Asymmetric Flows Driven by a Rotating Solid in a Fluid Layer

being a Thesis submitted for the Degree of Doctor of Philosophy in the University of Hull

by

Petr Valerievich Denissenko
B.Sc. (Novosibirsk), M.Sc. (Novosibirsk)

May 2003
# Contents

1 Introduction ........................................ 1

1.1 Precessing non-axisymmetric structures in the axisymmetric geometry . . . 2

1.1.1 Zhukovskij’s study on the inviscid flow with a vortex source . . . 2

1.1.2 Symmetry breaking in the viscous vortex-source flow ............... 3

1.1.3 Differentially rotating fluid in a circular domain ................. 4

1.1.4 Vorticity evolution in the two-dimensional domain ................. 7

1.1.5 Taylor-Couette setup with discrete azimuthal symmetry .......... 9

1.1.6 Miscellaneous related works .................................. 9

1.2 Essentially non-axisymmetric structures in rotating flow. Basic features . . . 13

1.2.1 Propeller-driven flow in the air gap ............................ 14

1.2.2 Cylinder-driven flow in a thin liquid layer with a free surface . . 16

1.2.3 Cylinder-driven flow in a deep liquid layer and a fluid flow in a gap

between rigid boundaries ........................................ 17

1.2.4 Fluid flow in a gap with differentially rotating boundaries .... 19

1.2.5 Typical flow pattern ........................................ 20

2 Experimental variations ................................ 22

2.1 Studies on the jet precession rate ............................ 22
2.1.1 Jet precession rate versus non-dimensional parameters .......... 23
2.1.2 Jet precession rate versus gap diameter ........................... 24
2.1.3 Jet precession rate and the amplitude of velocity perturbation .... 26
2.1.4 Precession of the pattern in a differentially rotating gap .......... 27
2.1.5 Setup with the jet precessing in negative direction ................ 27
2.2 PIV measurements for the stationary axisymmetric flow .......... 28
2.3 Free surface flow with two precessing jets. PIV measurement #168 . . 31
2.4 Miscellaneous experimental observations ............................. 36
2.5 Setups, in which the jet-like structures do not appear ............... 36
2.5.1 Propeller-driven air gap with the sealed outer edge ............... 36
2.5.2 Setup with the cylinder filling the whole air gap height ........... 37
2.6 Miscellaneous experiments where precessing structures arise ......... 38
2.6.1 Circular shear layer with the faster rotating fluid located further from the axis ......................................................... 38
2.6.2 Non-axisymmetric structure in the rarified granular flow ......... 40
2.6.3 Source flow in a circular gap ......................................... 40

3 Models ............................................................................. 42
3.1 Flow in the air gap at large Reynolds numbers. Darcy friction .... 43
3.1.1 Basic equations .......................................................... 44
3.1.2 Axisymmetric stationary solution .................................... 45
3.1.3 Linearized problem ..................................................... 48
3.1.4 Linear shooting procedure ............................................ 50
3.1.5 Numerical realization .................................................. 51
3.1.6 Neutral curves and parameters of some neutral modes . . . . . . . . 56

3.2 Layers of constant vorticity in the long wave approximation . . . . . . 61
3.2.1 Basic equations . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 62
3.2.2 Asymptotic model . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 63
3.2.3 Main assumptions and relations . . . . . . . . . . . . . . . . . . . . . . . 64
3.2.4 Shallow water equations . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 65
3.2.5 Azimuthally propagating discontinuities . . . . . . . . . . . . . . . 69

3.3 Flow of ideal fluid, initiated by a bulk of decaying vortices . . . . . 73

3.4 Axisymmetric flow in a narrow gap at moderate Reynolds numbers . . 75

4 Conclusions 81
4.1 Two-dimensional perturbations of the unstable velocity profile . . . . 81
4.2 Not yet answered questions . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 87
4.3 Prospective theoretical approaches . . . . . . . . . . . . . . . . . . . . . . . 87
4.4 Prospective experimental arrangements . . . . . . . . . . . . . . . . . . . . 89

Bibliography 92

Figures 99

Tables 174
Acknowledgements

This thesis is the result of experimental and theoretical research on the essentially non-axisymmetric flow patterns arising in the axisymmetric thin fluid layers, started in September 1997 at the Hong Kong University of Science and Technology and continued at the University of Hull from January 2001. I thank these universities for supporting the work.

I am grateful to my supervisor, Professor V.A. Vladimirov, for introducing me to this new research direction and for his encouragement during the work.

I thank Professor M.Yu. Zhukov for his essential help in dealing with mathematical models. I also thank Professor V.I. Yudovich and Professor A.I. Shnirerman for their attention to the work.

For their support and many useful conversations, I am grateful to my colleagues Dr. S.V. Kudriakov, I.S. Moskalev, Dr. K.I. Ilin, Dr. S.N. Lukaschuk, Prof. M.T. Montgomery and my wife and colleague S.G. Zakhidova.

Finally, I want to thank my father Professor V.V. Denissenko for initiating my interest in the study of science.
Abstract

This work is devoted to the essentially non-axisymmetric flows induced by a rotating solid located in the centre of a thin axisymmetric fluid layer. In spite of the axial geometry of the setup, the non-axisymmetric flow patterns appear when the solid rotates fast enough. The characteristic feature of the flow is the presence of azimuthally propagating radial jets on the background of slow motion of the remaining mass of fluid. Characteristic parameters of the appearing jet-like structures can differ significantly from parameters of the experimental setup: the length of the jets can be up to 10 times larger than the radius of the rotating solid, up to 10 times larger than the jet width and up to 100 times larger than the fluid layer thickness. The period of the non-axisymmetric structure revolution exceeds the period of the driving solid revolution by up to 1000 times. The tendency to concentration of the flow and formation of the jets is strikingly robust against the changes of experimental conditions. The jet-like structures appear both in layers confined between two rigid boundaries and in layers with a free surface. The flow can be driven by an axisymmetric cylinder, by a propeller possessing discrete angular symmetry or even by a solid of irregular shape. In one series of experiments, the jets look well ordered and rather laminar, while in the others turbulent jet-like structures emerge on the laminar background. In this work, interpretation of observed non-axisymmetric flow pattern is given, an extensive experimental study on the flow is presented and three models describing the non-axisymmetric patterns are suggested. Precession rate of the
modelled non-axisymmetric structures is compared with precession rate of the structures observed experimentally.

