Entanglement:

Warm-up:

Exercise 4.11. Are each of the following states a product state or entangled state? If it is a product state, give the factorization.

(a) $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$.

(b) $\frac{1}{\sqrt{2}}(|10\rangle + i|11\rangle)$.

Question 1:

10. Suppose we have a 2-bit input $x = x_0x_1$ and a phase-query that maps

$$O_{x,\pm} : |b\rangle \mapsto (-1)^{x_b}|b\rangle \quad \text{for } b \in \{0,1\}.$$

(a) Suppose we run the 1-qubit circuit $HO_{x,\pm}H$ on initial state $|0\rangle$ and then measure (in the computational basis). What is the probability distribution on the output bit, as a function of x?

(b) Now suppose the query leaves some workspace in a second qubit, which is initially $|0\rangle$:

$$O'_{x,\pm} : |b,0\rangle \mapsto (-1)^{x_b}|b,b\rangle \quad \text{for } b \in \{0,1\}.$$

Suppose we just ignore the workspace and run the algorithm of (a) on the first qubit with $O'_{x,\pm}$ instead of $O_{x,\pm}$ (and $H \otimes I$ instead of H, and initial state $|00\rangle$). What is now

the probability distribution on the output bit (i.e., if we measure the first of the two bits)?

Comment: This exercise illustrates why it's important to "clean up" (i.e., set back to $|0\rangle$) workspace qubits of some subroutine before running it on a superposition of inputs: the unintended entanglement between the address and workspace registers can thwart the intended interference effects.

Question 2:

12. Suppose Alice and Bob are not entangled. If Alice sends a qubit to Bob, then this can give Bob at most one bit of information about Alice.[9] However, if they share an EPR-pair, $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, then they can transmit *two* classical bits by sending one qubit over the channel; this is called *superdense coding*. This exercise will show how this works.

    (a) They start with a shared EPR-pair, $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Alice has classical bits a and b. Suppose she does an X-gate on her half of the EPR-pair if $a = 1$, followed by a Z-gate if $b = 1$ (she does both if $ab = 11$, and neither if $ab = 00$). Write the resulting 2-qubit state for the four different cases that ab could take.

    (b) Suppose Alice sends her half of the state to Bob, who now has two qubits. Show that Bob can determine both a and b from his state, using Hadamard and CNOT gates, followed by a measurement in the computational basis.