \leqslant \ell \left(\frac{1}{r} \left(\lambda_1 M_1 \otimes \cdots \otimes \lambda_m M_m\right)^r\right) + \frac{\eta}{2} \ell \ .$$

Let g and h be two indices. Let j be one of the indices such that $E_{g,j} = E_{j,h} = 0$, and $(\lambda_1 M_1 \otimes \cdots \otimes \lambda_m M_m)_{j,j}$ is minimal. Let us call μ this quantity. We know that $\mu = (\lambda_1 M_1)_{j,j} + \ldots + (\lambda_m M_m)_{j,j}$.

Since a product of uniform matrices is uniform, we have:

$$\begin{aligned}
(\ell_1 M_{i_1} \otimes \cdots \otimes \ell_k M_{i_k})_{g,h} &= (E \otimes \ell_1 M_{i_1} \otimes \cdots \otimes \ell_k M_{i_k} \otimes E)_{g,h} \\
&\leqslant E_{g,j} + (\ell_1 M_{i_1})_{j,j} + \ldots + (\ell_k M_{i_k})_{j,j} + E_{j,h} \\
&\leqslant \ell\mu \ .
\end{aligned}$$

Finally, we show that $\ell\mu \leqslant \ell \left(\frac{1}{r} \left(\lambda_1 M_1 \otimes \cdots \otimes \lambda_m M_m\right)^r\right) + \frac{\eta}{2}\ell$. For s large enough (namely greater than $\frac{n(a+1)}{\gamma}$), the computation of

$$\left(\lambda_1 M_1 \otimes \cdots \otimes \lambda_m M_m\right)^s_{g,h}$$

only involves a finite number of transient pairs and uses

$$\mu = \left(\lambda_1 M_1 \otimes \cdots \otimes \lambda_m M_m\right)_{j,j}$$

instead of

$$\left(\lambda_1 M_1 \otimes \cdots \otimes \lambda_m M_m\right)_{e,e}$$

for $e \neq j$, since μ is minimal. Then

$$(s-n)\mu \leqslant \left(\lambda_1 M_1 \otimes \cdots \otimes \lambda_m M_m\right)^s_{g,h} \leqslant s\mu.$$

Then,

$$\ell\mu - \ell n \mu r^{-1} \leqslant \ell \left(\frac{1}{r} \left(\lambda_1 M_1 \otimes \cdots \otimes \lambda_m M_m\right)^r\right)$$

, that is why

$$\ell\mu \leqslant \ell \left(\frac{1}{r} \left(\lambda_1 M_1 \otimes \cdots \otimes \lambda_m M_m\right)^r\right) + \frac{\eta}{2} \ell.$$

□

4 Comparing distance automata

In this section, we consider the problem of comparing the functions computed by distance automata. In particular, we establish Theorem 3, and we reduce Theorem 4 to our core theorem, Theorem 5.

We start by describing distance automata, and their relationship with matrices over the tropical semiring.

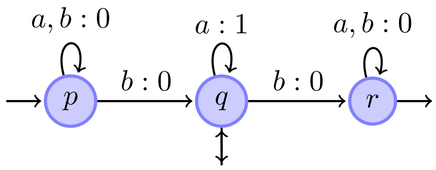
4.1 Distance automata

An *alphabet* is a finite set. The set of *words* over an alphabet $\mathbb{A}$ is denoted $\mathbb{A}^*$. A *distance automaton* is a tuple $(\mathbb{A}, Q, I, F, T)$, where Q is a finite set of *states* (that we can assume to be $\{1, \ldots, n\}$) where I (resp. T) is a row-vector (*resp.* column-vector) indexed by Q, and F is a morphism from words to $\mathcal{M}_{n,n}(\overline{\mathbb{N}})$. The function f computed by a distance automaton $(\mathbb{A}, Q, I, F, T)$ over an input word u is:

$$\begin{aligned} f : \mathbb{A}^* &\to \overline{\mathbb{N}} \\ u &\mapsto I \otimes F(u) \otimes T \ . \end{aligned}$$

We assume from now on that the initial and final vectors I, T of distance automata only range over $\{0, \infty\}$. The theorems are equally true without this assumption, but this simplifies slightly the proof. In practice the theorems without this restriction can be obtained by simple reductions to this case.

