Last week:

- Vector spaces, linear combinations, linear independence
- Basis and dimension
- Linear transforms and matrices
- Inner products and norms

This week:

- Unitary transforms ✓
- Outer product ✓ (+ eigenvalues and eigenvectors ✓)
- Tensor (Kronecker) product
- Conditional and total probability

Theorem 4. *Any of the following four conditions on a matrix U are equivalent:*

1. *U preserves distances, that is, $||U|v\rangle|| = |||v\rangle||$ for all $|v\rangle \in V$.*
2. *$U^\dagger = U^{-1}$, so $U^\dagger U = I$, the identity matrix*[23].
3. *The columns of U form an orthonormal basis.*
4. *U preserves inner products, that is, $\langle u|U^\dagger U|v\rangle = \langle u|v\rangle$ for all $|u\rangle, |v\rangle \in V$.*

Examples:

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad (\text{for all } \theta)$$

$$\begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix}.$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (= H)$$

Outer product: $|\psi\rangle, |\phi\rangle \in \mathbb{C}^n$

$$\Downarrow$$

$$\langle\psi|\phi\rangle = \langle\psi|\cdot|\phi\rangle \in \mathbb{C}$$

$$|\psi\rangle\langle\phi| = |\psi\rangle\cdot\langle\phi| \in \mathbb{C}^{n\times n}$$

Exampes: $$|i\rangle\langle j| = \begin{array}{cc} \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & 1 & \vdots \\ 0 & \cdots & 0 \end{bmatrix} & \rightarrow i^{th}\text{ row} \\ \uparrow j^{th}\text{ col} & \end{array}$$

$$H = |+\rangle\langle 0| + |-\rangle\langle 1|$$

$$H|0\rangle = |+\rangle\langle 0|\cdot|0\rangle + |-\rangle\langle 1|\cdot|0\rangle = |+\rangle$$

For any matrix M, with $|\psi_i\rangle = M|i\rangle$,

$$M = \sum_{i=0}^{n-1} |\psi_i\rangle\langle i|$$

$$M|k\rangle = \sum |\psi_i\rangle\underbrace{\langle i|k\rangle}$$

Eigenvalues/vectors: when $M|\psi\rangle = \lambda|\psi\rangle$ with $\lambda \in \mathbb{C}$, $|\psi\rangle$ is eigenvector with eigenvalue λ.

Examples: $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has eigenvectors $\{|0\rangle, |1\rangle\}$ with eigenvalue 1

$I = |0\rangle\langle 0| + |1\rangle\langle 1|$
$= |+\rangle\langle +| + |-\rangle\langle -|$

$Z|0\rangle = |0\rangle$
$Z|1\rangle = -|1\rangle$

$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$

$X|+\rangle = |+\rangle$
$X|-\rangle = -|-\rangle$

$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = |+\rangle\langle +| - |-\rangle\langle -|$

Tensor product: "multiplication" of vector spaces

V with basis $\{|\psi_i\rangle : 0 \le i < n\}$
W with basis $\{|\phi_j\rangle : 0 \le j < m\}$
$\Downarrow$
$V \otimes W$ with basis $\{|\psi_i\rangle \otimes |\phi_j\rangle\}$

$$\left(\dim(V\otimes W) = \dim V \cdot \dim W\right)$$

Kronecker product: $A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix}$, $B = \begin{bmatrix} B_{11} & \cdots & B_{1m} \\ \vdots & \ddots & \vdots \\ B_{m1} & \cdots & B_{mm} \end{bmatrix}$

$$\Downarrow$$

$$: A\otimes B = \begin{bmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & \ddots & \vdots \\ A_{n1}B & \cdots & A_{nn}B \end{bmatrix}$$

Very important property:

$$\boxed{(A\otimes B)\cdot(|\psi\rangle\otimes|\phi\rangle) = (A|\psi\rangle)\otimes(B|\phi\rangle)}$$

Examples: $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \otimes \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1\cdot\begin{bmatrix} 3 \\ 4 \end{bmatrix} \\ 2\cdot\begin{bmatrix} 3 \\ 4 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \\ 8 \end{bmatrix}$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\qquad I\otimes Z = ?$

$\Rightarrow$

$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\qquad Z\otimes I = ?$

In $\mathbb{C}^{2^n}$, $|i\rangle = |bin(i)\rangle$
(where $|b_i b_j\rangle = |b_i\rangle \otimes |b_j\rangle$)

Conditional probability: let $D \in \{1, 2, \ldots 6\}$ be a uniform random variable and consider the two events:

$$E = [D \leq 3]$$
$$F = [D \text{ is even}]$$

Let's compute:

$$Pr[E] = \frac{1}{2}$$
$$Pr[F] = \frac{1}{2}$$

$$Pr[E \cap F] = 1/6$$
$$Pr[F \mid E] = 1/3 = \frac{Pr[E \cap F]}{Pr[E]}$$

Total probability: $\Pr[F] = \Pr[F|E]\cdot\Pr[E]$

$$+\Pr[F|\bar{E}]\cdot\Pr[\bar{E}]$$

$$= \Pr[F\cap E] + \Pr[F\cap \bar{E}]$$

Consider a 2^n-sided die X labeled by bit strings.

Compute:

$$\Pr[X_1X_2X_3 = 010] = \frac{1}{8} = \frac{2^{n-3}}{2^n}$$

$$\Pr[X_1 = 1 \mid X_1X_2 \neq 00] = 2/3$$