Problem 1

Formally state the following postulates of quantum mechanics:

1. State space (superposition).
3. Unitary evolution.

Problem 2

Consider the following unitary matrices, known as the Pauli matrices.

\[ X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

For each one of the Pauli matrices, find out where the transformation maps the states \( |0\rangle, |1\rangle, |−\rangle, |+\rangle \).

(That is, compute \( X |0\rangle, X |1\rangle, X |−\rangle, \ldots, Z |−\rangle, Z |+\rangle \).)

Problem 3

Consider the state \( |\psi\rangle = \sqrt{\frac{3}{5}} |0\rangle + \sqrt{\frac{2}{5}} |1\rangle \).

1. Compute the inner product \( \langle \psi | \psi \rangle \).
2. Compute the matrix \( |\psi\rangle \langle \psi| \).
3. What will be the outcome of measuring \( \psi \) in the computational \( \{|0\rangle, |1\rangle\} \) basis?
4. What will be the outcome of measuring \( \psi \) in the \( \{|+\rangle, |−\rangle\} \) basis?

Problem 4

Consider the rotation gate

\[ R_\epsilon = \begin{pmatrix} \cos(\epsilon) & -\sin(\epsilon) \\ \sin(\epsilon) & \cos(\epsilon) \end{pmatrix}. \]

1. If we measure \( R_\epsilon |0\rangle \) in the \( \{|0\rangle, |1\rangle\} \)-basis, what is the probability we will observe \( |1\rangle \)?
2. If we measure \( R_\epsilon |1\rangle \) in the \( \{|+\rangle, |−\rangle\} \)-basis, what is the probability we will observe \( |+\rangle \)?
Problem 5

Suppose we only have access to a small pool of quantum gates and would like to construct new gates from them. Specifically, we start with the set \{H, cZ\}, where \( H \) is the (1-qubit) Hadamard gate and \( cZ \) is the (2-qubit) controlled-\( Z \) gate, that maps \(|11\rangle \mapsto -|11\rangle \) and acts as the identity on the other computational basis states.

1. Show how to construct a CNOT from \( H \) and \( cZ \) (you can apply 1-qubit gates on either or both qubits: \( I \otimes H, H \otimes I \) and \( H \otimes H \) are all allowed).

2. Show how to use Hadamard and CNOT gates to construct a modified CNOT’ that inverts the target and control qubits, i.e., CNOT’ maps \(|a\rangle \otimes |b\rangle \mapsto |a \oplus b\rangle \otimes |b\rangle \) for all \( a, b \in \{0, 1\} \).

3. A SWAP gate interchanges two qubits, mapping \(|a\rangle |b\rangle \mapsto |b\rangle |a\rangle \) for all \( a, b \in \{0, 1\} \). Show how a SWAP gate can be obtained from a few CNOTs.

4. (*) Imagine that the gate set we have access to is not \{\( H, cZ \)\}, but rather the set of all 1-qubit unitaries. Show that it is impossible to obtain a CNOT from any combination of such gates.