We have defined so far distance automata in terms of matrices. One can see this object in a more "automaton" form as follows. There is a transition labelled (a, x) from state p to state q if $x < \infty$ and $x = F(a)_{p,q}$. A state p is *initial* if $I_{1,p} = 0$. It is *final* if $T_{i,1} = 0$. An example of distance automaton is as follows:



One can redefine the function computed by a distance automaton as follows. A *run* of an automaton over a word $a_1 \ldots a_k$ is a sequence $p_0, \ldots, p_k$ of states. The *weight of a run* is the sum of the weights of its transitions, *i.e.*, $F(a_1)_{p_0,p_1} + \cdots + F(a_k)_{p_{k-1},p_k}$. Remark that if there is some non-existing transition in this sequence, say from p_{i-1} to p_i, this means that $F(a_i)_{p_{i-1},p_i} = \infty$, and as a consequence the run has an infinite weight. A run is *accepting* if p_0 is initial and p_k is final. One defines the function accepted by the automaton as:

$$\begin{aligned} f : \mathbb{A}^* &\to \overline{\mathbb{N}} \\ u &\mapsto \inf\{\text{weight}(\rho) \ : \ \rho \text{ accepting run over } u\} \ . \end{aligned}$$

This definition is equivalent to the matrix version presented above.

For instance, the function computed by the above automaton associates to each word $u = a^{n_0} b a^{n_1} \ldots b a^{n_k}$ the value $\min(n_0, \ldots, n_k)$.

4.2 Superior limits

In this section, we present Theorem 7. This result, that is a refinement of known proofs concerning distance automata, will prove useful for further reductions.

In order to define the superior limit of a set of matrices, a topology is required. The matrices over $\overline{\mathbb{N}}$ are equipped with the following topology. When two matrices are distinct, their distance is $1/n$ where n is the maximal positive integer such that the entries that carry values at most n are the same in both matrices. If no such integer exists, the distance is 1.

Given $X \subseteq \mathcal{M}_{n,n}(\overline{\mathbb{N}})$, a matrix N belongs to the *superior limit* of X if:

- N is the limit of some sequence of matrices from X,
- there exists no $M \in X$ such that $M > N$.

Let us call $\limsup(X)$ the set of matrices in the superior limit of S.

Theorem 7 (consequence of [4, 8])**.** *Given a set $X \subseteq \mathcal{M}_{n,n}(\overline{\mathbb{N}})$, $\limsup(X)$ is finite. Furthermore, there is a PSPACE algorithm which, given a morphism F from $\mathbb{A}^*$ to $\mathcal{M}_{n,n}(\overline{\mathbb{N}})$, and a language $L \subseteq \mathbb{A}^*$ enumerates $\limsup(F(L))$.*

The first part of the statement is a consequence of Higman's lemma. The second part is an adaptation of Leung's proof of decidability of limitedness for distance automata [8] (it subsumes this result). We are not aware of any similar statement in the literature, though it can be deduced from previous works.

4.3 A first reduction: the theorem of affine domination

Our goal in this section is to establish the theorem of affine domination (Theorem 3). This will be the opportunity to introduce some notations used in the subsequent section.

Let us fix ourselves two distance automata over the same alphabet $\mathbb{A}$. The first one, $\mathcal{A}_f = (\mathbb{A}, Q_f, F, I_f, T_f)$ calculates a function f. The second one, $\mathcal{A}_g = (\mathbb{A}, Q_g, G, I_g, T_g)$ calculates a function g.

Define $R_{p,0,q} \subseteq \mathbb{A}^*$ to be the set of words over which there is a run of $\mathcal{A}_g$ of weight 0 from state p to state q. Let ℓ be a non-null weight occurring in some transition of $\mathcal{A}_g$, and p, q be states in Q_g. Define $R_{p,\ell,q} \subseteq \mathbb{A}^*$ to contain the words over which there is a run of $\mathcal{A}_g$ from state p to state q which uses one transition of weight ℓ, and otherwise only transitions of weight 0. We will reuse this languages in the next section.

Proof of theorem 3. Define K to be the largest number that occurs in one of $\limsup(F(R_{p,\ell,q}))$ for some states p, q and weight of a transition ℓ (such a number exists since by Theorem 7 it is the maximum of finitely many numbers). Given a matrix M, call an *m-expansion* of M a matrix $M' \geqslant M$ such that for all i, j, $M_{i,j} > K$ implies $M'_{i,j} \geqslant m$. We first show a claim concerning expansions.

Claim. For all $M \in F(R_{p,\ell,q})$ and all m there exists an m-expansion $M' \in F(R_{p,\ell,q})$ of M.