The first chapter contains a brief review of related effects observed by other authors, both experimentally and in theory. Next, our experimental observations are described and the basic features of observed non-axisymmetric structures are listed. The second chapter is devoted to an extensive experimental study of the jet-like structures appearing in flows with a wide range of Reynolds numbers and in a variety of axisymmetric setups. Velocity measurements with the use of Particle Image Velocimetry are presented together with studies on the flow pattern precession rate and a number of miscellaneous observations. In the third chapter, three two-dimensional models leading to formation of the azimuthally propagating structures are developed. The first model contains a friction term in Darcy form, i.e. the friction term proportional to the fluid velocity. The azimuthal force imposed at a certain radius forms the basic axisymmetric flow. Linear stability of the basic flow and properties of the neutrally stable perturbations are studied. The second model describes the discontinuity propagation in the system of vortex layers in circular geometry. A hyperbolic system of equations for evolution of the vortex layers is constructed. Stable discontinuities and their properties are studied. The third model leads to the non-axisymmetric pattern formation while numerically solving a system of ordinary differential equations which describe the evolution of a system of decaying vortices in the ideal fluid. Finally, an asymptotic model describing the three-dimensional axisymmetric flow for moderate Reynolds numbers is presented. The fourth chapter contains general conclusions and proposals for further studies.
Nomenclature

Axisymmetric = possessing rotational symmetry

Non-axisymmetric = lacking rotational symmetry

\(a\)  inner radius of the domain, radius of the rotating cylinder if it fills the whole gap height

\(b\)  outer radius of the domain

\(A, B\)  the inner and the outer radii of the vortex layer in Section 3.2

\(c\)  propeller half-span, radius at which the external force is applied in Section 3.1

\(h\)  height of the rotating propeller in axial direction

\(H\)  height of the gap in axial direction

\(m\)  angular order of the non-axisymmetric structure

\(Q\)  the total azimuthal fluid flux in Section 3.2

\(r\)  radial coordinate

\(Re = \Omega c H \nu^{-1}\), Reynolds number, based on the maximum speed of rotating solid and the gap height in the three-dimensional case

\(Re = V_{0max} c \nu^{-1}\), Reynolds number, based on the maximum fluid speed in the basic flow and the radius of forcing \(c\) in the two-dimensional model in Section 3.1

\(t\)  time

\(u, v, w\)  radial, azimuthal and axial velocity components

\(V, V_0\)  azimuthal velocity, when independent of azimuthal coordinate

\(\alpha = A^2\)  in Section 3.2
\( \beta = B^2 \) in Section 3.2

\( \gamma = \frac{m\Phi}{\nu \tau} \) in Section 3.1

\( \Gamma \) strength of a vortex in Section 3.3

\( \varepsilon \) friction coefficient in Section 3.1

\( \varphi \) azimuthal coordinate

\( \phi \) stretched azimuthal coordinate in Section 3.2

\( \Phi \) the amplitude of the driving force in Section 3.1

\( \psi, \Psi \) stream function

\( \nu \) Newtonian viscosity

\( \sigma \) coefficient of exponential growth in \( e^{\sigma t} \) in Section 3.1

\( \tau \) stretched time in Section 3.2

\( \omega \) angular velocity of the jet(s) precession in experiments,
  angular velocity of the perturbation precession \( \omega = \text{Im}(\sigma)/m \) in Section 3.1,
  speed of the discontinuity propagation in azimuthal direction in Section 3.2

\( \Omega \) angular velocity of rotating solid in experiments,
  maximum angular velocity of the fluid in the basic flow \( \Omega = V_{0\text{max}}/c \) in Section 3.1

\( \Omega_0 \) constant vorticity within the vortex layer in Section 3.2

\( \Omega_{\text{in}} \) angular velocity of the inner boundary rotation in Section 3.1
Chapter 1

Introduction

Essentially non-axisymmetric (lacking rotational symmetry) flow patterns often appear in the fluid embedded into the axisymmetric domain with the axisymmetric boundary conditions. By the essentially non-axisymmetric flow we mean the flow, in which the fluid velocity deviation from the axisymmetric is of the same order as the velocity itself in a significant part of the flow domain. A circular domain whose size in axial direction is much smaller than its radius is considered. A rotating solid is placed to the centre of the domain (Fig.1). The solid is either rotationally symmetric or rotates fast enough to enable its influence on the flow to be considered rotationally symmetric. The well-defined non-axisymmetric structures, travelling in azimuthal direction, are observed when the solid rotation speed exceeds a certain value (Figs.9-22).

In this chapter, the general information about the phenomenon we deal with is presented. The first section contains a brief review of works by different authors, devoted to the non-axisymmetric flow structures appearing in axisymmetric setups. In the second section, distinguishing features of the flow regimes we are interested in are listed. Then, the four major classes of experiments, in which the essentially non-axisymmetric flow patterns appear in the axisymmetric setups, are described. To highlight the relevance of
subjects discussed to what is observed in reality, references to the experimental figures that have not yet been described are used throughout the text.

