Last week:

- Principle of deferred measurement

This week:

- Review of Problem Set 1

Problem 1 (10 marks)

Consider the following unitary matrices, known as the Pauli matrices.

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

1. We can write X as a linear combination of outer products: $X = |1\rangle\langle 0| + |0\rangle\langle 1|$. Write Y and Z as a linear combination of outer products.

$$X = |1\rangle\langle 0| + |0\rangle\langle 1|$$
$$= |+\rangle\langle +| - |-\rangle\langle -|$$
$$Y = i|0\rangle\langle 1| - i|1\rangle\langle 0|$$
$$= |i\rangle\langle i| - |-i\rangle\langle -i|$$
$$Z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

$$|0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$
$$|0\rangle\langle 1| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$|1\rangle\langle 0| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
$$|0\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sum \lambda_i |\varphi_i\rangle\langle \varphi_i|$$

$$|i\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$$

$$|-i\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}}$$

2. For each one of the Pauli matrices, use Dirac notation to find where each transformation maps the states $|0\rangle, |1\rangle, |-\rangle, |+\rangle$. (That is, compute $X|0\rangle$, $X|1\rangle$, $X|-\rangle$, $\ldots$, $Z|-\rangle$, $Z|+\rangle$.)

$X|0\rangle = |1\rangle$ $\quad Y|0\rangle = i|1\rangle$ $\quad Z|0\rangle = |0\rangle$

$X|1\rangle = |0\rangle$ $\quad Y|1\rangle = -i|0\rangle$ $\quad Z|1\rangle = -|1\rangle$

$X|+\rangle = |+\rangle$ $\quad Y|+\rangle = -i|-\rangle$ $\quad Z|+\rangle = |+\rangle$

$X|-\rangle = -|-\rangle$ $\quad Y|-\rangle = i|+\rangle$ $\quad Z|-\rangle = -|-\rangle$

$$Y\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) = \frac{-i|1\rangle - i|0\rangle}{\sqrt{2}}$$

Problem 2 (15 marks)

Consider the state $|\psi\rangle = \sqrt{\frac{3}{5}}\,|0\rangle + \sqrt{\frac{2}{5}}\,|1\rangle$.

1. Compute the inner product $\langle\psi|\psi\rangle$.

$$\langle\psi|\psi\rangle = \left(\sqrt{\tfrac{3}{5}}\langle 0| + \sqrt{\tfrac{2}{5}}\langle 1|\right)\left(\sqrt{\tfrac{3}{5}}|0\rangle + \sqrt{\tfrac{2}{5}}|1\rangle\right)$$

$$= \frac{3}{5}\underbrace{\langle 0|0\rangle}_{1} + \cancel{\frac{\sqrt{6}}{5}\langle 0|1\rangle} + \cancel{\frac{\sqrt{6}}{5}\langle 1|0\rangle} + \frac{2}{5}\underbrace{\langle 1|1\rangle}_{1}$$

$$= 1$$

2. Compute the **matrix** $|\psi\rangle\langle\psi|$.

$$\left(\sqrt{\tfrac{3}{5}}|0\rangle + \sqrt{\tfrac{2}{5}}|1\rangle\right)\left(\sqrt{\tfrac{3}{5}}\langle 0| + \sqrt{\tfrac{2}{5}}\langle 1|\right)$$

$$= \frac{3}{5}|0\rangle\langle 0| + \frac{\sqrt{6}}{5}|0\rangle\langle 1| + \frac{\sqrt{6}}{5}|1\rangle\langle 0| + \frac{2}{5}|1\rangle\langle 1|$$

$$= \begin{pmatrix} \frac{3}{5} & \frac{\sqrt{6}}{5} \\ \frac{\sqrt{6}}{5} & 2/5 \end{pmatrix}$$

3. What will be the outcome of measuring $|\psi\rangle$ in the computational $(\{|0\rangle, |1\rangle\})$ basis?

$$|\psi\rangle = \sqrt{\tfrac{3}{5}}|0\rangle + \sqrt{\tfrac{2}{5}}|1\rangle$$

$$\Downarrow$$

$$\begin{cases} |0\rangle & \text{w.p. } 3/5 \\ |1\rangle & \text{w.p. } 2/5 \end{cases}$$