Indeed, by definition of the superior limit, there is some $L \in \limsup(F(R_{p,\ell,q}))$ such that $L \geqslant M$. Furthermore, by choice of K, whenever $M_{i,j} > K$, $L_{i,j} = \infty$. Finally, still by definition of the superior limit, L is the limit of a sequence of matrices in $F(R_{p,\ell,q})$. Hence, for all m, there exists a matrix M' in this sequence which is sufficiently close to L that it is an m-expansion of M. This proves the claim.

Let us turn now to the core of the proof. Our goal is to prove that if f is dominated by g, (*i.e.,* there exists $\alpha : \mathbb{N} \to \mathbb{N}$ extended with $\alpha(\infty) = \infty$ such that $f \leqslant \alpha \circ g$), then $f \leqslant K(1+g)$. The proof is by contraposition. Thus, assume $f \not\leqslant K(1+g)$. This means $f(u) > Kg(u) + K$ for some word u. We have to prove that f is not dominated by g.

The first case is $g(u) = 0$. This means that $u \in R_{p,0,q}$ with p initial and q final. Using the above claim, one can chose for all m a word $v^{(m)} \in R_{p,0,q}$ such that $F(v^{(m)})$ is an m-expansion of $F(u)$. Since $f(u) > K$, this means that for all initial state r and all final state s of $\mathcal{A}_f$, $F(u)_{r,s} > K$. This means that for all such r, s, $F(v^{(m)})_{r,s} \geqslant m$. It follows that $f(v^{(m)}) \geqslant m$. Hence over the sequence $(v^{(m)})_m$, g is bounded and f tends to infinity. This forbids the existence of a function α such that $f \leqslant \alpha \circ g$, f is not dominated by g.

Assuming $g(u) \neq 0$, the argument is similar. Remark first that $g(u)$ is finite since $f(u) > Kg(u) + K$. This means one can find $p_0, \ldots, p_k$ with p_0 initial, p_k final, and such that:

$$u = u_1 \ldots u_k, \quad u_1 \in R_{p_0,\ell_1,p_1}, \ldots, u_k \in R_{p_{k-1},\ell_k,p_k} \ ,$$

where $\ell_1, \ldots, \ell_k$ are all non-null and of sum $g(u)$. By the above claim, for all $i = 1 \ldots k$, and all m, one can select $v_i^{(m)}$ in R_{p_{i-1},ℓ_i,p_i} such that $F(v_i^{(m)})$ is an m-expansion of $F(u_i)$. Consider now the word $v^{(m)} = v_1^{(m)} \ldots v_k^{(m)}$. Clearly $g(v^{(m)}) = g(u)$. For the sake of contradiction, assume now that $f(v^{(m)}) < m$ for some m. This means that there exists $q_0, \ldots, q_k$ such that q_0 is initial, q_k is final, and $F(v_i^{(m)})_{q_{i-1},q_i} < m$ for all $i = 1 \ldots k$. Since $F(v_i^{(m)})$ is an m -expansion of $F(u_i)$, this implies $F(u_i)_{q_{i-1},q_i} \leqslant K$. It follows that $f(u) \leqslant Kk \leqslant Kg(u)$. A contradiction. Hence $f(v^{(m)}) \geqslant m$. Thus, g is bounded over $(v^{(m)})_m$ while f is not. As a consequence, f is not dominated by g. □

4.4 The reduction construction

We reuse definitions and notations of automata $\mathcal{A}_f$ and $\mathcal{A}_g$ given in the preceding section. In particular, we use the sets $R_{p,\ell,q}$ again.

Our goal is to construct a finite set of weighted matrices X that captures the relationship between f and g. The key ideas behind this reduction are the following. Each matrix (M, ℓ) in X corresponds to a set of runs of g, that start in a given state p and end in a given state q, and use exactly one transition of non-null weight ℓ. The corresponding matrix M is in charge of (a) simulating the behaviour of F over some word corresponding to such a run (there may be infinitely many such runs, but only the finitely many matrices of the superior limit need to be considered), and (b) keeping information concerning the first and last state of the run of $\mathcal{A}_g$ for being able to check that pieces of run of g are correctly concatenated.

One also needs to define the part of the matrix in charge of controlling the validity of the run of $\mathcal{A}_g$. The construction behind Lemma 17 below is the one of a deterministic automaton, that reads words over the alphabet Q_g^2, and accepts a word $(p_1, q_1) \ldots (p_k, q_k)$ if, either p_1 is not initial, or q_k is not final, or if $q_{i-1} \neq p_i$ for some i. One can verify that this language is accepted by a deterministic and complete automaton of states $Q_g \uplus \{i, \bot\}$. The unique initial state is i, and, when reading the word $(p_1, q_1) \ldots (p_k, q_k)$, the automaton reaches state $\bot$ if p_1 is not initial or $q_{i-1} \neq p_i$ for some i, otherwise it reaches state q_k. The final states are the one not in T_g plus $\bot$ plus possibly i if there are no states that are both initial and final in g. Translated in matrix form, this yields Lemma 17.