1.1 Precessing non-axisymmetric structures in the axisymmetric geometry

The appearance of the non-axisymmetric structures in circular geometry is studied by a number of authors. Commonly, the three-dimensional flows in a narrow gap (or thin layer) are considered to be essentially two-dimensional with the extra-terms introduced to account for the influence of the gap walls (or the layer bottom and the free surface). A number of experimental setups is constructed to study the quasi-two-dimensional flows. This work follows a series of experiments which were started by V.A. Vladimirov in 1990th and resulted in the report by Vladimirov and Pedley (1993). The authors observed precessing jet-like structures in experiments similar to ones performed by Zhukovskij.

1.1.1 Zhukovskij’s study on the inviscid flow with a vortex source

The existence of well-defined non-axisymmetric structures in turbulent rotating flows in a gap between two parallel plates was first reported by Zhukovskij (Zhukovsky, Joukovskii) (1914). The aim of Zhukovskij’s experiment was to model a source of fluid with a given non-zero vorticity. His idea was that stationary axisymmetric two-dimensional flow of ideal fluid with a non-zero radial flux and a constant non-zero vorticity cannot exist, because it contradicts conservation of the velocity circulation around expanding liquid contours. Zhukovskij showed that, due to the angular momentum conservation, velocity circulation is the same along any two circular contours with different radii, which is in contradiction to the presence of fluid with the non-zero vorticity between these radii. Thus,
some kind of non-stationary or non-axisymmetric flow is to be observed. In Zhukovskij’s experiment, air was injected to the centre of the gap along the axis of the rotating propeller, which was designed as a right-angled cross with four rectangular vertical blades (Fig. 2). To visualize the flow, a number of small flags was suspended in the gap at several locations. Zhukovskij discovered that the flow is non-axisymmetric and unsteady. Interpreting observations, he suggested that the main element of non-axisymmetric flow structure represents a rotating (precessing) jet. At the time Zhukovskij performed these experiments (1911), no technique to measure the velocity field was available, so the angular precession rate of the jet has been measured. To explain the flow observed, Zhukovskij proposed a simple analytical solution: a plane inviscid incompressible fluid flow with velocity discontinuities. The flow consists of two areas separated by the vortex line. One area contains potential flow and the other area contains the flow with constant non-zero vorticity. Kinematic and dynamic matching conditions (absence of normal to the boundary velocity component and absence of the pressure discontinuity across the boundary) are satisfied.

If discussing relevance of Zhukovskij’s description to the general case of the essentially non-axisymmetric flows observed in narrow gaps, disadvantages of the model suggested are the absence of precession of the jet, presence of the non-zero total radial fluid flux and the absence of account for viscosity.

1.1.2 Symmetry breaking in the viscous vortex-source flow

Flows that can be related to our case are described by Goldshtik, Shtern and Yavorski (1989) and Goldshtik, Hussan and Shtern (1991). They studied the 2-dimensional vortex
source flow and found out that it bifurcates into a countable set of the angular-dependent steady solutions. It is shown that the solutions with zero radial flow rates can only have 1 or 2 azimuthal oscillations. The soliton-like (in azimuthal direction) flow is described: the spiral structure with \( m \) jet-like branches is precessing in azimuthal direction. Azimuthal coordinate of the maximum of radial velocity, in accordance with Goldshtik, Shtern and Yavorski (1989), is

\[
\varphi = -\frac{V_0}{2(U_0 + 4)} \ln \frac{\nu t}{r^2},
\]

where \( \nu \) is the fluid viscosity,

\[
U_0 = \frac{1}{2\pi\nu} \int_{0}^{2\pi} u \ r \ d\varphi, \quad V_0 = \frac{1}{2\pi\nu} \int_{0}^{2\pi} v \ r \ d\varphi.
\]

Here \( u \) and \( v \) are radial and azimuthal velocities. Note, that the angular speed \( d\varphi/dt \) of the structure propagation depends on time \( t \). An interesting feature is that the sign of \( d\varphi/dt \) is different from the sign of \( V_0 \), i.e. that the direction of the structure precession is opposite to the direction of velocity circulation. Similar effect is observed in our experiments (Fig.35).

Described model can be relevant to the structures we observe. However, presence of the non-zero radial flux and absence of any effects, related to the boundaries, restricting the flow domain in axial direction, require separate consideration.

1.1.3 Differentially rotating fluid in a circular domain

One way to establish the fluid differential rotation is to set the corresponding boundary conditions. Nezlin and Snezhkin (1990) built a setup with the differentially rotating parabolic bottom to study the circular shear flow in a thin fluid layer (Fig.3a,b). Precessing structures, visually resembling ones observed in our experiments, are studied.
particular, authors observed what they call a solitary Rossby wave and the systems of Rossby waves (Fig. 3c,d). The change of the layer depth plays the principal role in the effects considered and the shallow water approximation is used to describe the results. One of the experiments performed by authors is worth mentioning in the context of numerical study of the non-axisymmetric structures development from the axisymmetric vorticity distribution. The authors form a triangular mono-polar vortex with the help of distorted cylinder inserted and then removed from the fluid layer (Fig. 3c). The forming system of three vortex pairs resembles three thermal convective plumes, propagating in the radial direction. It would be interesting to understand the set of conditions, under which the initial perturbation transforms either to the triangular vortex consisting of the core and three satellites (like in Fig. 6c,d, obtained by Carnevale and Kloosterziel (1994)) or to the three vortex pairs. Spiral structures, that are claimed to be of similar nature with the spirals appearing in astronomical observations, are also detected by Nezlin and Snezhkin in the form of azimuthally propagating surface waves, arising in the experimental setup with the differentially rotating bottom.

Similarity of Nezlin and Snezhkin experiments to ours is more or less obvious. In both cases thin layer of liquid, central part of which is rotating faster than the periphery, is considered. However, variation of the layer thickness is crucial in experiments by Nezlin and Snezhkin, while we observe the non-axisymmetric structures even in the layer, restricted by rigid plates.

