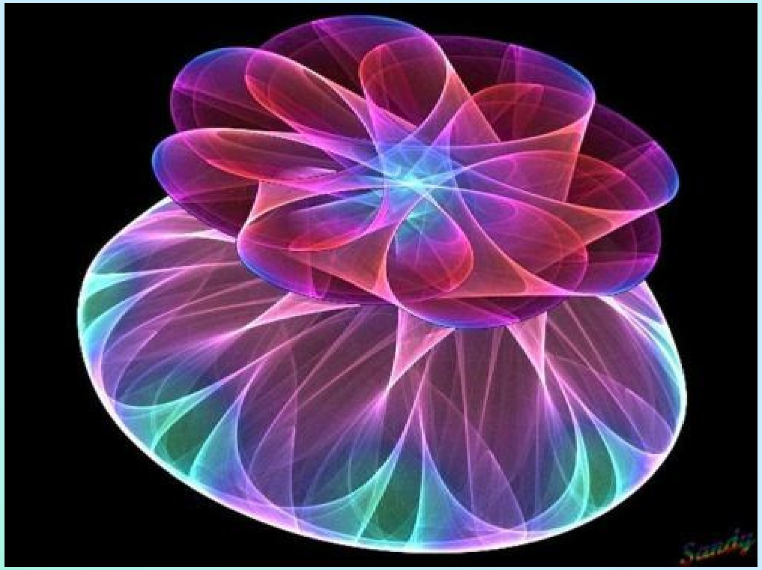
Large fluctuations in chaotic systems



Large Fluctuations in Chaotic Systems

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Outline

- Large Fluctuational Approach and Model Reduction
- Escape in quasi-hyperbolic systems
- Escape in non-hyperbolic systems
- Conclusions

Deterministic Chaos and Noise: Environment

Environment induces

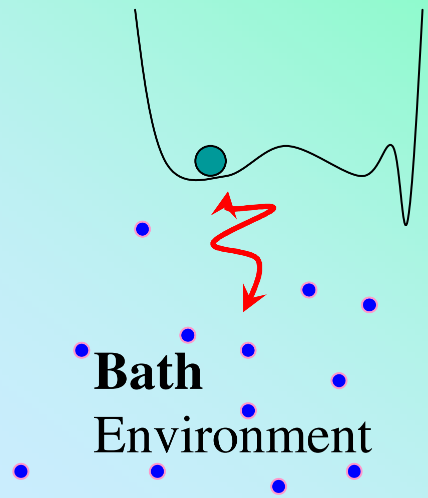
Dissipation and Fluctuations

$$H = H_S + H_B + H_{SB}$$

H_S System Hamiltonian

H_B Bath (Environmental) Hamiltonian

H_{SB} Hamiltonian of interaction



Elimination of the environmental degrees of freedom leads to

(***external*** or ***internal*** degrees of freedom)

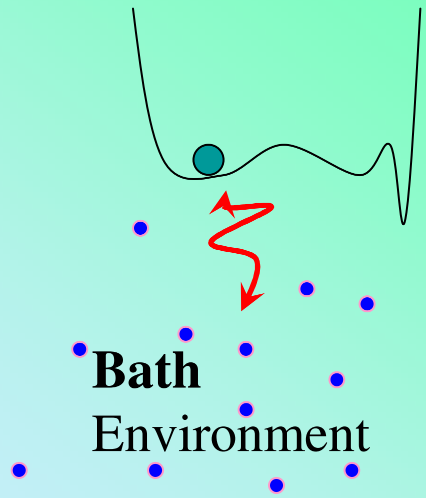
- Dissipation and
- Fluctuations

Note: Elimination is ,as a rule, a challenge task and it is often phenomenological

Deterministic Chaos and Noise: Environment

Archetypical Example: Environment as a Collection of Linear Oscillators

$$H = H_S + H_B + H_{SB}$$



$$H_S = \frac{p^2}{2m} + V(q;t)$$

The system is a model of a particle in potential

$$H_B = \sum_{n=1}^{N}\left(\frac{p_n^2}{2m_n} + \frac{m_n}{2}\omega_n^2 x_n^2\right)$$

The collection of harmonic oscillators

$$H_{SB} = -q\sum_{n=1}^{N} c_n x_n + q^2 \sum_{n=1}^{N} \frac{c_n^2}{2m_n\omega_n^2}$$

Linear coupling between system and bath

Elimination leads to

$$m\ddot{q} + 2\gamma m\dot{q} + \frac{\partial V}{\partial q} = \xi(t)$$

Damping (dissipation)

Fluctuations noise

Dissipation and Fluctuations have the same origin

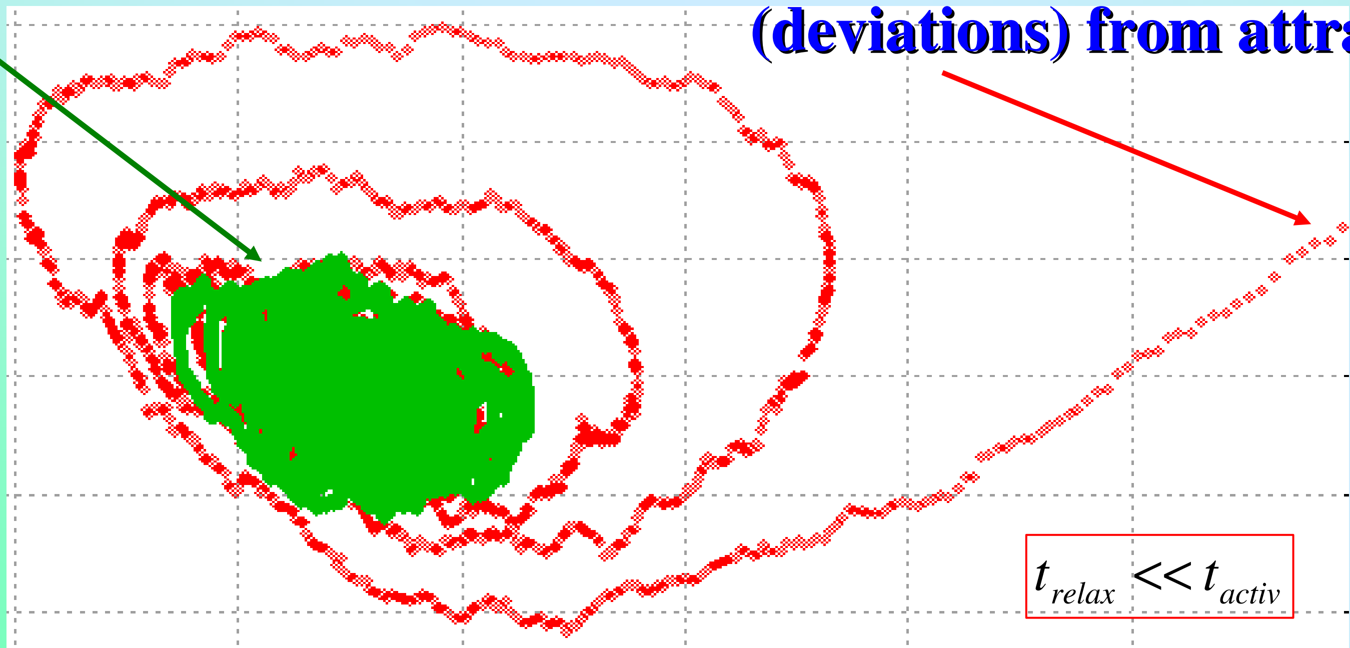
Chaos and Noise: Large Fluctuations

The simplification of dynamics: considering dynamics related to Large Fluctuations

Different manifestations of fluctuations:

Diffusion in a vicinity of attractor

Large fluctuations (deviations) from attractors



$t_{relax} << t_{activ}$

Chaos and Noise: Large Fluctuation Approach

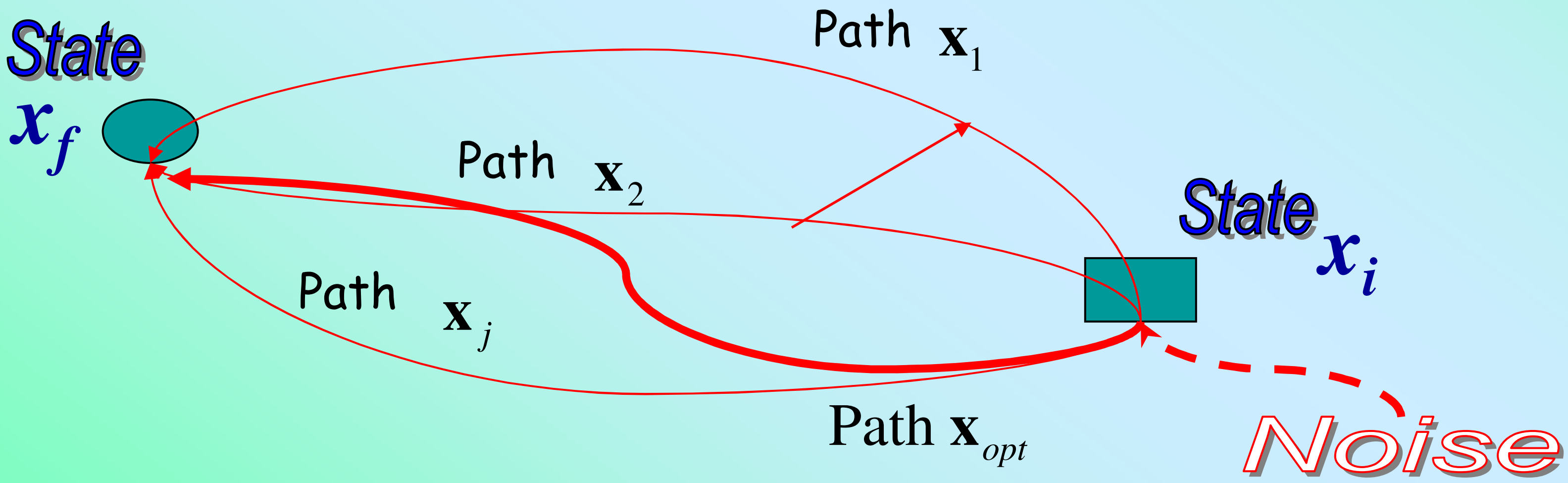
The system described by Langevin equations:

$$\dot{\mathbf{x}} = \mathbf{K}(\mathbf{x},t) + \mathbf{Q}\xi(t),$$
$$\langle \xi_\alpha \rangle = 0, \langle \xi_\alpha(t)\xi_\beta(s) \rangle = \mathbf{Q}\delta(t-s)$$