4. What will be the outcome of measuring ψ in the $\{|+\rangle, |-\rangle\}$ basis?

$$\begin{cases} |+\rangle \text{ w.p. } \frac{1}{2} + \frac{\sqrt{6}}{5} \\ |-\rangle \text{ w.p. } 1 - \left(\frac{1}{2} + \frac{\sqrt{6}}{5}\right) = \frac{1}{2} - \frac{\sqrt{6}}{5} \end{cases}$$

$$\langle + | \psi \rangle = \frac{1}{\sqrt{2}} \left(\underline{\langle 0|} + \underline{\langle 1|} \right) \left(\sqrt{\frac{3}{5}} \underline{|0\rangle} + \sqrt{\frac{2}{5}} \underline{|1\rangle} \right)$$

$$= \sqrt{\frac{3}{10}} \langle 0|0\rangle + \frac{1}{\sqrt{5}} \langle 1|1\rangle = \sqrt{\frac{3}{10}} + \frac{1}{\sqrt{5}}$$

$$\left(\sqrt{\frac{3}{10}} + \frac{1}{\sqrt{5}} \right)^2 = \frac{3}{10} + 2\sqrt{\frac{3}{50}} + \frac{1}{5} = \frac{1}{2} + \frac{\sqrt{6}}{5}$$

Problem 3 (15 marks)

Consider the the rotation gate

$$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

1. If we measure $R_\theta |0\rangle$ in the $\{|0\rangle, |1\rangle\}$-basis, what is the probability we will observe $|1\rangle$? $\sin^2 \theta$.

$$R_\theta |0\rangle = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$= \cos\theta |0\rangle + \boxed{\sin\theta} |1\rangle$$

2. If we measure $R_\theta \left|1\right\rangle$ in the $\{\left|+\right\rangle, \left|-\right\rangle\}$-basis, what is the probability we will observe $\left|+\right\rangle$?

$$R_\theta|1\rangle = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

$$= -\sin\theta|0\rangle + \cos\theta|1\rangle$$

$$|\langle -|R_\theta|1\rangle|^2 = \frac{1}{2} + \sin\theta\cos\theta$$

$$\langle +|R_\theta|1\rangle = \frac{1}{\sqrt{2}}\left(\langle 0| + \langle 1|\right)\left(-\sin\theta|0\rangle + \cos\theta|1\rangle\right)$$

$$= \frac{-\sin\theta + \cos\theta}{\sqrt{2}}$$

$$|\langle +|R_\theta|1\rangle|^2 = \frac{(-\sin\theta + \cos\theta)^2}{2} = \frac{\sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta}{2}$$

$$= \frac{1}{2} - \sin\theta\cos\theta$$

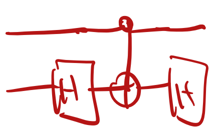
Problem 5 (20 marks)

This problem considers the task of designing quantum circuits that implement a given unitary using a restricted set of gates.

1. The controlled-Z gate is a 2-qubit gate with the following matrix representation:

$$\mathsf{cZ} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Show how to implement cZ using H and CNOT gates.



$$\text{CNOT} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right) = \begin{pmatrix} I & 0 \\ 0 & X \end{pmatrix} \neq I \otimes X$$

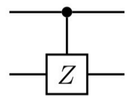
$$cZ = \begin{pmatrix} I & 0 \\ 0 & Z \end{pmatrix}$$

$$(I\otimes H)\left(|0\rangle\langle 0|\otimes I + |1\rangle\langle 1|\otimes X\right)(I\otimes H) = |0\rangle\langle 0|\otimes H^2 + |1\rangle\langle 1|\otimes HXH$$

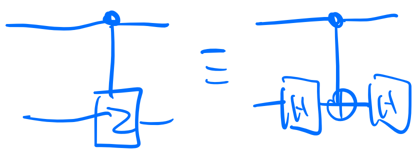
$$= |0\rangle\langle 0|\otimes I + |1\rangle\langle 1|\otimes Z$$

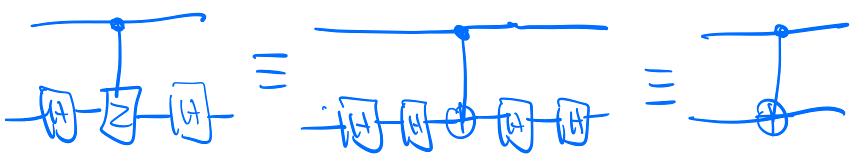
$I|0\rangle\langle 0|\otimes HI$ $\qquad$ $I|1\rangle\langle 1|\otimes HX$

$$(A\otimes B)(C\otimes D) = (AC)\otimes(BD)$$

2. The controlled-Z gate is written as follows: 

Show how to implement CNOT using H and controlled-Z gates.