Flow in a narrow gap with the differentially rotating boundaries is studied by Rabaud and Couder (1983). A circular shear layer in a circular gap, narrow in axial direction, is formed by making inner and outer parts of the gap rotating independently (Fig. 4a).
Model for calculation of the angular velocity for the axisymmetric flow, similar to the model developed by Vladimirov, Yudovich, Zhukov, Denissenko (2001) (Section 3.4 from Chapter 3), is presented. Formation of systems of vortices, visually resembling balls in a bearing, have been experimentally studied, by visualizing the flow with a soap film spanned the mid-plane of the gap (Fig. 4b). Authors studied the modes appearing, transitions between these modes and hysteresis. Dependence of precessing speed of these structures on their angular order and rotation speeds of inner and outer parts of the gap has been derived from general principles, based on the finite size of observed vortices (in particular, on the experimentally measured minimum and maximum radii, between which the vortices are located), and successfully compared with the experimental observations. One flow regime the authors detected (Fig. 4c) and its schematic (Fig. 4d) can help us to explain the existence of structures, precessing in the direction, opposite to the inner cylinder rotation.

Anoda (1983) performed a series of experiments with the narrow in axial direction air gap between synchronously rotating circular plates. A series of the flow visualization is performed. As a result, precessing flow patterns consisting of areas of the radial inflow and outflow are observed. The author analytically investigated linear stability of inviscid and viscous models with the basic flow in the form of rigid body rotation and with the zero-pressure condition at the gap outer edge.

Another way to initiate the fluid differential rotation is to apply a body force. This is realized in Gledzer, Dolzhanskij, Obukhov (1981). Authors studied behaviour of conductive fluid in the annular domain in the presence of magnetic field. Radial current combined with the radially dependent axial magnetic field provides sinusoidal profile of
azimuthal velocity in the basic rotationally symmetric flow. As a result of developing instability, systems of stationary vortices arise for suitably chosen conditions (Fig. 5a). This system was modelled numerically by Ponomarev (1980) with the use of low-dimensional Galerkin approximation (Fig. 5b).

1.1.4 Vorticity evolution in the two-dimensional domain

The 2-dimensional flow structures, visually similar to the ones described above, have been observed experimentally and in numerical simulations by several groups of authors (will be counted later). They considered the evolution of initially axisymmetric vorticity distribution of the form

$$\omega = \left(1 - \frac{\alpha}{2} r^\alpha\right) \exp(-r^\alpha). \tag{1.1.1}$$

It corresponds to the localized spot of vorticity, exponentially decaying with radius. It is somehow relevant to the case of axisymmetric flow in a narrow gap, induced by a rotating cylinder (Section 3.4), where the azimuthal velocity is concentrated near the rotating cylinder and exponentially decreases with the radius. The governing equation for the vorticity evolution is

$$\frac{D\omega}{Dt} = -\nu_0 \omega + \nu_2 \nabla^2 \omega - \nu_4 \nabla^4 \omega - \nu_8 \nabla^8 \omega. \tag{1.1.2}$$

The first term in the right hand side corresponds to the so-called Rayleigh friction used to simulate the damping due to appearance of Ekman layer near the vessel bottom, the second term depicts Newton viscosity and the last two terms correspond to the so-called hyperviscosity, used as an instrument to prevent numerical instabilities in the finite-element simulation.
In 1992, Orlandi and van Hejst studied evolution of the axisymmetric distribution with imposed random perturbation into the tripolar vortex (Fig.6a,b) in the rotating viscous fluid of the finite depth. Structures with the angular wavenumber 3 are mentioned in their work.

The structures with the angular wavenumber 3 are also considered by Carnevale and Kloosterziel (1994), who described the stable vorticity formations they call ‘Triangular Vortices’, i.e. the core of single-signed vorticity, surrounded by three semi-circular satellites of oppositely signed vorticity (Fig.6c,d). Authors apply numerical simulations to investigate the equation for vorticity of the fluid in a rotating tank, including Rayleigh friction and the hyper-viscosity terms. Point-vortices models are also considered. Possibility of appearance of square and pentagonal vortices is also mentioned in the work by Carnevale and Kloosterziel.

The evolution (growth from the axisymmetric flow pattern and saturation) of the tripolar vortices is described by Carton and Legras (1994). The authors studied tripoles growth, oscillation and the self-induced rotation by investigating equations for vorticity with the hyper-viscosity terms. Patterns, visually similar to the ones experimentally observed, are obtained by numerical simulation (Fig.6e,f).

A good review on the two-dimensional and quasi-two-dimensional vortices and the study on the vortices in a stratified fluid are presented in the published PhD thesis by Marcel Beckers (1999). In his work, density stratification suppresses the fluid vertical motion, and thus the role of stratification is common to the role of rigid boundaries of the gap.

Revising described class of works, we highlight the presence of the self-induced rotation
(precession) of the non-axisymmetric vorticity distributions in ideal fluid. Possibly, we deal with the similar case, where the losses due to viscous dissipation are compensated by a rotating cylinder, that plays a role of the core vortex. A model with the non-zero $\nu_0$ and $\nu_2$ in (1.1.2) and a model with the non-zero $\nu_0$ will be considered in Chapter 3.

### 1.1.5 Taylor-Couette setup with discrete azimuthal symmetry

One phenomenon, that can be closely related to the jet-like structures we observe, has been studied by Kobine and Mullin (1994). Authors considered the short in axial direction Couette-Taylor setup with the outer boundary of rectangular shape. The structure in the form of small vortex, appearing near the rotating cylinder and moving outwards is observed. The shape of this small vortex somehow depends on the azimuthal coordinate and, in spite of the discreteness of azimuthal symmetry, can be associated with the ‘exploding vortex’, showing up in our experiments when observing the flow from the side (Fig.17). In other words, periodicity of the small vortex to appear can be due to the angular precession of certain structure. Measured frequency of the small vortex appearance weakly depends on the Reynolds number, that looks to be similar to our experimental observation (Fig.50a). Results of the perturbation amplitude measurements versus Reynolds number for the rectangular setup suggests that the phenomenon corresponds to the Hopf bifurcation, that is not contradictory to our measurements in the small aspect ratio setup (Fig.51).