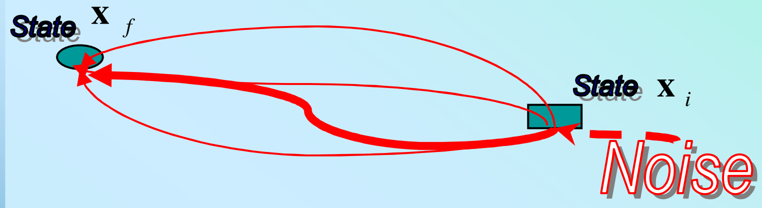
Transition probability via fluctuations paths

$$\rho(\mathbf{x}_f, t_f \mid \mathbf{x}_i, t_i) = \sum_j \rho[\mathbf{x}(t)_j] \approx \rho[\mathbf{x}(t)_{opt}]$$

The selection of the most probable (optimal) path



Deterministic chaos and noise: optimal path approach
deterministic pattern of fluctuations



$$\dot{\mathbf{x}} = \mathbf{K}(\mathbf{x},t) + \boldsymbol{\xi}(t),$$

$$\langle \xi_\alpha \rangle = 0, \langle \xi_\alpha(t)\xi_\beta(s) \rangle = D\mathbf{Q}\delta(t-s)$$

The probability of fluctuational path $\rho[\mathbf{x}(t)_j]$ is related to the probability $\rho[\boldsymbol{\xi}(t)_j]$ of random force to have a realization $\boldsymbol{\xi}(t)_j$

For Gaussian noise: $\rho[\boldsymbol{\xi}(t)_j] = C\exp\left(-\frac{1}{2}\int_{t_i}^{t_f} \boldsymbol{\xi}(t)_j^{\,2}\, dt\right) = C\exp\left(-\frac{1}{2}S\right)$

Since the exponential form, the most probable path has a minimal $S=S_{min}$

Changing to dynamical variables:

$$\text{Action } S = S[\boldsymbol{\xi}(t)] \xrightarrow[\boldsymbol{\xi}(t) = \dot{\mathbf{x}} - \mathbf{K}(\mathbf{x},t)]{\dot{\mathbf{x}} = \mathbf{K}(\mathbf{x},t) + \boldsymbol{\xi}(t)} S = [x(t)]$$

In the limit $D \to 0$, $\rho(\mathbf{x}_f;\mathbf{x}_i) = \rho\left(\mathbf{x}(t)_{opt}\right) = Const \times \exp\left(-\frac{S[\mathbf{x}(t)_{opt}]}{D}\right)$

$$S_{\min} = S[x_{opt}(t)] = \min \int dt\left(\dot{\mathbf{x}} - \mathbf{K}(\mathbf{x},t)\right)^2$$

Deterministic minimization problem

Large fluctuations and Model reduction

The initial model: the Hamiltonian for the system, the bath and coupling
In general case the dimension is infinite.

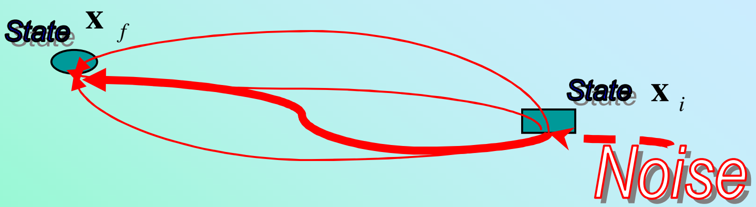
$$H = H_S + H_B + H_{SB}$$

Langevin model reduction: finite dimensional system with noise terms,
The dimension is infinite

$$\dot{\mathbf{x}} = \mathbf{K}(\mathbf{x},t) + \boldsymbol{\xi}(t),$$

$$\langle \xi_\alpha \rangle = 0, \langle \xi_\alpha(t)\xi_\beta(s) \rangle = D\mathbf{Q}\delta(t-s)$$

Large fluctuations reduction leads to a specific object: the optimal path as a solution of boundary value problem of the finite dimensional Hamilton system



$$S_{\min} = S[x_{opt}(t)] = \min \int dt(\dot{\mathbf{x}} - \mathbf{K}(\mathbf{x},t))^2$$

Initial state:

$$\mathbf{q}(t_i) = \mathbf{x}_i, \mathbf{p}(t_i) = 0, \quad t_i \to -\infty;$$

Final state:

$$\mathbf{q}(t_f) = \mathbf{x}_f, \mathbf{p}(t_f) = 0, \quad t_f \to \infty.$$

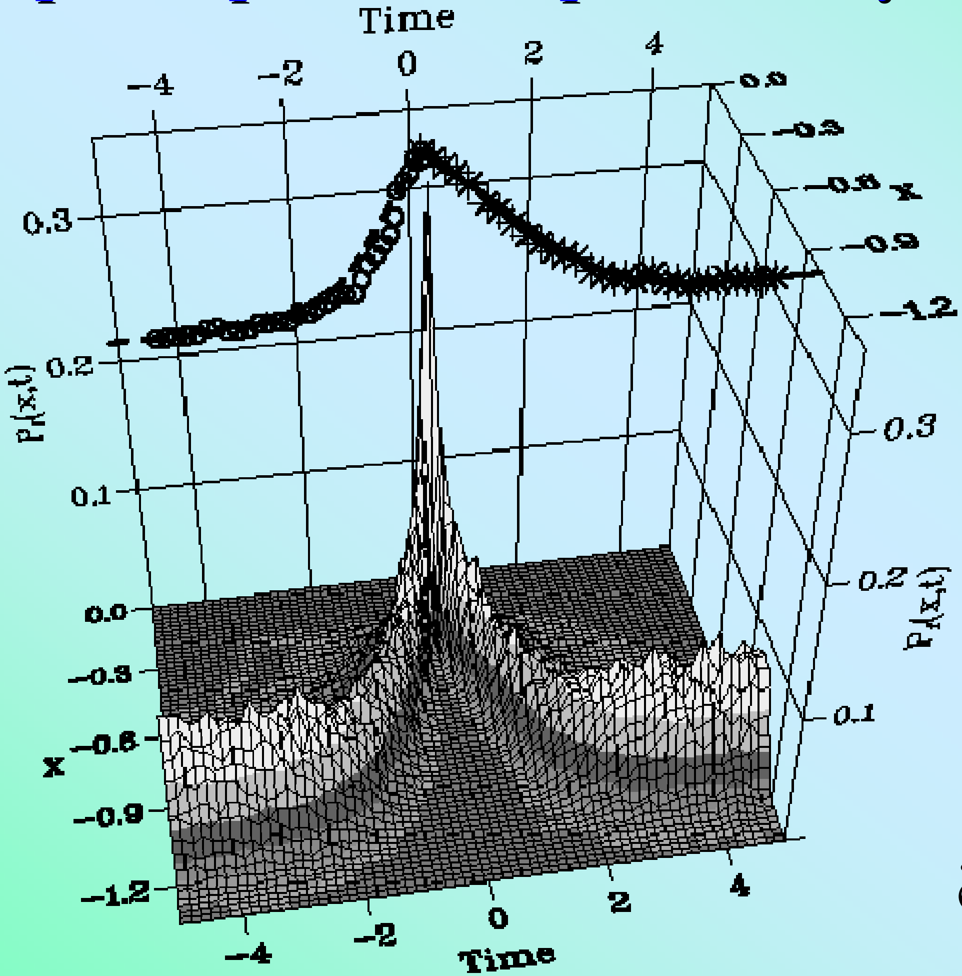
Formally the deterministic minimization problem can be formulated in the Hamiltonian form:

$$H = \frac{1}{2}\mathbf{pQp} + \mathbf{pK}(\mathbf{q},t);$$

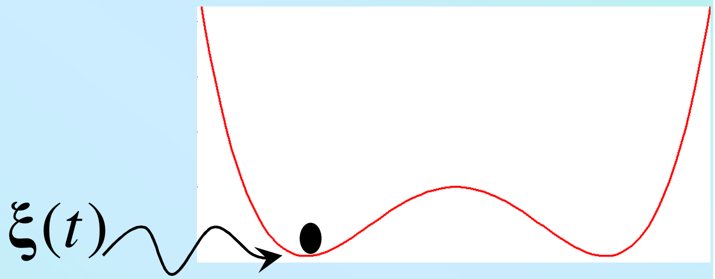
$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}, \quad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}},$$

Optimal path approach
deterministic pattern of fluctuations

Optimal paths are experimentally observable (Dykman'92)



The prehistory probability of transition between states of bistable oscillator (electronic experiment)