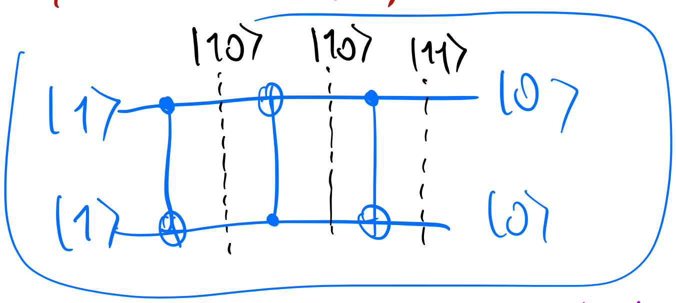




3. A SWAP gate interchanges two qubits, mapping $|ab\rangle \mapsto |ba\rangle$ for all $a, b \in \{0, 1\}$. Show how to implement a SWAP gate using CNOT gates. (Hint: either qubit can be the control qubit!)

If two linear transforms M, M' act the same way on a basis $\Rightarrow M = M'$

$(M|\psi_i\rangle = M'|\psi_i\rangle \ \forall i)$



SWAP: $|00\rangle \mapsto |00\rangle$

$|10\rangle \mapsto |01\rangle$

$|01\rangle \mapsto |10\rangle$

$|11\rangle \mapsto |11\rangle$

Our circuit: $|00\rangle \longmapsto |00\rangle$

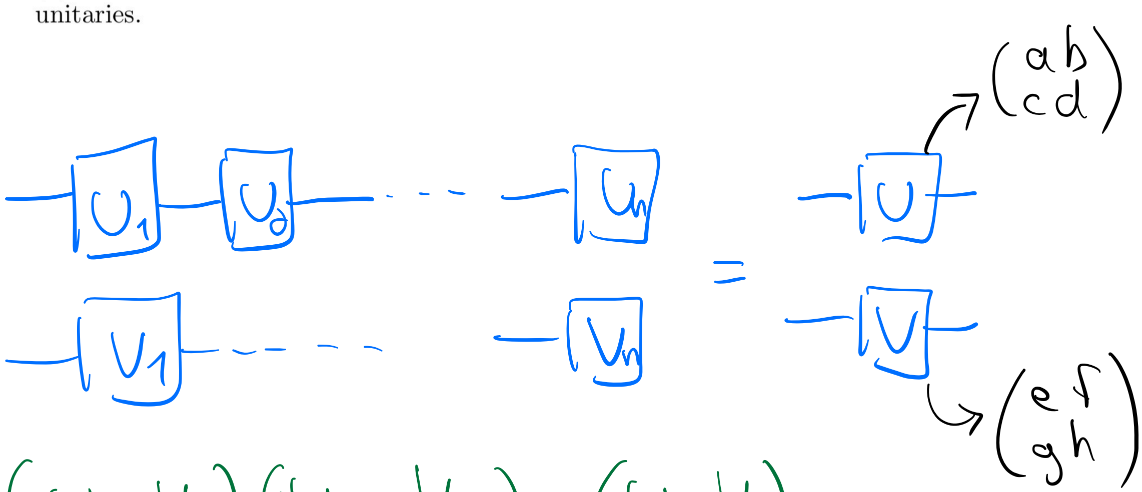
$|01\rangle \longmapsto |10\rangle$

$|10\rangle \longmapsto |01\rangle$

$|11\rangle \longmapsto |11\rangle$

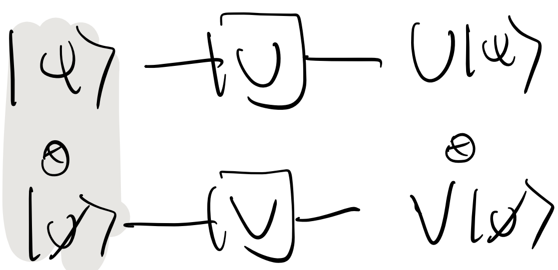
$$\mathrm{CNOT} \cdot \mathrm{CNOT}' \cdot \mathrm{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

4. (*) Show that it is impossible to implement the CNOT gate from *any* combination of single-qubit unitaries.



$$(U_n\otimes V_n)(U_{n-1}\otimes V_{n-1})\cdots(U_1\otimes V_1)$$

$$= \underbrace{(U_n U_n \cdots U_1)}_{U}\otimes\underbrace{(U_n V_{n-1}\cdots V_1)}_{V}$$



$$CNOT|+\rangle|0\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$$