### 1.1.6 Miscellaneous related works

Patterns, appearing in the setup with rotating grid of blades, is considered by Belyanovski and Kurzin (1983) as a case related to the axial compressors. Inhomogeneity of the
velocity field, which precesses with the speed around 50 times less than the speed of the grid rotation, is studied. Authors proposed a model for the motion of vortex source in the flow, distorted by a grid. Existence of the limiting cycle for the vortex source motion in the form of a circle is shown.

Bifurcation phenomena in the Couette-Taylor setup with a small (down to 0.3) aspect ratio is considered by Pfister, Schmidt, Cliffe and Mullin (1991). The flow transition from the two-cell structure to the single-cell structure is observed and modelled numerically. The similar phenomenon was described by Benjamin and Mullin (1981) for the more general case, when the number of Taylor vortices is greater than two.

Bifurcation from two Taylor vortices to one Taylor vortex is relevant to our experiments, because often it precedes formation of the jet. Initially, it was supposed to consider the jet-like structures as the second bifurcation of the small aspect ratio Couette-Taylor Flow (bifurcation, following transition from two to one Taylor vortex). However, regime with the jet is also observed in the free surface flow, where the only vortex exists from the very beginning. It leads to conclusion that the bifurcation from 2 vortices to 1 vortex is not crucial for the jets formation. Also, the jet-like structure, appearing in the two-cell regime and symmetrical with respect to the middle-gap plane is experimentally observed. It occurs if the rotation speed of the cylinder is increased rapidly enough, so that the transition from two to one Taylor vortex does not happen.

An interesting work is written by Kida (1981). He theoretically studied the vortex filaments that travel without changing their shape in ideal fluid. The closed filaments, rotating (precessing) with respect to certain axis, are discovered. Existence of the soliton-like filament, though the straight one, is shown. Described structures, possessing discrete
azimuthal symmetry, resemble the vortex structures observed in our experiments (Fig.15). It may be possible to describe a solitary disturbance of the circular vortex, travelling in the azimuthal direction, either by considering a filament of the large enough radius or by embedding the flow to an appropriate circular domain.

In 1997 Zhukov (not published) studied the two-dimensional model for the turbulent flow in the gap, where the action of rotating propeller is imitated by the constant body force and the resistance effect of the gap walls is imitated by the body force proportional to the fluid velocity, so that the total force imposed on the fluid is

\[ f(r) = -\varepsilon V + c_r f_0 r e^{-\alpha (r-r_0)^2}. \]  

(1.1.3)

Because to the form of resistance force, the model was conventionally named as ‘Darcy’ model. A study on the case with zero newtonian viscosity is performed. Only the neutral modes with the azimuthal wave numbers 1 or 2 are detected. In this work, a study on the viscous flow is presented (Section 3.1).

Zhukov and Petrovskaya (2000) applied low-order Galerkin method with the basis functions similar to ones used by Ponomarev (1980) to study the 2-dimensional unsteady viscous flow between two cylinders. Two types of precessing non-axisymmetric structures with the angular order 1 are observed numerically. One type is the vortex-like structure (Fig.7a) and the other is the jet-like structure (Fig.7b,c,d). The vortex-like structure appears as a bifurcation from the basic Couette flow, while the jet-like structure appears from the ‘pure air’ at certain value of the inner cylinder rotation speed. While providing patterns, visually similar to the ones experimentally observed, the approach has one disadvantage. The critical Reynolds numbers, for which the vortex-like and the jet-like structures occur increases with the increase of the number of basis functions.
One obvious approach to description of the circular shear flow is to associate it with the certain vorticity profile.

In the plain (not circular) two-dimensional case, evolution of a free surface layer of inviscid fluid with the uniform vorticity is studied by Miroshnikov (2002). Solitary waves are found under the shallow water approximation. The flow separation near the crest appears, so that the streamlines visually resemble ones observed by Nezlin and Snezhkin (1990) in the regime they called Rossby soliton (Fig. 3c).

Lyapidevskij (articles (1994-2001)) and Ovsiannikov et al (1985) considered evolution of the system consist of the layer of constant non-zero vorticity, embedded between two irrotational layers in a straight channel. Long waves approximation is in use. The authors described conditions, under which the system is hyperbolic and studied the propagation of hydraulic jumps along the channel. An extension of Lyapidevskij’s model to the case of curved channel is presented in Section 3.2.
1.2 Essentially non-axisymmetric structures in rotating flow. Basic features

This work is devoted to fluid flows in experimental setups with the two common features:

(sf.1) The flow domain is axisymmetric and its radius is at least several times greater than its dimension in the axial direction. It is either a thin fluid layer with the flat bottom or a narrow gap between two flat plates.

(sf.2) The source of motion in the system is a solid, of the size at least several times smaller than the flow domain radius, located in the centre of the gap, and rotating with respect to the axis of the flow domain. Rotating solid is either axisymmetric or it is rotating fast enough, so that its influence on the flow is essentially axisymmetric.

We concentrate on flow regimes with the four common features:

(ff.1) The flow contains spatially concentrated areas with high positive radial velocity (jets), leaving the rest of circumference to the areas with low negative radial velocity (inflow areas), as the total radial flux must be zero.

(ff.2) Length of the jet(s) can be an order larger than the layer depth or the gap height.

(ff.3) The flow structure is steadily precessing with respect to the flow domain axis with the rate at least an order lower than that of the solid rotation. In other words, the flow is stationary being observed from the slowly rotating frame of reference. Typically, precession direction coincides with the one of the solid rotation.