Lemma 17. *There are $(|Q_g| + 2, |Q_g| + 2)$-matrices $(C^{p,q})_{p,q \in Q_g}$ over $\mathbb{B}$ and vectors I_C*

and T_C *such that for all* $p_1, q_1, \ldots, p_k, q_k \in Q_g$,

$$I_C \otimes C^{p_1,q_1} \otimes \cdots \otimes C^{p_k,q_k} \otimes T_C = \begin{cases} \infty & \text{if } p_1 \in I_g,\ q_1 = p_2, \ldots, q_{k-1} = p_k \text{ and } q_k \in T_g, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This is implemented in matrix form as follows. For each p, q where $p, q \in Q_g$, set the matrix $C^{p,q}$ that has indices in $Q_g \cup \{i, \bot\}$, to be such that:

$$(C^{p,q})_{p',q'} = \begin{cases} 0 & \text{if } p' = i,\ p \in I_g \text{ and } q' = q, \\ 0 & \text{if } p' = i,\ p \notin I_g \text{ and } q' = \bot, \\ 0 & \text{if } p' = p \text{ and } q' = q, \\ 0 & \text{if } p' \neq i \text{ and } p' \neq p \text{ and } q' = \bot, \\ \infty & \text{otherwise.} \end{cases}$$

Define furthermore I_C be the vector with all entries ∞ but i which is 0, and let T_C be the vector with all entries equal to 0 except T_g and i if there is a state both initial and final in $\mathcal{A}_g$. □

We can now construct the set X as follows:

$$X = \left\{ \left(\begin{pmatrix} M & \infty \\ \infty & C^{p,q} \end{pmatrix}, \ell \right) \ :\ M \in \limsup(F(R_{p,\ell,q})) \right\}$$

and the vectors

$$I = (I_f\ I_C) \qquad \text{and} \qquad T = \begin{pmatrix} T_f \\ T_C \end{pmatrix} .$$

The following lemma shows the validity of the construction, and more particularly how it relates the comparison of distance automata to the computation of the closure of a set of weighted matrices.

Lemma 18. *For all* $\beta > 0$, $f \leqslant \beta g$ *if and only if for all* $(W, \ell) \in \langle X \rangle$, $I \otimes W \otimes T \leqslant \beta \ell$.

Proof. Assume first $f \not\leqslant \beta g$, which means $f(u) > \beta g(u)$ for some u. Then clearly, $g(u)$ is finite and hence, there is an accepting run ρ of g over u. This means that one can find $p_0, \ldots, p_k$ with p_0 initial, p_k final, such that:

$$u \in R_{p_0,\ell_1,p_1} R_{p_1,\ell_2,p_2} \ldots R_{p_{k-1},\ell_k,p_k} ,$$

where $\ell_1, \ldots, \ell_k$ are all non-null and of sum $\ell = g(u)$. For all $i = 1 \ldots k$, set M_i to be some matrix in $\limsup(F(R_{p_{i-1},\ell_i,p_i}))$ such that $F(u_i) \leqslant M_i$. Let also C_i be C^{p_{i-1},p_i}. Clearly, the weighted matrix

$$(W_i, \ell_i) \qquad \text{with} \quad W_i = \begin{pmatrix} M_i & \infty \\ \infty & C_i \end{pmatrix}$$

belongs to X. Hence (W, ℓ) belongs to $\langle X \rangle$, where $W = W_1 \otimes \cdots \otimes W_k$. We then have $I \otimes W \otimes T = \min(x_f, x_C)$ with

$$x_f = I_f \otimes M_1 \otimes \cdots \otimes M_k \otimes T_f \qquad \text{and} \qquad x_C = I_C \otimes C_1 \otimes \cdots \otimes C_k \otimes T_C .$$

By choice of the M_i's, $x_f \geqslant I_f \otimes F(u) \otimes T_f = f(u)$. Furthermore, by Lemma 17, $x_C = \infty$. It follows that $I \otimes W \otimes T \geqslant f(u) > \beta g(u) = \beta\ell$.