Optimal paths are essentially deterministic trajectories

Chaos and Noise: Quasi-hyperbolic Attractor

Lorenz system $\sigma = 10, \quad b = 8/3, \quad r = 24.08$

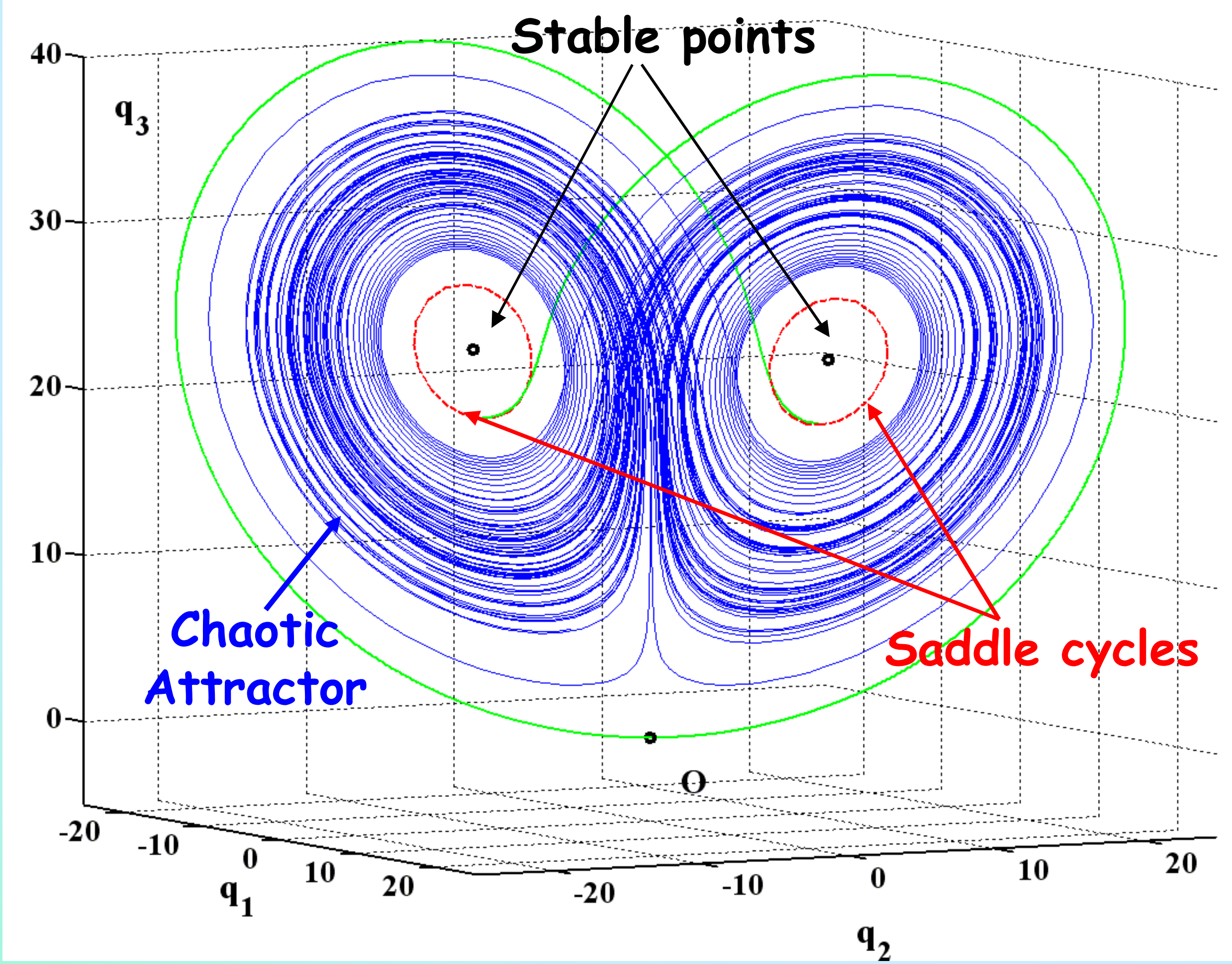
$$\dot{q}_1 = \sigma(q_2 - q_1)$$

$$\dot{q}_2 = rq_1 - q_2 - q_1 q_3$$

$$\dot{q}_3 = q_1 q_2 - bq_3 + \sqrt{2D}\,\xi(t)$$

Consider noise-induced escape from the chaotic attractor to the stable point in the limit $D \to 0$

The task is to determine the most probable (optimal) escape path

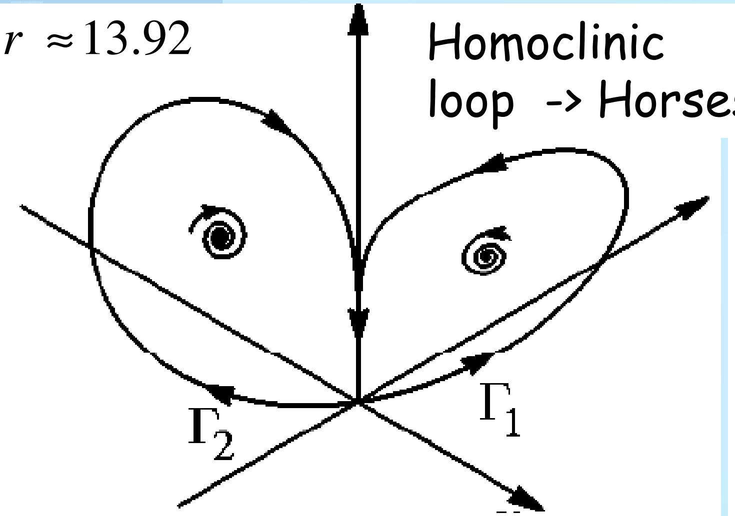


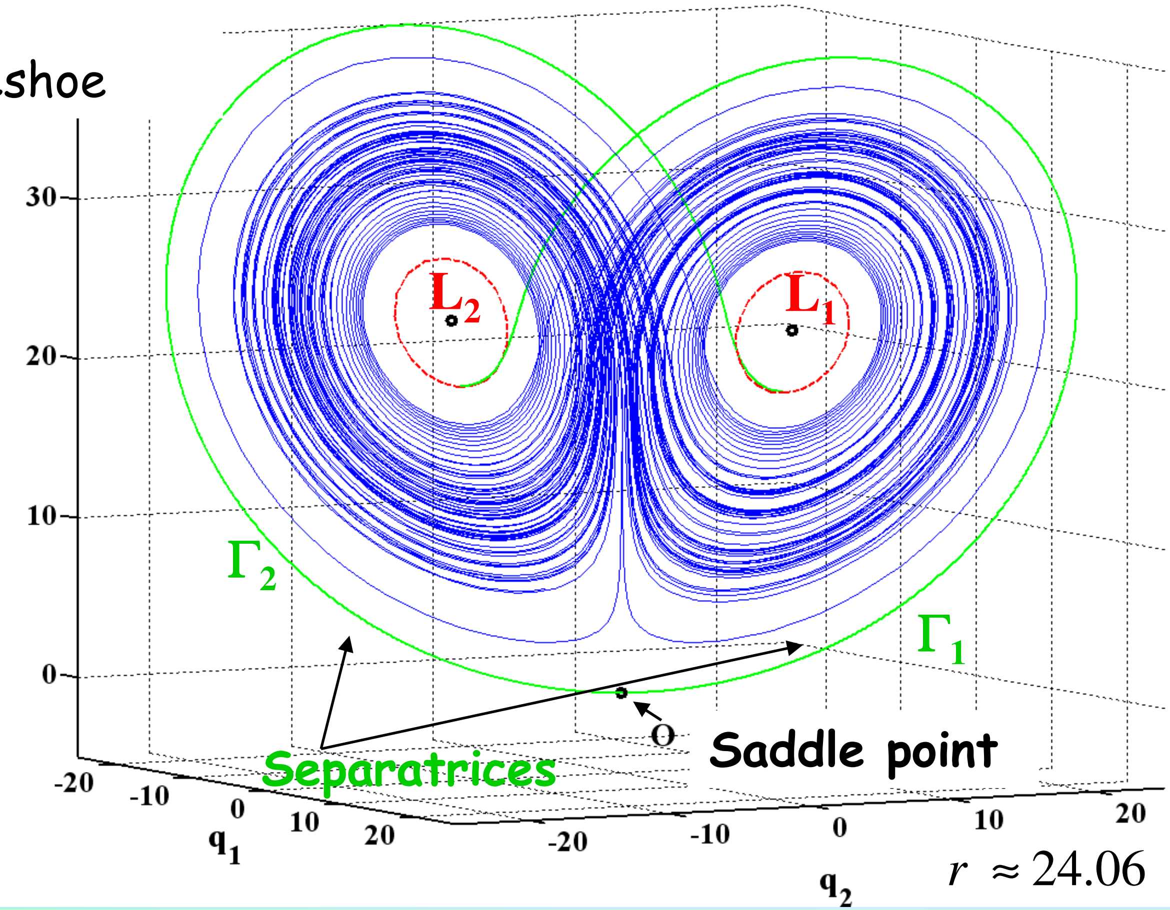
Chaos: Quasi-hyperbolic Attractor

Lorenz Attractor

$r \approx 13.92$

Homoclinic loop -> Horseshoe



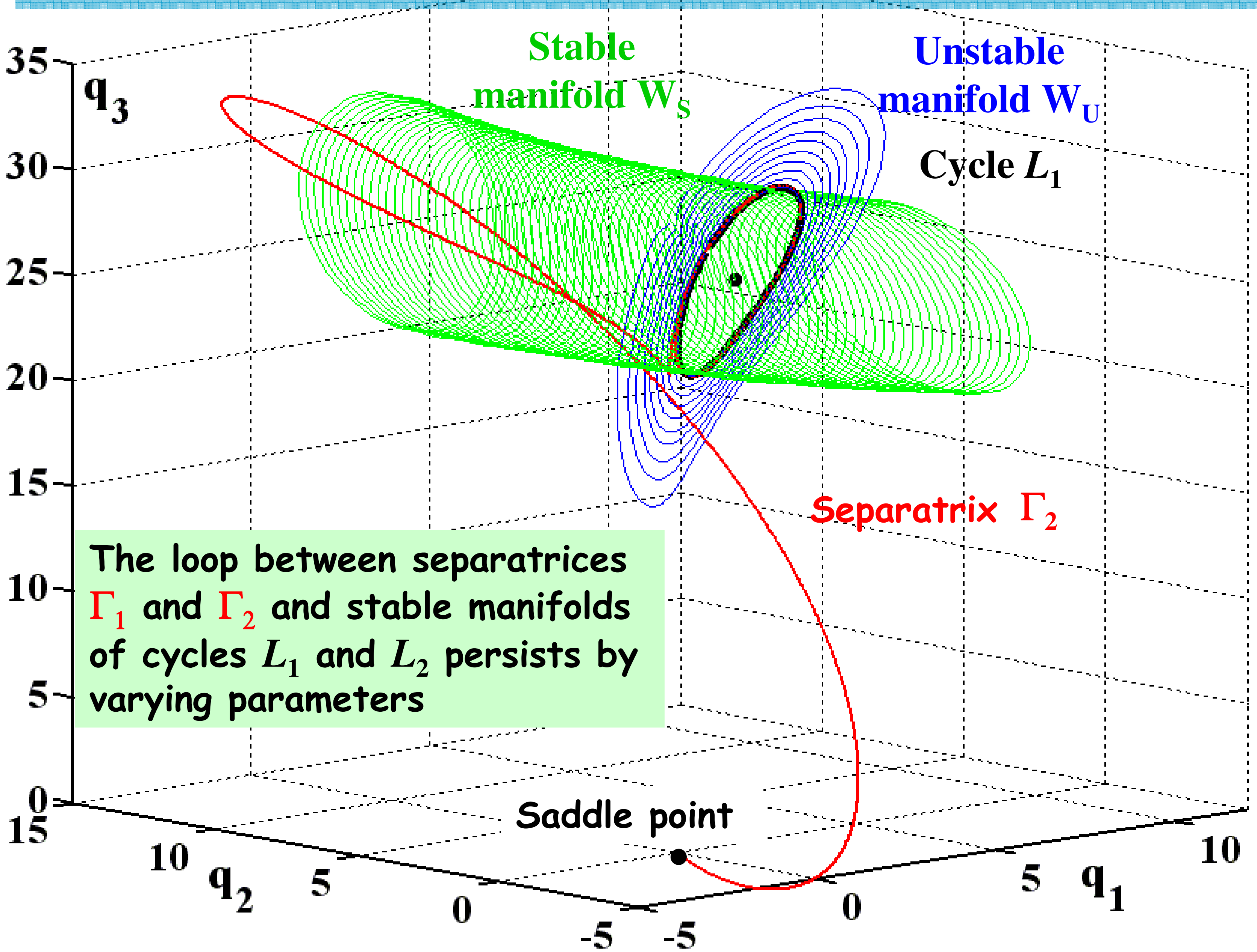


$r \approx 24.06$

The saddle point and its separatrices belong to chaotic attractor and form "bad set" or non-hyperbolic part of the attractor

Loops between separatrices Γ_1 and Γ_2 and stable manifolds of cycles L_1 and L_2 generate
The Lorenz attractor – quasi-hyperbolic attractor

Chaos: Quasi-hyperbolic Attractor



The loop between separatrices Γ_1 and Γ_2 and stable manifolds of cycles L_1 and L_2 persists by varying parameters