(ff.4) The flow structure is robust. It exists in the wide range of parameters and is preserved under not too dramatic changes of experimental conditions.

Further, four major classes of experimental setups we deal with are presented. Exact
parameters of the experimental facilities are listed in the figure captions.

1.2.1 Propeller-driven flow in the air gap

The experimental setup consists of two parallel circular plates and a four-blades cross-shaped propeller, suspended in the centre of the gap (Fig.8a). The propeller axis is perpendicular to the plates and coincides with the gap axis. The upper plate is transparent and a small source of oil fog is introduced into the gap to visualize the flow. The fog generator consists of the heater evaporating the engine oil. Mixing with the air, the oil vapor cools down and condensates, forming the white fog. It is believed that the hot oil fog does not affect the flow significantly. It has been tested by measuring the jet precession rate while varying intensity of the fog source. The outer edge of the gap is either open to atmosphere (Fig.8a) or sealed (Fig.8b). The setup with the open outer edge is more convenient for the flow visualization, because the oil fog used does not fill the whole flow domain, being washed out by the jet. Also the jet behaviour is more regular when the gap outer edge is open, so this case is chosen for further investigation.

As the result of extensive program of the flow visualization (setup as in Fig.8a), two stationary non-axisymmetric flow regimes have been observed: a single-jet regime (Fig.9) and a two-jets regime (Fig.10). No stable regimes with more than two jets are observed. Visually, the flow can be divided into two areas: the area disturbed by rotating propeller and the area containing air coming from the outside of the gap. For various sets of parameters, Reynolds number (propeller blades tips speed times gap height divided by air kinematic viscosity) belongs to the interval from $10^3$ to $10^5$, so that the flow in the disturbed area is turbulent. The Mach number (the ratio of propeller tips velocity to the
speed of sound in the air) does not exceed 0.1 in our experiments, so the air compressibility is not to be accounted. Side view of the single-jet flow regime (Fig.9b) suggests existence of the boundary layers near the plates. Observed structures precess with the angular velocity approximately 100 times lower than the propeller rotation speed.

Obtained flow regimes turned out to be amazingly robust. They are not destroyable by impermeable objects placed not too close to the rotating propeller even if the obstacle obstructs the whole gap height. The jet-like structures do not disappear if the gap walls become inclined; the only effect is that the jet(s) precession rate become dependent on the jet(s) azimuthal position. The qualitative picture does not significantly change if the propeller is shifted along the gap axis to the one of the gap walls or if propeller’s blades are twisted in the manner it is done with the ventilator or the aircraft propeller, so that it drives air towards one of the plates. Principal flow structure does not noticeably depend on the propeller shape as well, propeller shape affects the jet(s) precession rate only. Propellers with 1, 2, 3, 4 and 16 blades have been used to test this fact (Fig.8j).

In all described experiments, propeller rotation speed is large in the sense that the propeller revolution period is much smaller (20 to 1000 times) than the jet revolution period. Thus, it sounds natural to simulate the rotating propeller with some kind of continuous force. Experimentally, the propeller has been replaced with an axisymmetric solid. A rotating disk, of a thickness less than the gap height, is placed into the gap instead of propeller (Fig.8e) or is flush-mounted to one of the plates (Fig.8h) to observe precessing jet-like structures (Fig.11 and Fig.12 respectively). No jet-like structures are observed in the air gap where the disk thickness is equal to the gap height (Fig.8f, Section 2.5.2). Similarly to experiments with a propeller, no stable structures with more than
two turbulent jets are observed when the rotating solid is axisymmetric. The structure with three arms is detected as a transition stage from rest to the fully-developed 2-jets regime (Fig.12c).

To make sure that studied phenomenon is not caused by some features specific for the air gap experiments, visualization of the water flow in the similar setup is performed. Ink is injected to the centre of a propeller-driven gap (similar to the one in Fig.8b) filled with water. Appearing flow pattern (Fig.13) does not noticeably differ from the one observed in the air gap (Fig.9). It is worth noticing that the shape of precessing jet does not (visually) change when the propeller rotation speed changes almost five times (Fig.13b-f).

1.2.2 Cylinder-driven flow in a thin liquid layer with a free surface

Typical flow in a thin layer with a free surface, driven by rotating cylinder, suspended above the bottom (as in Fig.8k), is shown in Fig.14. Well-developed precessing jet-like structures are observed. The fluid layer is thin, so deviations of the free surface from horizontal are comparable with the layer thickness: fluid depth varies from zero in the cavities adjacent to the rotating cylinder to a doubled average layer thickness within the jets. The flow is subject to hysteresis, e.g. structures with 2, 3 and 4 arms are observed at the same disk rotation speed (Fig.15). Flows with 2, 3, 4, 5, 6 and 7 arms spread around circumference appear in the vessel described (as in Figs.8i, k) for different disk rotation speeds. For structures with the angular order 4, 5, 6, 7, arms length is less than the disk radius, so they are not as spectacular as the 3-jet structures. Typically, higher numbers of jets appear at lower disk rotation speed, and the ratio of jets precession rate to disk rotation speed is higher for larger number of jets. This ratio decreases with the increase
of disk rotation speed, and for high enough disk rotation speed patterns with 2 and 3 jets start to precess in the opposite direction (e.g. the pattern in Fig. 15a). When the jets are precessing in the direction opposite to the direction of disk rotation, visual impression is that the jets are moving upstream not continuously but with the small jumps.

In experiments with a thin free surface layer, the flow domain is rather sophisticated. Say, to obtain regimes presented in Figs. 14, 15, rotating disk is suspended 3 mm above the rigid boundary (Fig. 8k). Strong cavities on the free surface make the mathematical description of phenomenon even more tricky. Each non-axisymmetric flow regime exists for a narrow range of parameters that makes it hard to study general properties of the flow. Thus, no systematic studies on the free surface flow in a thin layer are performed and the case is used mainly for visual demonstration of phenomenon.

