CS4192

THE UNIVERSITY OF WARWICK

MEng Examinations: Summer 2020

Quantum Computing

Time allowed: 3 hours.

Answer **THREE** questions: **Question 1** from Section A and **TWO** questions from Section B.

Read carefully the instructions on the answer book and make sure that the particulars required are entered on **each** answer book.

Calculators are not allowed.

Section A Answer the following question

1. (a) **(6 points)** State the formal definition of a quantum state.

   (b) **(6 points)** Given a 2-qubit state $|\psi\rangle = \alpha_1 |00\rangle + \alpha_2 |01\rangle + \alpha_3 |10\rangle + \alpha_4 |11\rangle$ and an orthonormal basis $\{|v_1\rangle, |v_2\rangle, |v_3\rangle, |v_4\rangle\}$. What is the probability of measuring $|v_i\rangle$ in the aforementioned basis?

   (c) **(18 points)** For each of the following statements, indicate whether it is **true** or **false**. You will be awarded 3 points for each correct answer and -3 points for each incorrect answer. (Should the sum of awarded points be negative, you will receive 0 points for this part of the question.)

   i. Measurement in any orthonormal basis $\{|v_1\rangle, \ldots, |v_n\rangle\}$ can be emulated by a unitary transformation and measurement in the computational basis.

   ii. The state $|1\rangle$ is indistinguishable from the state $i\,|1\rangle$.

   iii. The state $|+\rangle$ is indistinguishable from the mixed state that is $|0\rangle$ with probability 0.5 and $|1\rangle$ with probability 0.5.

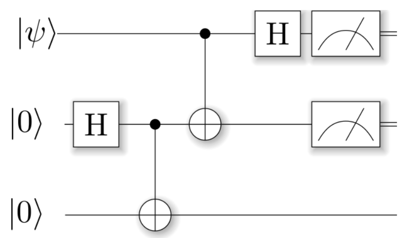
   iv. A quantum state $|\psi\rangle$ can only evolve to a quantum state $|\psi'\rangle$ via multiplication by a Hermitian matrix A; i.e., $|\psi'\rangle = A\,|\psi\rangle$.

   v. There exists a quantum query algorithm that can compute the parity of string $x \in \{0,1\}^n$ using a single query in superposition.

   vi. Shor's algorithm and Grover's algorithm rely on Fourier transforms on *different* groups.

Section B Answer **TWO** questions (out of three)

1. Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, and consider the following circuit.



   (a) **(15 points)** Determine the state of the three qubits after applying each one of the gates.
   (b) **(5 points)** Determine the probabilities of the possible outcomes of the measurements of the top two qubits.
   (c) **(10 points)** What state does the third qubit collapse to in each of the cases above?
   (d) **(5 points)** Describe at least one application that this circuit can be used for.

2. (a) **(10 points)** Prove that the Bell state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$ is entangled.
   (b) **(10 points)** Let U_f be a unitary transformation that **XOR**-implements a function $f : \{0,1\}^n \to \{0,1\}$; that is,

$$U_f(|x\rangle \otimes |a\rangle) = |x\rangle \otimes |a \oplus f(x)\rangle .$$

   Show how to obtain a sign-implementation of f (include all calculations); that is

$$U_f^{\pm}|x\rangle = (-)^{f(x)}|x\rangle .$$

   (Hint, plug-in $|a\rangle = |-\rangle$, expand the expression that U_f outputs, and omit ancillas at the end.)
   (c) **(15 points)** Let $U_f^{\pm}$ be a unitary transformation that **sign**-implements a function $f : \{0,1\}^n \to \{0,1\}$, and let $\{\hat{f}(S)\}_{S\subseteq[n]}$ be the Fourier spectrum of f. Design a quantum algorithm that samples a random $S \subseteq [n]$ with probability $\hat{f}(S)^2$. Analyse the state of the system after each gate.

3. (a) **(5 points)** Provide a formal and complete definition of a mixed state and its density.

   (b) **(15 points)** Compute the density matrix of the following mixed state:

$$\begin{cases} |1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} & \text{with probability } 1/2 \\ \frac{1}{3}|1\rangle + \frac{2}{3}|2\rangle + \frac{2i}{3}|3\rangle = \begin{pmatrix} 1/3 \\ 2/3 \\ 2i/3 \end{pmatrix} & \text{with probability } 1/2 \end{cases}.$$

   (c) **(5 points)** Prove that the states $|0\rangle$ and $-|0\rangle$ are indistinguishable.

   (d) **(10 points)** Prove that the mixed state that the CHSH game yields:

$$\begin{cases} |0\rangle & \text{with probability } 1/2 \\ |1\rangle & \text{with probability } 1/2 \end{cases} \quad \text{and} \quad \begin{cases} |+\rangle & \text{with probability } 1/2 \\ |-\rangle & \text{with probability } 1/2 \end{cases}$$

   are indistinguishable.