Large Fluctuations: the prehistory probability

The prehistory approach

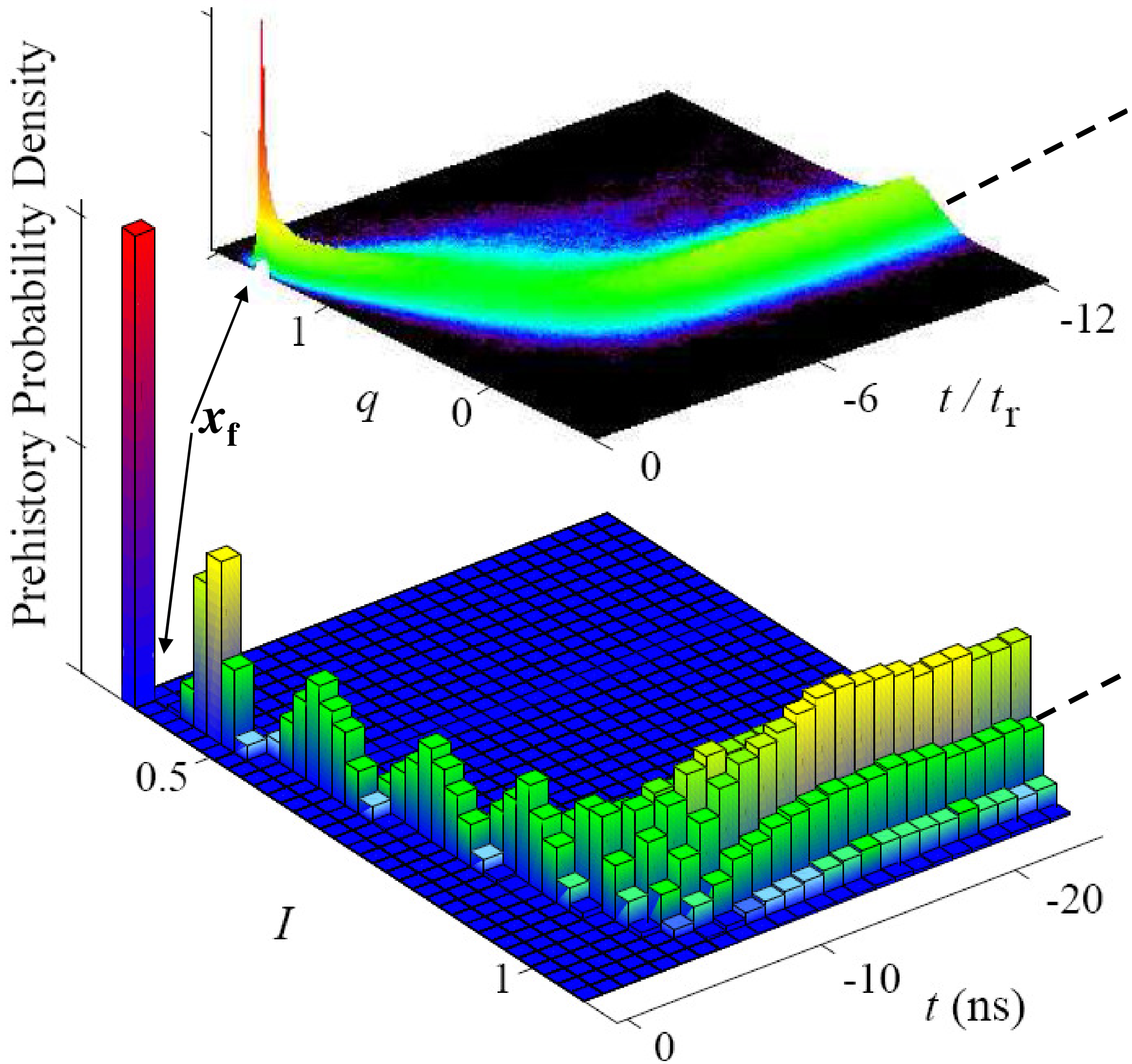
$\dot{\mathbf{x}} = \mathbf{K}(\mathbf{x},t) + \xi(t),$

$\langle \xi_\alpha \rangle = 0, \langle \xi_\alpha(t)\xi_\beta(s) \rangle = D\mathbf{Q}\delta(t-s)$

1. Select the regime $D \to 0$ i.e. rare large fluctuations

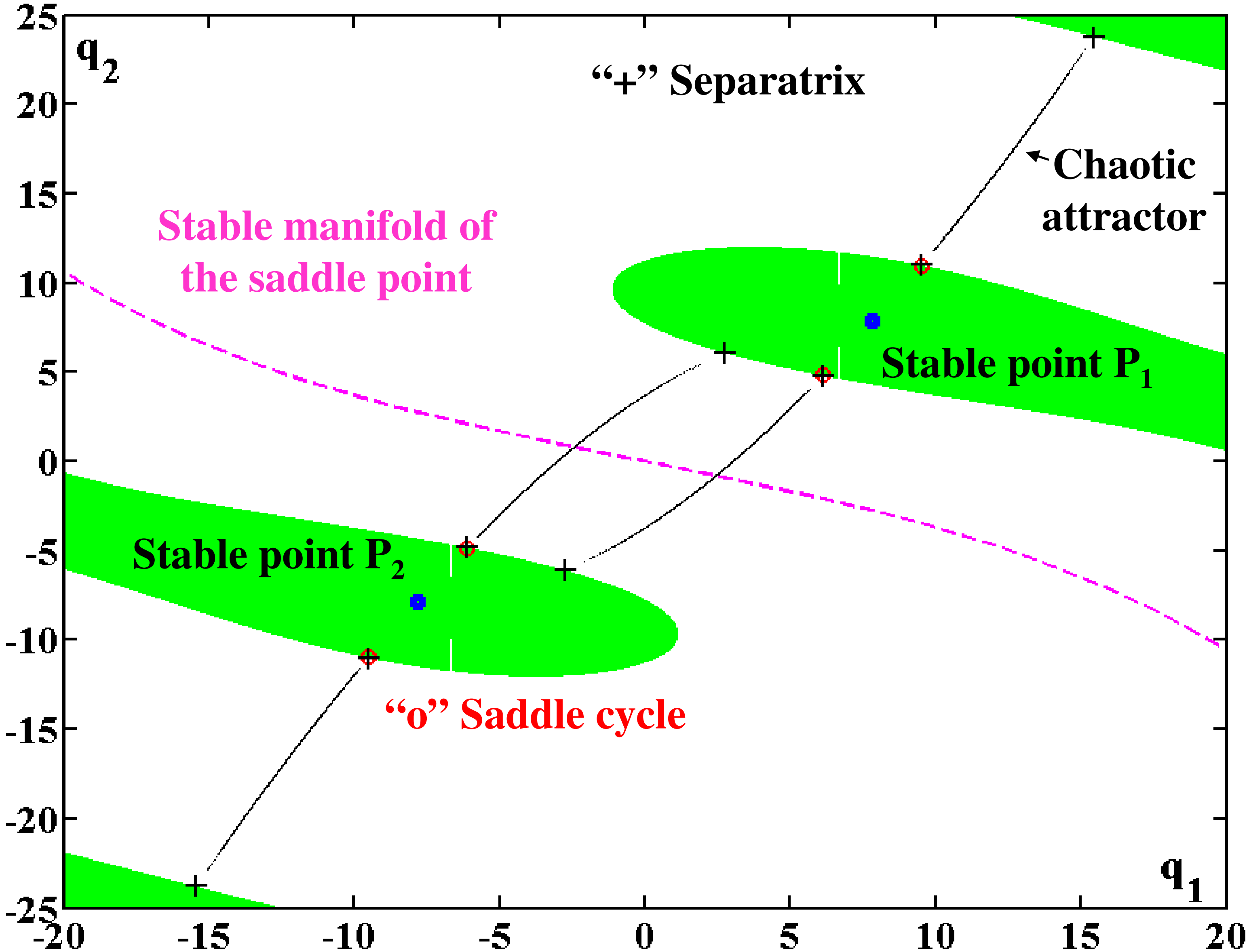
$$t_{relax} << t_{activ}$$

2. Record all trajectories $\mathbf{x}_j(t)$ arrived to the final state and build the prehistory probability density $p_h(\mathbf{x},t)$
The maximum of the density corresponds to the **most probable (optimal) path**
3. Simultaneously noise realizations $\xi(t)$ are collected and give us the **optimal fluctuational force**