Assume now that $f \leqslant \beta g$. Consider some $(W,\ell) \in \langle X\rangle$, it is obtained as $(W,\ell) = (W_1,\ell_1) \otimes \cdots \otimes (W_k,\ell_k)$ with $(W_i,\ell_i) \in X$ for all i. By definition of X, each of the W_i's can be written, for some $p_i, q_i \in Q_g$, as

$$W_i = \begin{pmatrix} M_i & \infty \\ \infty & C^{p_i,q_i} \end{pmatrix} \qquad \text{with} \quad M_i \in \limsup F(R_{p_i,\ell_i,q_i}).$$

Once more, one has $I \otimes W \otimes T = \min(x_f, x_C)$ with

$$x_f = I_f \otimes M_1 \otimes \cdots \otimes M_k \otimes T_f \qquad \text{and} \qquad x_C = I_C \otimes C_1 \otimes \cdots \otimes C_k \otimes T_C \ .$$

Remark first that if $x_C = 0$, clearly, $I \otimes W \otimes T = 0 \leqslant \beta\ell$. Hence, let us assume that $x_C = \infty$. This means by Lemma 17 that p_1 is initial, q_k is final, and $p_i = q_{i-1}$ for all $i = 2 \dots k$. One needs to prove $x_f \leqslant \beta\ell$.

Assume for the sake of contradiction that $x_f > \beta\ell$. By continuity of the product, and using the definition of the superior limit, there exist words $u_1, \dots, u_k$ such that for all $i = 1 \dots k$, $u_i \in R_{p_i,\ell_i,q_i}$, and $I_f \otimes F(u_1) \otimes \cdots \otimes F(u_k) \otimes T_f > \beta\ell$. Furthermore, by definition of the sets R_{p_i,ℓ_i,q_i}, the fact that p_1 is initial, that q_k is final, and that $q_{i-1} = p_i$ for all $i = 2 \dots k$, it follows that $g(u_1 \dots u_k) = \ell$. It follows that $f(u_1 \dots u_k) > \beta g(u_1 \dots u_k)$. A contradiction. □

We are now ready to establish the main theorem of the paper.

Proof of Theorem 4. Let us consider two functions f and g computed by distance automata and some $\varepsilon > 0$. The algorithm works as follows. It computes the set X of weighted matrices as defined in this section, as well as the corresponding vectors I, T. Using Theorem 5, it computes a finitely presented set Y of weighted matrices such that $Y \approx_{\frac{\varepsilon}{2}} \langle X\rangle$. Then it tests the existence in Y of a weighted matrix (M,ℓ) such that $I \otimes \frac{1}{\ell}M \otimes T > 1 - \frac{\varepsilon}{2}$. This is easy to do for finitely presented sets. If such a weighted matrix exists, the algorithm answers "no". It answers "yes" otherwise. Let us show the correctness of this approach.

- Assume $f \leqslant (1-\varepsilon)g$, and that, for the sake of contradiction, the algorithm answers "no". This means that $I \otimes \frac{1}{\ell}M \otimes T > 1 - \frac{\varepsilon}{2}$ for some weighted matrix $(M,\ell) \in Y$. Furthermore, there exists $(M',\ell') \in \langle X\rangle$ such that $(M,\ell) \preccurlyeq_{\frac{\varepsilon}{2}} (M',\ell')$. This implies $\frac{1}{\ell}M \leqslant \frac{1}{\ell'}M' + \frac{\varepsilon}{2}$. It follows that $I \otimes M' \otimes T > (1-\varepsilon)\ell'$. This contradicts Lemma 18.

- Assume $f \not\leqslant g$, then by Lemma 18, there exists a matrix $M \in \langle X\rangle$ such that $I \otimes \frac{1}{\ell}M \otimes T > 1$. Furthermore, there exists $M' \in Y$ such that $(M,\ell) \preccurlyeq_{\frac{\varepsilon}{2}} (M',\ell')$. This implies $\frac{1}{\ell}M \leqslant \frac{1}{\ell'}M' + \frac{\varepsilon}{2}$, and hence $I \otimes \frac{1}{\ell'}M' \otimes T > 1 - \frac{\varepsilon}{2}$. The algorithm answers "no".

□

5 Conclusion and further remarks

In this paper, we have provided an algorithm for deciding the approximate comparison of distance automata. This algorithm involves the computation of the closure under product of sets of–what we call–weighted matrices. This result can be of independent interest.

The main open question is the complexity of the problem. It is clear that the problem is at least PSPACE hard. A correct implementation of the arguments in this paper shows that EXSPACE is an upper bound. We do not know what is the exact complexity.

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