CS419: Quantum Computing – Assignment 1

Problem 1

Formally state the following postulates of quantum mechanics:

1. State space (superposition).
2. Measurement (in a general basis).
3. Unitary evolution.

Problem 2

Consider the following unitary matrices, known as the Pauli matrices.

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For each one of the Pauli matrices, find out where the transformation maps the states $|0\rangle, |1\rangle, |-\rangle, |+\rangle$. (That is, compute $X|0\rangle$, $X|1\rangle$, $X|-\rangle$, $\ldots$, $Z|-\rangle$, $Z|+\rangle$.)

Problem 3

Consider the state $|\psi\rangle = \sqrt{\frac{3}{5}}|0\rangle + \sqrt{\frac{2}{5}}|1\rangle$.

1. Compute the inner product $\langle\psi|\psi\rangle$.
2. Compute the **matrix** $|\psi\rangle\langle\psi|$.
3. What will be the outcome of measuring ψ in the computational ($\{|0\rangle, |1\rangle\}$) basis?
4. What will be the outcome of measuring ψ in the $\{|+\rangle, |-\rangle\}$ basis?

Problem 4

Consider the the rotation gate

$$R_\epsilon = \begin{pmatrix} \cos(\epsilon) & -\sin(\epsilon) \\ \sin(\epsilon) & \cos(\epsilon) \end{pmatrix}.$$

1. If we measure $R_\epsilon|0\rangle$ in the $\{|0\rangle, |1\rangle\}$-basis, what is the probability we will observe $|1\rangle$?
2. If we measure $R_\epsilon|1\rangle$ in the $\{|+\rangle, |-\rangle\}$-basis, what is the probability we will observe $|+\rangle$?

Problem 5

Suppose we only have access to a small pool of quantum gates and would like to construct new gates from them. Specifically, we start with the set $\{H, cZ\}$, where H is the (1-qubit) Hadamard gate and cZ is the (2-qubit) *controlled-Z* gate, that maps $|11\rangle \mapsto -|11\rangle$ and acts as the identity on the other computational basis states.

1. Show how to construct a CNOT from H and cZ (you can apply 1-qubit gates on either or both qubits: $I \otimes H$, $H \otimes I$ and $H \otimes H$ are all allowed).

2. Show how to use Hadamard and CNOT gates to construct a modified CNOT' that inverts the target and control qubits, i.e., CNOT' maps $|a\rangle \otimes |b\rangle \mapsto |a \oplus b\rangle \otimes |b\rangle$ for all $a, b \in \{0, 1\}$.

3. A SWAP gate interchanges two qubits, mapping $|a\rangle |b\rangle \mapsto |b\rangle |a\rangle$ for all $a, b \in \{0, 1\}$. Show how a SWAP gate can be obtained from a few CNOTs.

4. (*) Imagine that the gate set we have access to is not $\{H, cZ\}$, but rather the set of *all 1-qubit unitaries*. Show that it is impossible to obtain a CNOT from any combination of such gates.