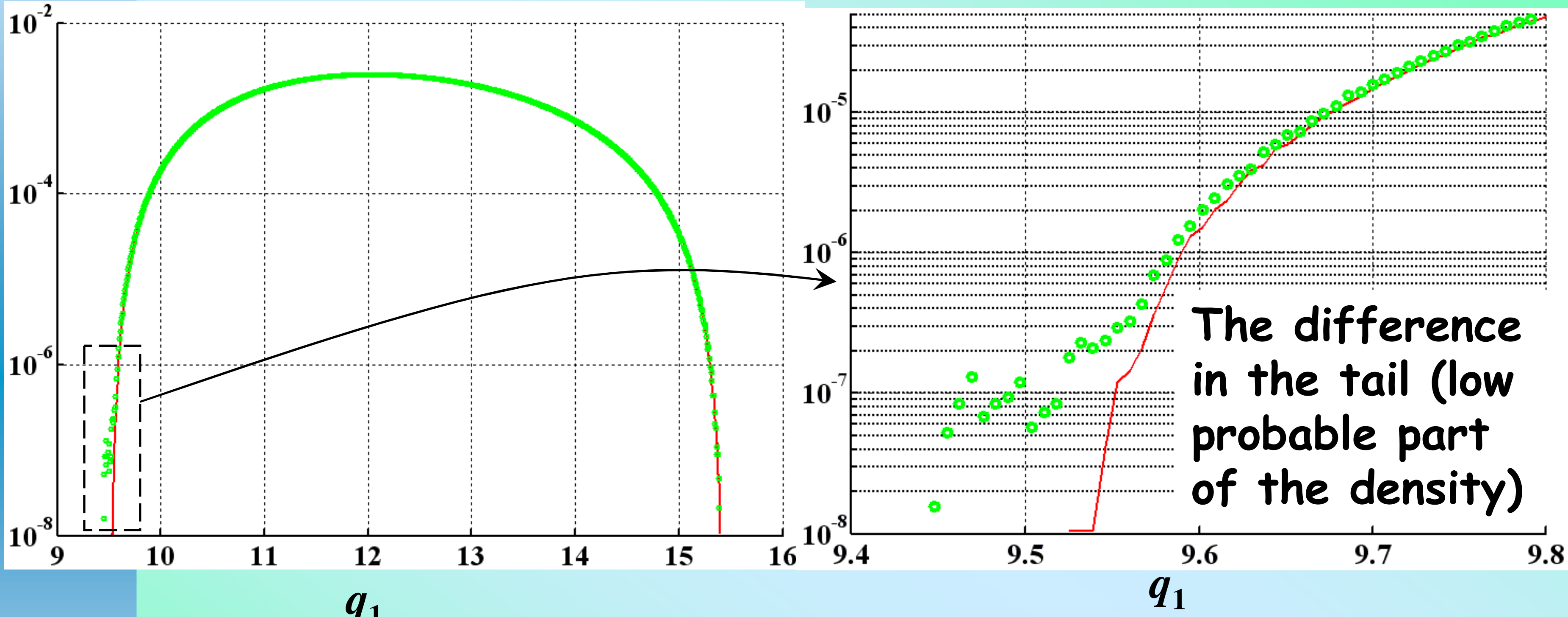
Chaos: Quasi-hyperbolic Attractor

The Poincaré section $q_3 = r - 1$



Chaos and Noise: Quasi-hyperbolic Attractor

Probability density $p(q_1)$ for Poincaré section $q_3 = r - 1$



D=0 In the absence of noise

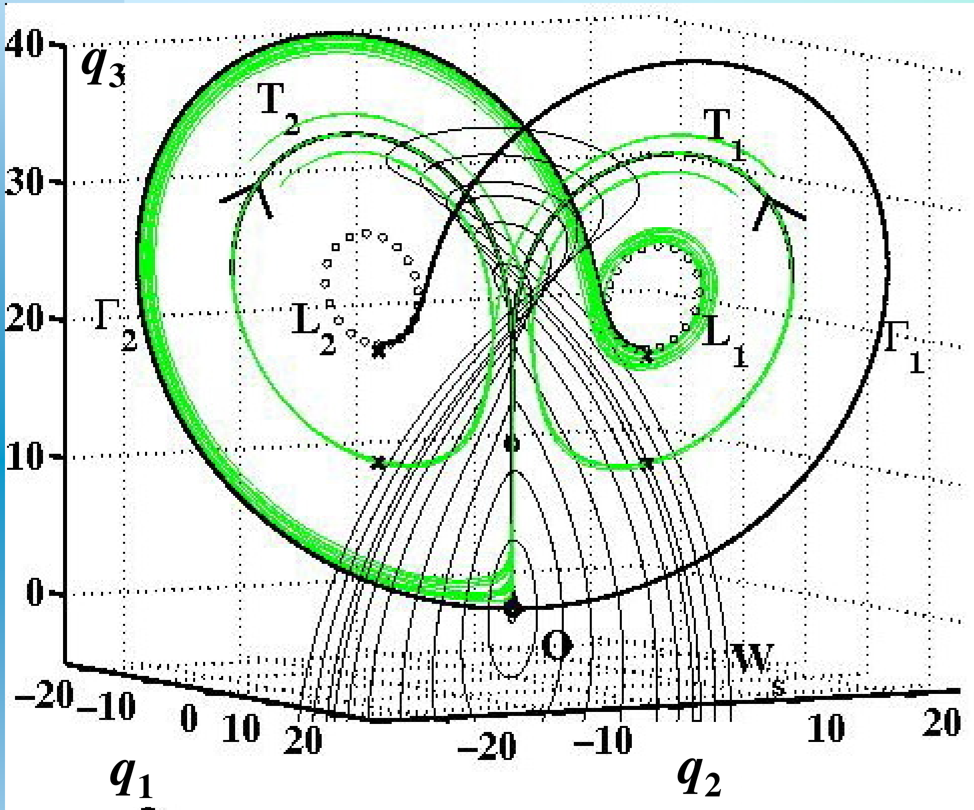
D=0.001 In the presence of noise

Noise does not change significantly the probability density

Are noise-induced tail important?

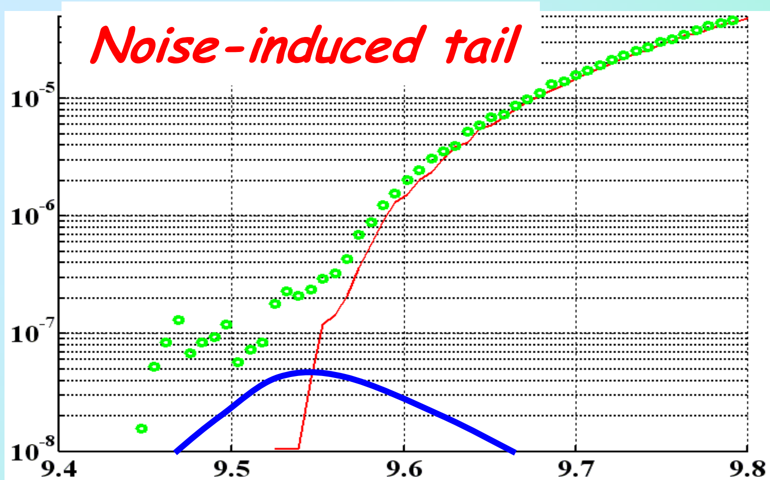
Chaos and Noise: Quasi-hyperbolic Attractor

Escape from quasi-hyperbolic attractor



Escape trajectories

$\mathbf{W_S}$ is the stable manifold and Γ_1 and Γ_2 are separatrices of the saddle point **O**
$\mathbf{L_1}$ and $\mathbf{L_2}$ are saddle cycles
$\mathbf{T_1}$ and $\mathbf{T_2}$ are trajectories which are tangent to $\mathbf{W_S}$

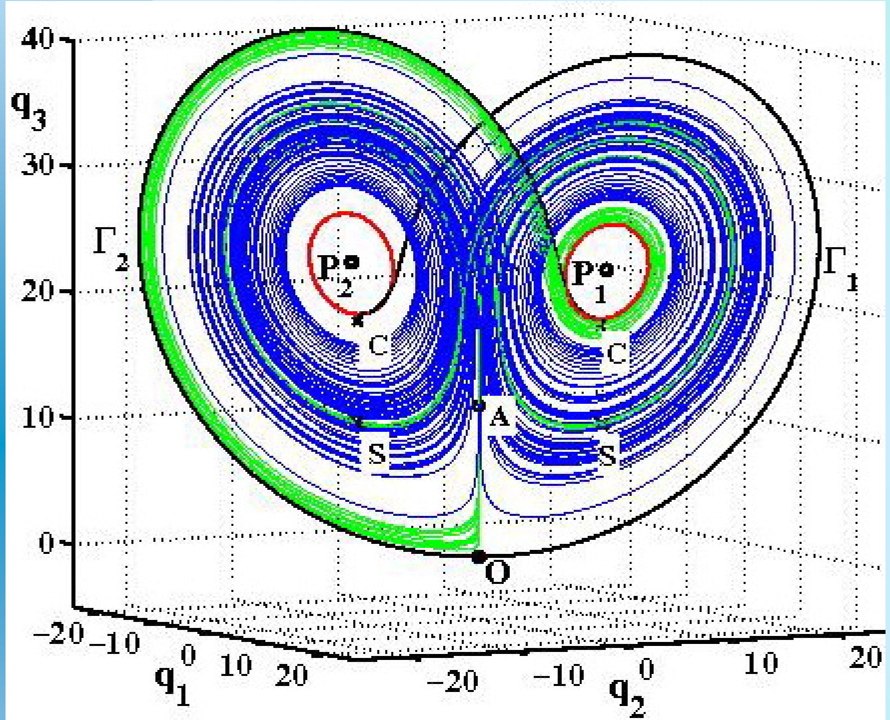


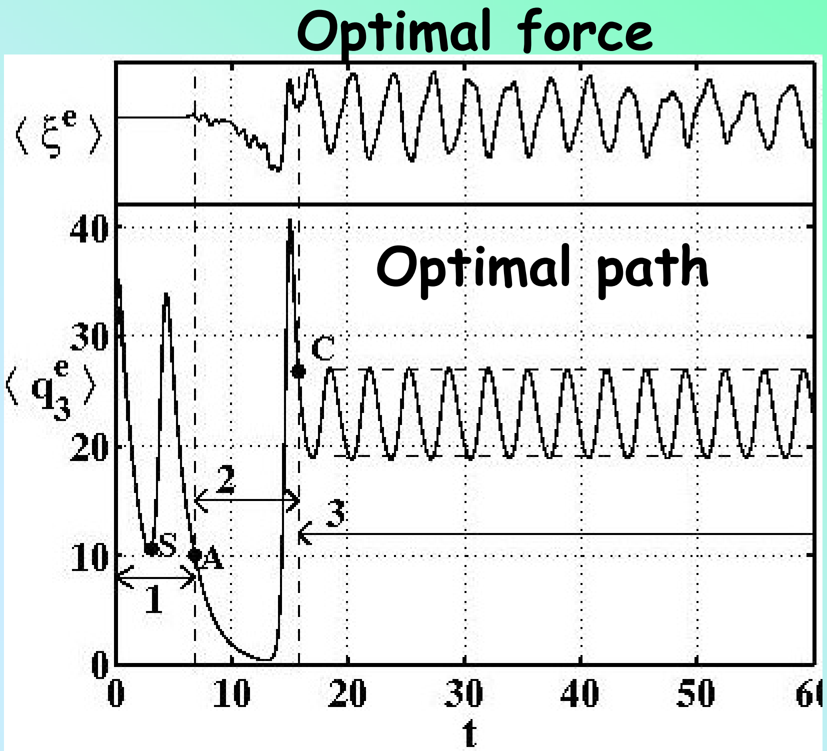
The distribution of escape trajectories (exit distribution)

The escape process is connected with the non-hyperbolic structure of attractor: stable and unstable manifolds of the saddle point

Chaos and Noise: Quasi-hyperbolic Attractor

The optimal path and fluctuational force from analysis of fluctuations prehistory





Three parts:

1) Deterministic part, the force equals to zero; The point A is the initial state $\mathbf{x_i}$

2) Noise-assisted motion along stable and unstable manifolds of the saddle point

3) Slow diffusion to overcome the deterministic drift of unstable manifold of the saddle cycle and cross the cycle

Chaos and Noise: Non-hyperbolic Attractor

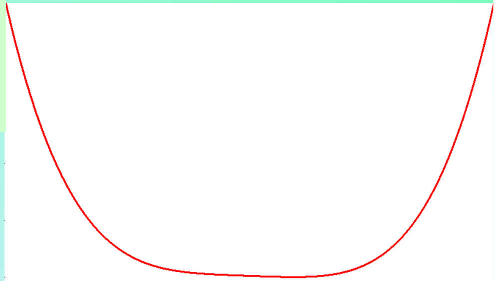
Non-autonomous nonlinear oscillator

$$\ddot{q} + \Gamma\dot{q} + \frac{\partial U(q,t)}{\partial q} = \sqrt{2D}\xi(t)$$