1.2.3 Cylinder-driven flow in a deep liquid layer and a fluid flow in a gap between rigid boundaries

Well-defined non-axisymmetric structures are also observed in deep layers, where the free surface deviation from horizontal is at least an order less than the layer depth. Double and triple-jet structures, observed in the vessel similar to the one in Fig. 8g, are shown in Fig. 16. In Fig. 16a, jets are dying out before reaching the vessel outer edge while in the case (b) jets are longer than the layer size in radial direction, so they are impinging the outer wall.

Essential feature of the flow in a deep layer is existence of the Taylor vortex in vicinity of rotating cylinder. Being observed from the side, flow gives an impression of exploding Taylor vortex (Fig. 17). While the image series is acquired, flow pattern is precessing, making its different cross-sections exposed to the camera, so that the observed time evo-
olution can be associated with the flow angular structure. The vortex, located near the
cylinder, grows by entraining the fluid from the outer part of the flow domain (the inflow
exists near the bottom in frames #1, #2, #3) and then explodes, forming a jet (frames
#5, #6, #7). The frame #8 corresponds to the time instant, when the vortex starts to
form again. The latter is, in fact, the cross-section of the ‘tail’ of broken 3-dimensional
Taylor vortex (see the sketch below Fig.17).

If the vertical lasersheet, used for flow visualization, is spaced from the cylinder edge
(as in Fig.45b), the travelling pair of areas with the oppositely signed radial vorticity com-
ponent is observed (Fig.18). This pair can be interpreted as the azimuthally propagating
radial protrusion of Taylor vortex.

The setup, in which the diameter of rotating cylinder is similar to the fluid depth,
is presented in Fig.19. Note that the characteristic length of the jet-like structure is an
order larger than the radius of rotating cylinder and the layer depth. A few drops of ink
are added to the free surface in Fig.19b to highlight the jet.

Development of the non-axisymmetric structure, following rapid increase of the cylin-
der rotation speed is shown in Fig.20. When the cylinder rotation speed increases, Taylor
vortex grows in size (b), loses stability (c,d), and then a pair of jet-like structures forms.
One interesting formation that exists in the flow is a pair of shoe-shaped vortices, residing
behind the jets and consisting of the fluid that is never entrained by the Taylor vortex.
Such pair, visualized by a few drops of ink on the layer surface, is shown in Fig.21. The
photo is done few minutes after the ink is added. Characteristic time of the ink leakage
from the shoe-like vortices due to either molecular diffusion or some other effect is about
an hour.
Non-axisymmetric structures in a gap (setup as in Fig. 8f) are shown in Fig. 22. Being less impressive visually, flows in a gap represent a convenient case for theoretical description because of the simple flow domain geometry and simple (the non-slip) boundary conditions.

1.2.4 Fluid flow in a gap with differentially rotating boundaries

While searching for the most convenient for the mathematical description system, the setup consisting of a gap with a rotating central area (two rotating disks flash-mounted to the gap walls as in Fig. 8m) is assembled. The gap height is 10 times smaller than the radius of its central (rotating) area. The gap is filled with the water-glycerol mixture to the extent, where the air bubble of the size less than the rotating area remains. When the non-axisymmetric flow regime occurs, bubble boundary shows the order of the flow angular symmetry. Apart from simplifying the flow visualization, the bubble was introduces to exclude the singular point $r = 0$ from the possible mathematical consideration. Also, the idea to study the travelling waves arising on the bubble boundary looks fruitful. Regimes with the angular order from 2 to 7 have been observed. Regimes with the triangular, square and pentagonal bubble are shown in Fig. 23. The gap outer edge, the edge of the rotating part of the gap and the non-circular bubble boundary are all visible. As in the case with a free surface, described in the previous section, flow patterns with the larger angular order are observed at lower disks rotation speed, and the ratio of the flow pattern precession rate to the disk rotation speed is larger for the patterns with larger angular order.
1.2.5 Typical flow pattern

Revising visualization experiments described above together with the number of similar observations, we can now sketch basic features of the flow observed. Structure (a set of pass lines) of the laminar, essentially three-dimensional flow, similar to the one shown in Figs. 16, 19, 21 (geometry as in Fig. 8g), is sketched in Fig. 24. Reaching the point, where the jet originates, fluid forming the Taylor vortex separates into two parts: one part stays in vicinity of the cylinder, while the other part leaves to the outer area, where it loses momentum to be entrained by the Taylor vortex again. Shoe-shaped areas of the fluid that is never entrained by the Taylor vortex, are located behind the jets. These areas correspond to vortices, whose upper ends lies on the free surface and the lower ends rest at the boundary between the cylinder and the bottom of the layer (indicated as spirals). So, there are two parts of the fluid that do not mix with each other: fluid in the Taylor vortex and Jets, and fluid in the Shoe-shaped vortices. If the outer edge is far enough, the third part of the fluid distinguishes, that is the one located far enough from the cylinder.

The structure of the flow, similar to the one shown in Fig. 10 (geometry as in Fig. 8a), and believed to have essentially two-dimensional structure, is sketched in Fig. 25. Coming from the outside of the gap within more or less laminar inflow areas, the fluid is entrained by rotating propeller and thrown out towards the gap outer edge in the form of spatially concentrated jets. If observing the flow from aside (Fig. 9b), well-defined boundary layers that form near the gap walls are seen. It is still not clear if they play a crucial role in the jets formation.

