THE UNIVERSITY OF WARWICK

MEng Examinations: Summer 2020

Quantum Computing

Time allowed: 3 hours.

Answer THREE questions: Question 1 from Section A and TWO questions from Section B.

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not allowed.
Section A   Answer the following question

1. (a) **(6 points)** State the formal definition of a quantum state.

(b) **(6 points)** Given a 2-qubit state \( |\psi\rangle = \alpha_1 |00\rangle + \alpha_2 |01\rangle + \alpha_3 |10\rangle + \alpha_4 |11\rangle \) and an orthonormal basis \( \{|v_1\rangle, |v_2\rangle, |v_3\rangle, |v_4\rangle\} \). What is the probability of measuring \( |v_i\rangle \) in the aforementioned basis?

(c) **(18 points)** For each of the following statements, indicate whether it is **true** or **false**. You will be awarded 3 points for each correct answer and −3 points for each incorrect answer. (Should the sum of awarded points be negative, you will receive 0 points for this part of the question.)

i. Measurement in any orthonormal basis \( \{|v_1\rangle, \ldots, |v_n\rangle\} \) can be emulated by a unitary transformation and measurement in the computational basis.

ii. The state \( |1\rangle \) is indistinguishable from the state \( |i\rangle |1\rangle \).

iii. The state \( |+\rangle \) is indistinguishable from the mixed state that is \( |0\rangle \) with probability 0.5 and \( |1\rangle \) with probability 0.5.

iv. A quantum state \( |\psi\rangle \) can only evolve to a quantum state \( |\psi'\rangle \) via multiplication by a Hermitian matrix \( A \); i.e., \( |\psi'\rangle = A |\psi\rangle \).

v. There exists a quantum query algorithm that can compute the parity of string \( x \in \{0, 1\}^n \) using a single query in superposition.

vi. Shor’s algorithm and Grover’s algorithm rely on Fourier transforms on **different** groups.
1. Let $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$, and consider the following circuit.

(a) **(15 points)** Determine the state of the three qubits after applying each one of the gates.
(b) **(5 points)** Determine the probabilities of the possible outcomes of the measurements of the top two qubits.
(c) **(10 points)** What state does the third qubit collapse to in each of the cases above?
(d) **(5 points)** Describe at least one application that this circuit can be used for.

2. (a) **(10 points)** Prove that the Bell state $|\Phi^+\rangle = \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle$ is entangled.
(b) **(10 points)** Let $U_f$ be a unitary transformation that XOR-implements a function $f : \{0, 1\}^n \rightarrow \{0, 1\}$; that is,
$$U_f(|x\rangle \otimes |a\rangle) = |x\rangle \otimes |a \oplus f(x)\rangle.$$  
Show how to obtain a sign-implementation of $f$ (include all calculations); that is
$$U_f^\pm |x\rangle = (-)^{f(x)} |x\rangle.$$  
(Hint, plug-in $|a\rangle = |\rangle$, expand the expression that $U_f$ outputs, and omit ancillas at the end.)
(c) **(15 points)** Let $U_f^\pm$ be a unitary transformation that sign-implements a function $f : \{0, 1\}^n \rightarrow \{0, 1\}$, and let $\{\hat{f}(S)\}_{S \subseteq [n]}$ be the Fourier spectrum of $f$. Design a quantum algorithm that samples a random $S \subseteq [n]$ with probability $\hat{f}(S)^2$. Analyse the state of the system after each gate.
3. (a) (5 points) Provide a formal and complete definition of a mixed state and its density.

(b) (15 points) Compute the density matrix of the following mixed state:

\[
\begin{cases}
|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ with probability } 1/2 \\
\frac{1}{3} |1\rangle + \frac{2}{3} |2\rangle + \frac{2i}{3} |3\rangle = \begin{pmatrix} 1/3 \\ 2/3 \\ 2i/3 \end{pmatrix} \text{ with probability } 1/2
\end{cases}
\]

(c) (5 points) Prove that the states \( |0\rangle \) and \(-|0\rangle \) are indistinguishable.

(d) (10 points) Prove that the mixed state that the CHSH game yields:

\[
\begin{cases}
|0\rangle \text{ with probability } 1/2 \\
|1\rangle \text{ with probability } 1/2 \\
\end{cases} \quad \text{and} \quad \begin{cases}
|+\rangle \text{ with probability } 1/2 \\
|-\rangle \text{ with probability } 1/2 \\
\end{cases}
\]

are indistinguishable.