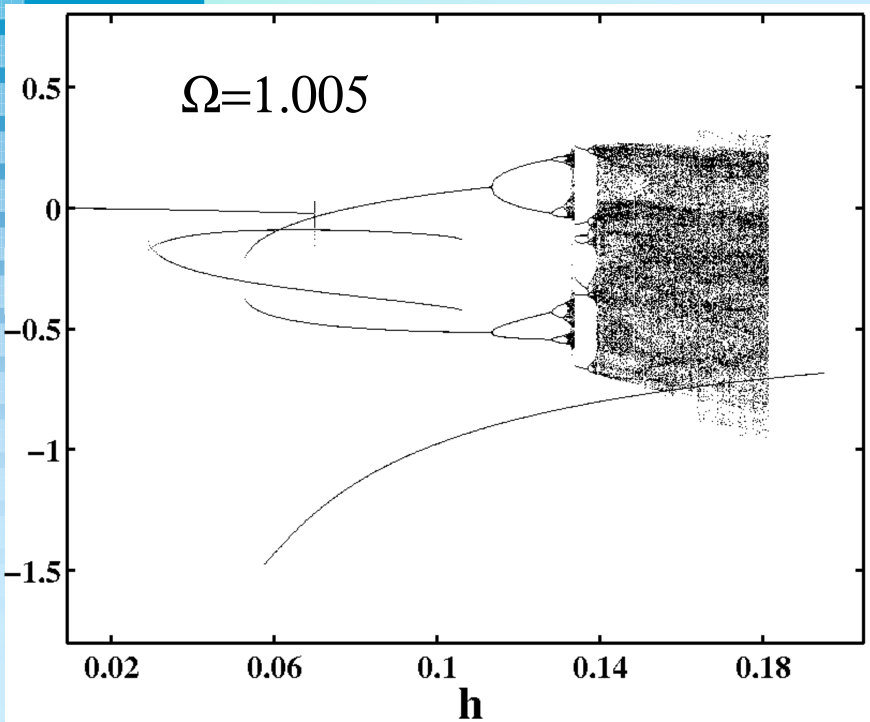
$$U(q,t) = \frac{\omega_0^2}{2}q^2 + \frac{\beta}{3}q^3 + \frac{\gamma}{4}q^4 + q\,h\sin\Omega t$$

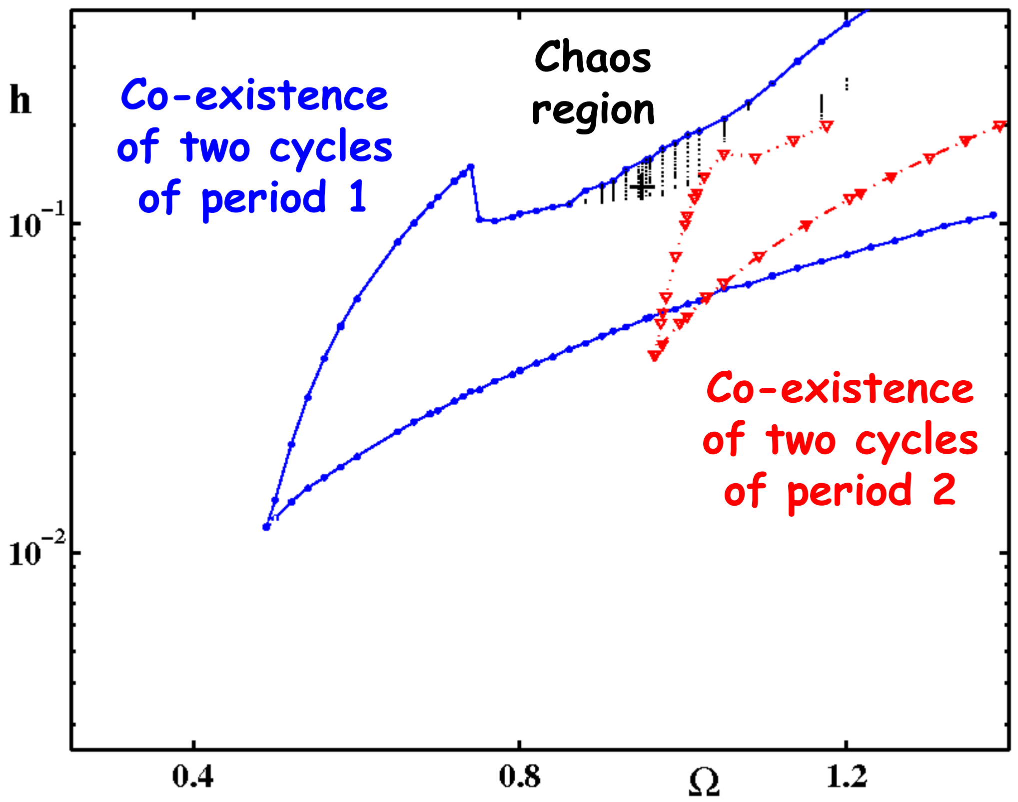
$$\Gamma = 0.05 \quad \omega_0 = 0.597 \quad \beta = \gamma = 1$$

The potential $U(q)$ is monostable



The motion is underdamped





Chaos and Noise: Non-hyperbolic Attractor

The co-existence of chaotic attractor and the limit cycle

$h = 0.13 \quad \Omega = 0.95$

Initial state: **?**

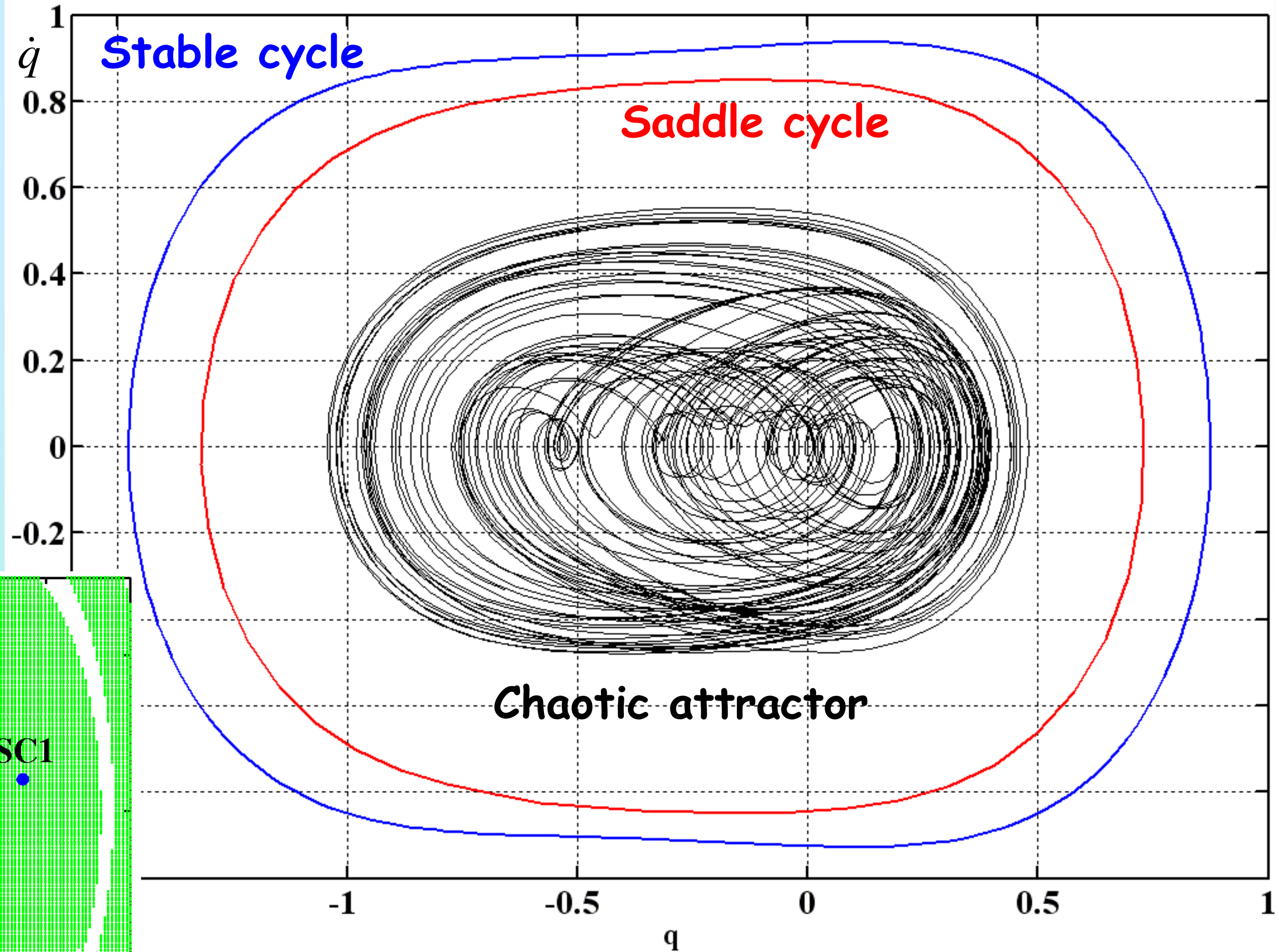
$\mathbf{q}(t_i) = \mathbf{x}_i, \mathbf{p}(t_i) = 0,$

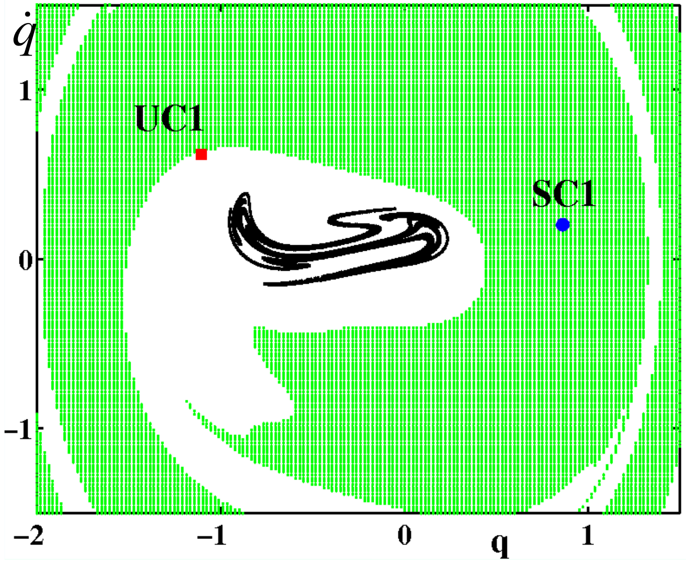
$t_i \to -\infty;$

Final state: The stable cycle

$\mathbf{q}(t_f) = \mathbf{x}_f, \mathbf{p}(t_f) = 0,$

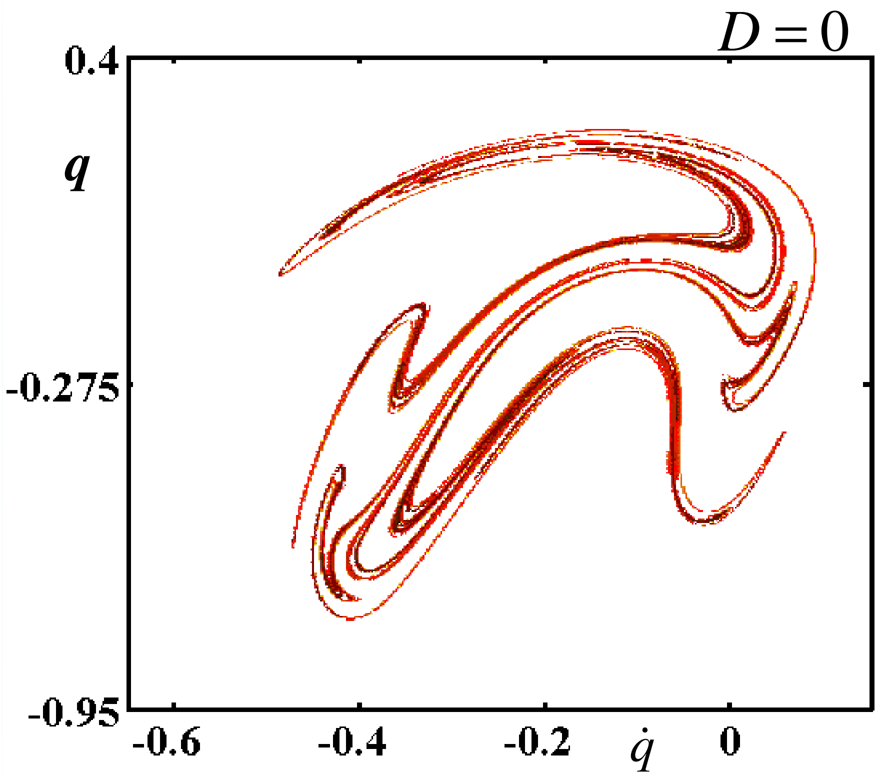
$t_f \to \infty.$

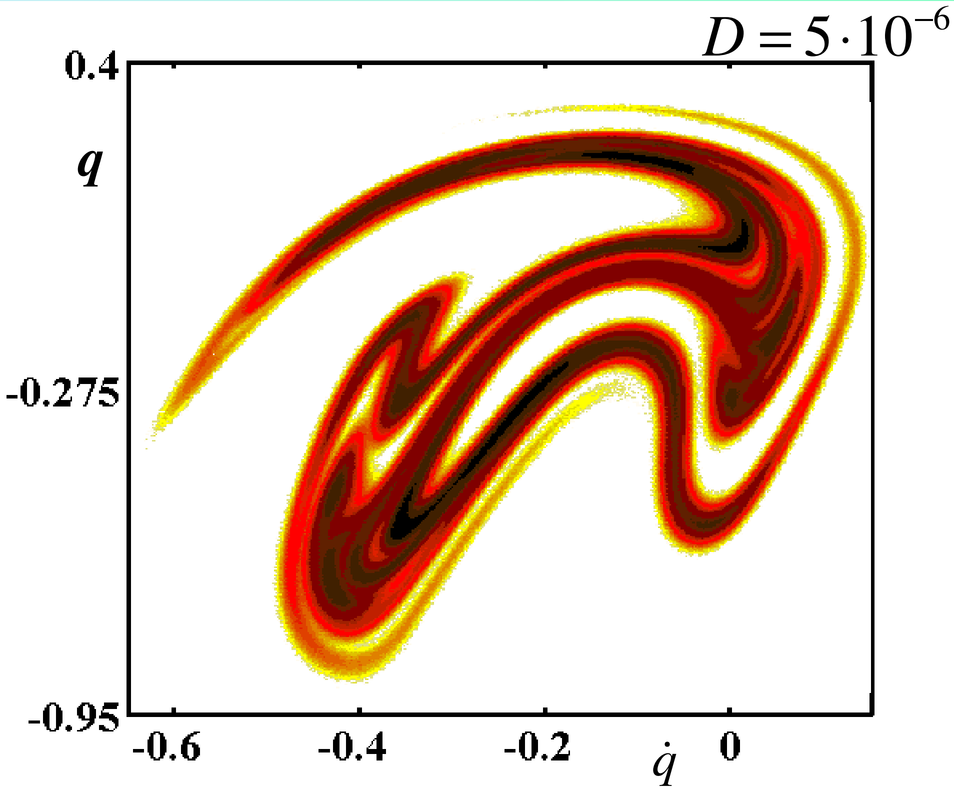




Chaos and Noise: Non-hyperbolic Attractor

Probability density $p(q,\dot{q})$ for Poincaré section $\Omega t = 0$



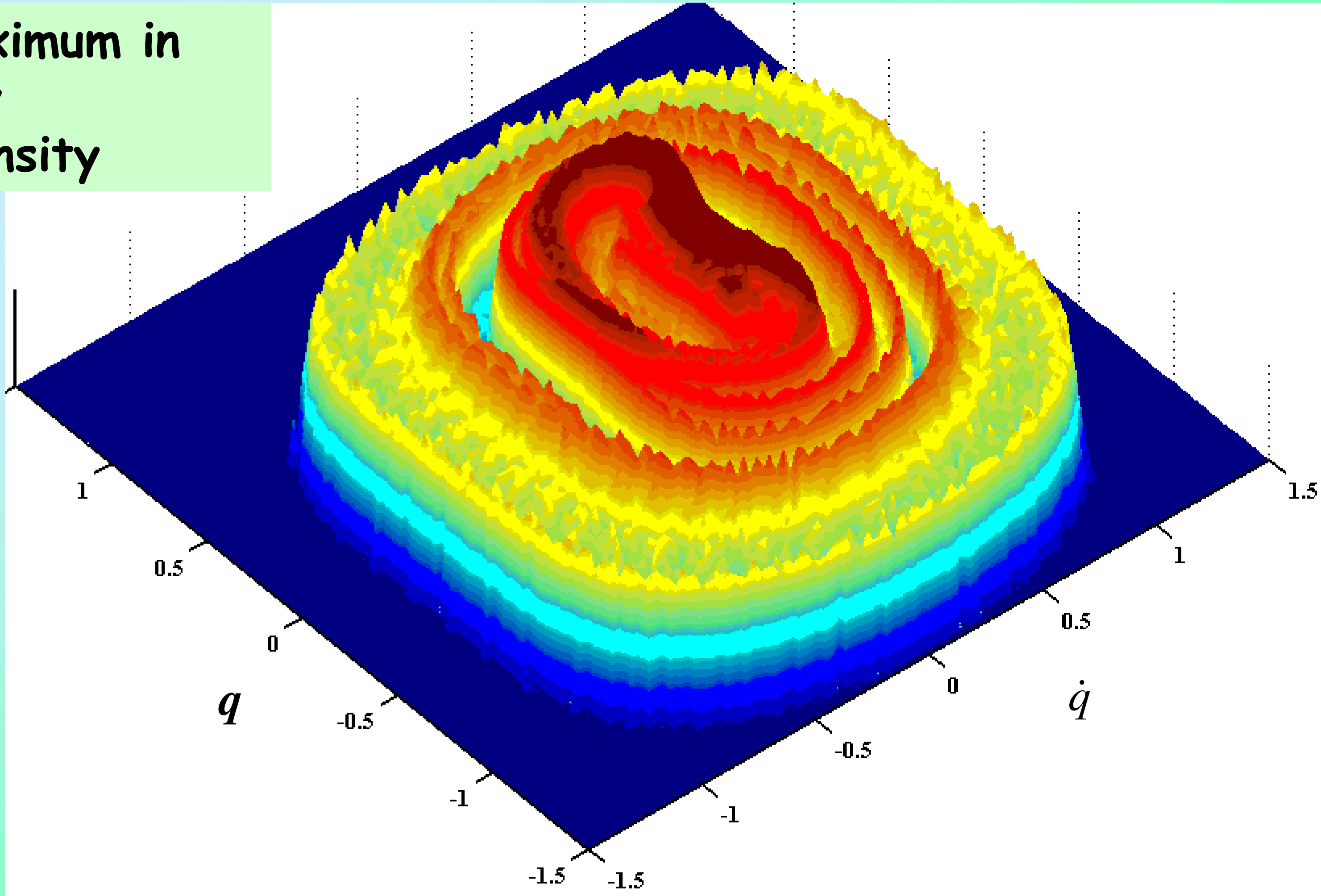


A weak noise significantly changes the probability density

Chaos and Noise: Non-hyperbolic Attractor

The prehistory probability density $p_h(q,\dot{q},t)$ $D = 5 \cdot 10^{-4}$

There is a maximum in the prehistory probability density

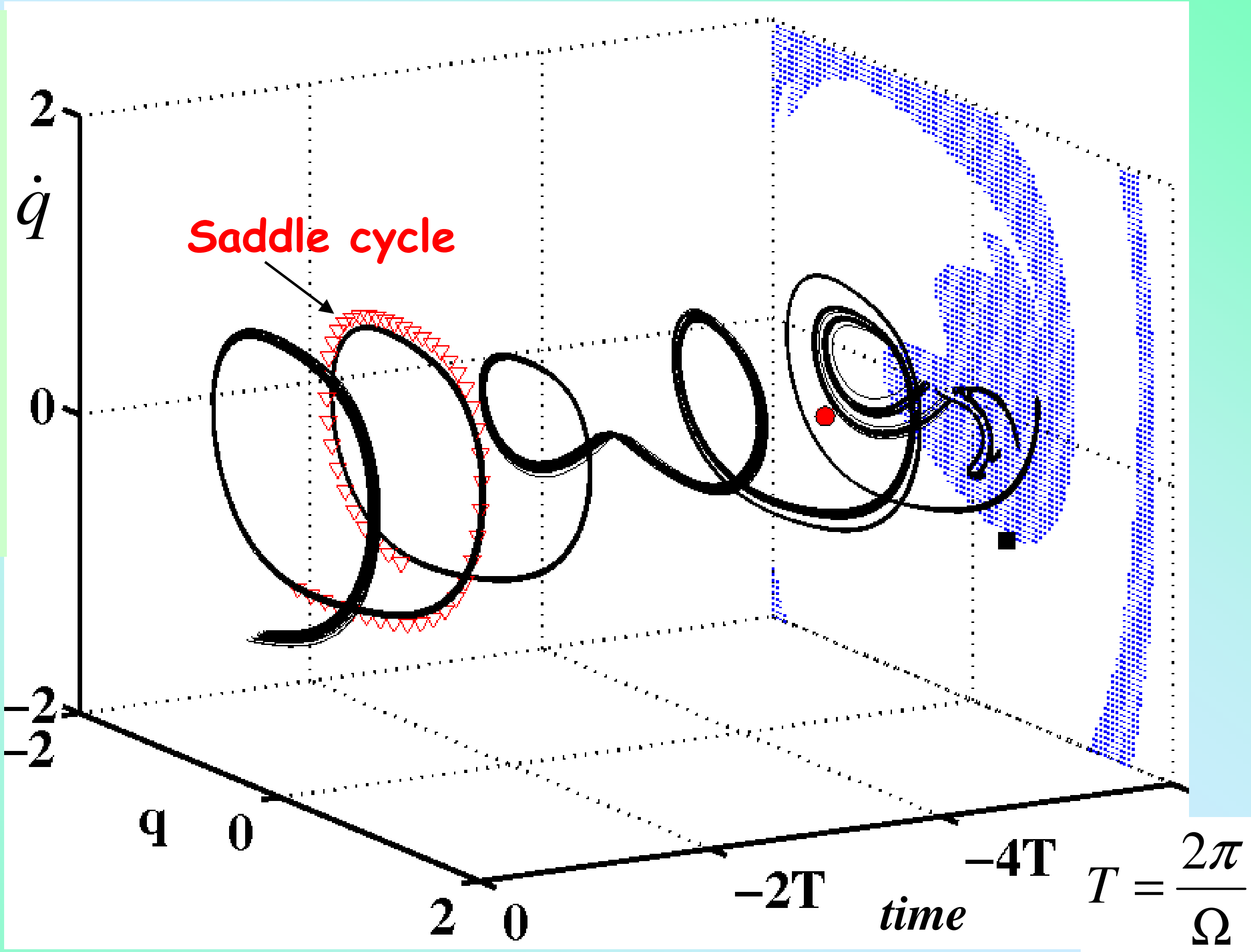


Chaos and Noise: Non-hyperbolic Attractor

Escape trajectories follow a narrow tube

Q: Do any sets form the escape path?

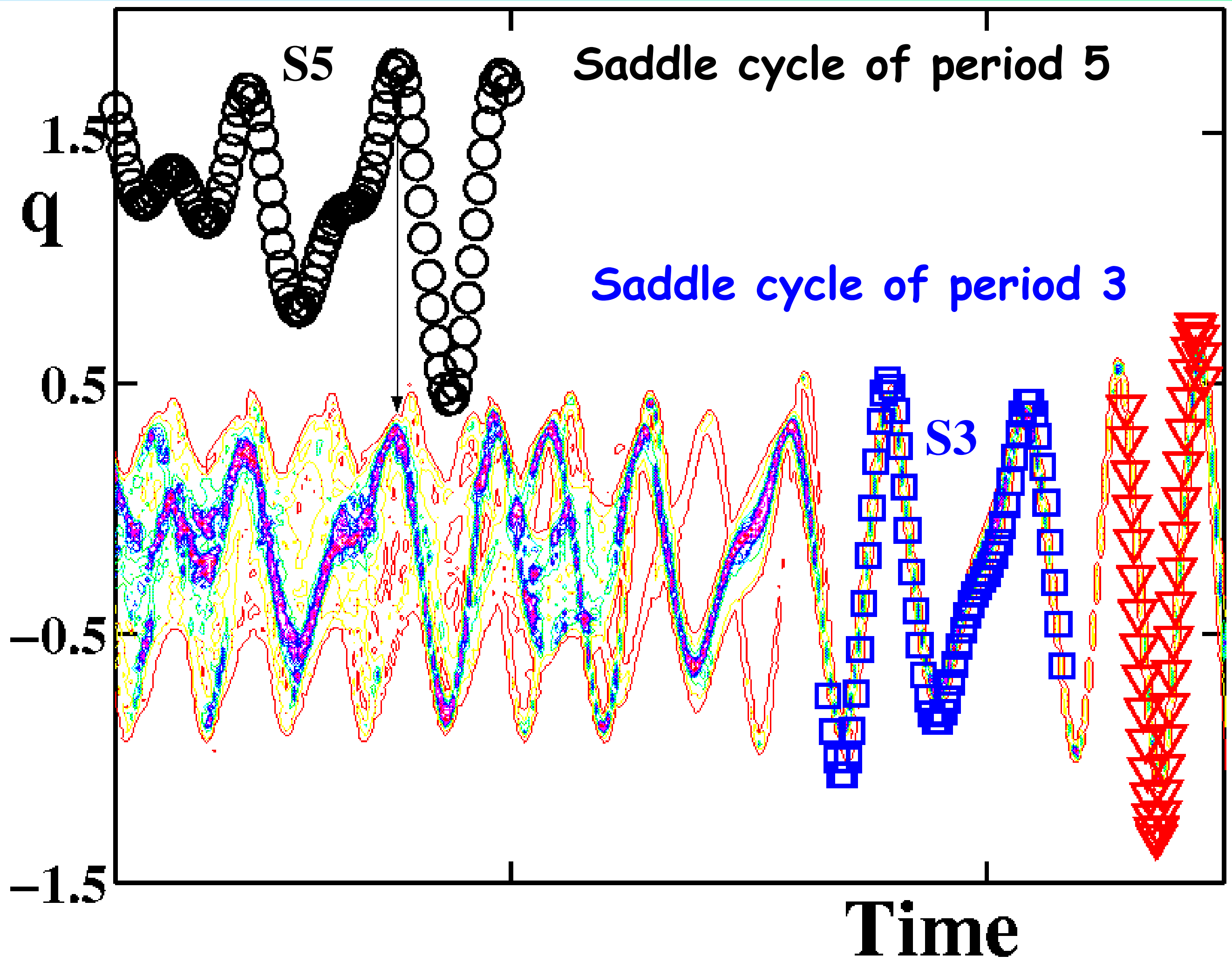
To answer we take initial conditions along the path and try to localize any sets



Chaos and Noise: Non-hyperbolic Attractor

The prehistory probability density $p_h(q,t)$

Saddle cycles form the escape path



Chaos and Noise: Non-hyperbolic Attractor

The escape is a sequence of jumps between saddle cycles.

Escape trajectory is a heteroclinic trajectories connected saddle cycles of Hamilton system.

$$H = \frac{1}{2}\mathbf{pQp} + \mathbf{pK}(\mathbf{q}, t);$$

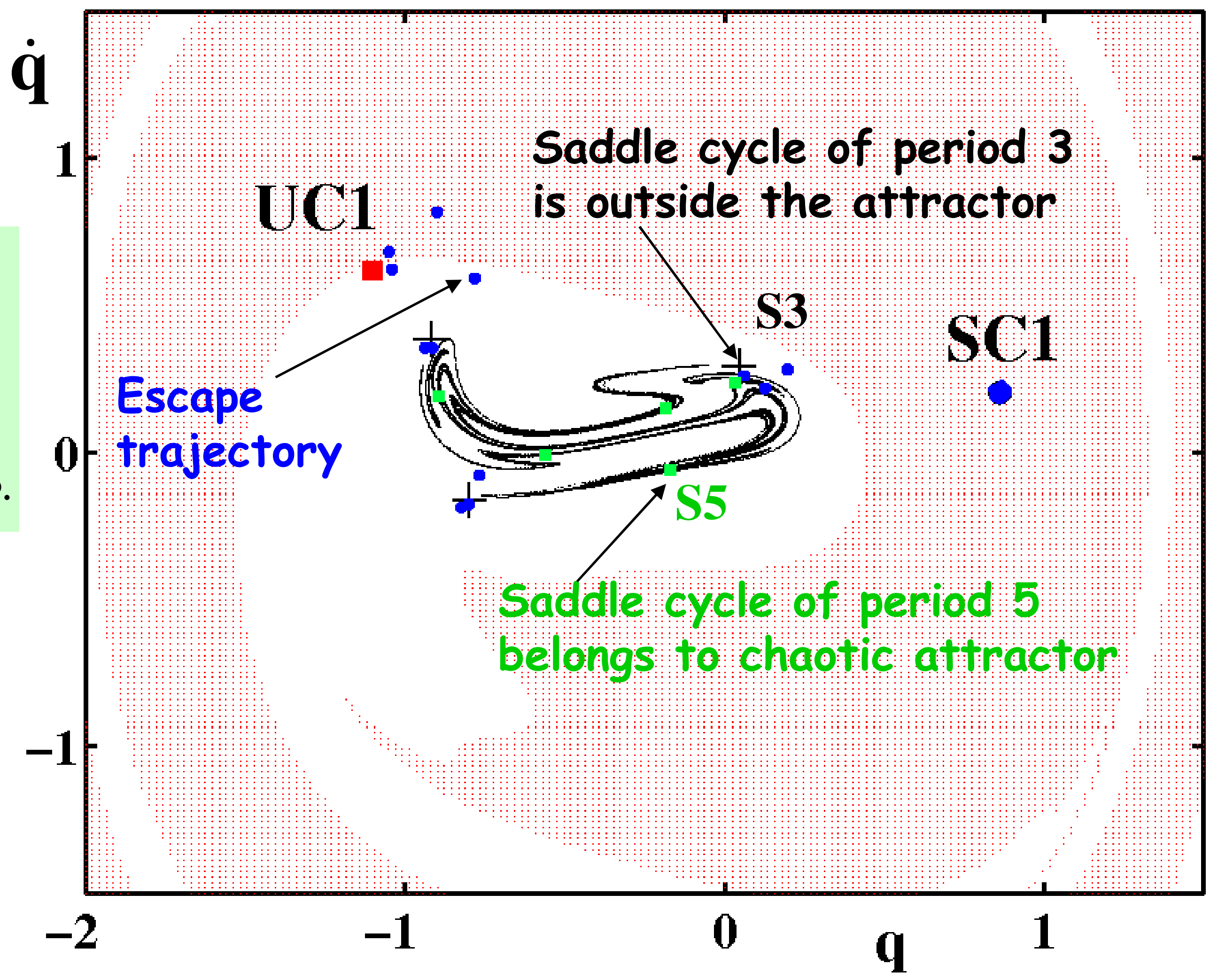
$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}, \quad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}},$$

Initial state: cycle S5

$$\mathbf{q}(t_i) = \mathbf{x}_i, \mathbf{p}(t_i) = 0, \quad t_i \to -\infty;$$

Final state: cycle UC1

$$\mathbf{q}(t_f) = \mathbf{x}_f, \mathbf{p}(t_f) = 0, \quad t_f \to \infty.$$



Large Fluctuations

Summary

For a quasi-hyperbolic attractor, its non-hyperbolic part plays an essential role in the escape process.

For a non-hyperbolic attractor, saddle cycles embedded in the attractor and basin of attraction are important. Escape from a non-hyperbolic attractor occurs in a sequence of jumps between saddle cycles.

For both types of chaotic attractor we can select specific sets which are connected with Large Fluctuations and the most probable paths.
Thus the description of large fluctuations is reduced to specification of a particular trajectory (the optimal path).