At this point, we assume that the phenomenon, meant by the essentially non-axisymmetric
flow patterns arising in thin axisymmetric layers, is qualitatively outlined. In the next chapter we will describe a number of quantitative measurements and experimental variations, aimed to provide the background for mathematical modelling.
Chapter 2

Experimental variations

While trying to understand the nature of the jet-like structures arising in axisymmetric setups, a number of variations of experiment described by Zhukovskij (Fig.2) is carried out (Fig.8). One common feature of setups assembled is existence of the gap between flat plates (or the layer with a flat bottom), such that the ratio of the gap height (or the layer depth) to the flow domain radius is small (not more than 1:5). Another feature is that the flow is driven by a rotating solid, not necessarily axisymmetric, located in the gap centre and oriented so that its axis of rotation coincides with the gap axis.

2.1 Studies on the jet precession rate

One of the most noticeable features of the jet-like structures is their precession. Precession rate of the flow pattern is an easily measurable parameter, and it will probably give us a hint to understanding of the mechanism of the jets formation. An extensive study on precession rate of the turbulent jet (setup as in Fig.8a) is performed in summer 2000 (Fluid Dynamics Laboratory, Department of Mathematics, HKUST, Hong Kong). Several experiments on the jet precession rate are conducted later using the improved experimental facility in 2002–2003 (Fluid Dynamics Laboratory, Department of Mathe-
matics and Statistics, Hull University, UK). Measurements for the laminar flows (setups Fig.8ℓ, m) also provided some interesting results, that will be reported below.

2.1.1 Jet precession rate versus non-dimensional parameters

The single-jet flow regime (similar to the one shown in Fig.9) in the air gap with the open outer edge is chosen for this study. A cross-shaped propeller of span 2c and of height h is suspended in the middle of the gap of diameter 2b and of height H. Jet precession rate is measured manually with the stop watch (time required for 3 to 10 jet revolutions is measured 3 times and then averaged). For this purpose, the jet is visualized in two ways. One way is by placing the tiny oil vapor source (heated 0.25 Wt resistor on which the oil drips) near the gap outer edge: it visualizes the flow direction without introducing significant disturbance to the flow structure. Another way is by mounting a sound detector (PC microphone) to the gap wall: being displayed on the oscilloscope screen, sound signal shows a typical increase of noise level when the jet passes by microphone location. One defect of the experimental arrangement is discovered after the whole series has been finished: there is an annular hole of approximately 2 cm² area in the place where propeller shaft enters the gap. It causes the non-zero average radial air flow, that can significantly affect the jet precession rate.

Experiments showed that the jet precession rate ω linearly depends on the propeller rotation speed Ω for a wide range of parameters. Also, the linear fit to the data points ω(Ω) does not necessarily go through the origin (Figs.26, 27). Data presented in mentioned figures together with a number of other similar measurements led us to the conclusion
that the jet precession rate can be expressed via setup parameters as

\[ \omega = \omega_0(h, H) + \alpha(b, c, h, H)(\Omega - \Omega_0(h, H)), \]  

(2.1.1)

Note that \( \omega_0 \) and \( \Omega_0 \) depend on the axial sizes of the setup \( h \) and \( H \) and are independent of the propeller half-span \( c \) and the gap radius \( b \). Results of measurements of \( \alpha \) for various geometrical parameters are presented in Table 4.5.

Since we claim that the flow is essentially turbulent, it is natural to try to express \( \alpha(b, c, h, H) \) as a function of some non-dimensional parameters appearing in the problem, say, \( \rho, \zeta \) and \( \eta \) which are

\[ \rho = \frac{c}{b}, \quad \zeta = \frac{h}{H}, \quad \eta = \frac{2c}{H}. \]  

(2.1.2)

We failed to find a function \( \alpha(\rho, \zeta, \eta) \). It could be caused by several reasons. First is existence of the laminar boundary layers near the gap walls (Fig. 9b). Second is presence of the non-turbulent regime in the inflow areas, i.e. areas where the calm air comes to the gap through the open outer edge. Third is the presence of a hole near the propeller shaft (this experimental fault is mentioned earlier). However, a general formula for dependence of the jet precession rate on setup parameters is constructed and tested. In Fig. 28, dependence of \( d\omega/d\Omega \) on dimensional combination

\[ \left( \frac{c}{H} \right)^{25/16} \left( \frac{h}{b} \right)^{5/4} \cdot h^{5/2} \propto \rho^{5/4} \zeta^{5/4} \eta^{5/16} \cdot h^{1/2} \]  

(2.1.3)

is presented. Scatter plot covers interval of \( d\omega/d\Omega \) from 0.003 to 0.06 (30-fold range) and can be interpreted as linear. No particular conclusions are done from this fact yet.

### 2.1.2 Jet precession rate versus gap diameter

An interesting dependence of the jet precession rate on the gap diameter is discovered in recent experiments with the setup, similar to the one considered in the previous section
(Fig. 8a), but with the sealed propeller shaft. Propeller fills the whole gap height now (Fig. 8d), that is aimed to exclude one non-dimensional parameter \( \zeta \) from the set (2.1.2).

Fixing all parameters of the setup but the radius of the gap (the gap outer edge is open to atmosphere), jet precession rate \( \omega \) is measured (Fig. 29a). As in previous experiments, dependence of the jet precession rate on the propeller rotation speed is approximated with linear fits. Note, that for large enough \( \Omega \), deviation of dependence \( \omega(\Omega) \) from linear is less than the markers size (of order of 1%). The slope \( d\omega/d\Omega \) is plotted versus gap diameter in Fig. 29b. Linear fit in the logarithmic scale suggests that

\[
\frac{d\omega}{d\Omega} \propto \ln \frac{b}{2c}.
\] (2.1.4)

Coefficient 2 in the denominator is chosen empirically, so that the value \( 2c \) can be interpreted as an ‘effective’ propeller span. No particular conclusions are made from this fact. However, relation (2.1.4) looks to be meaningful, because the linear dependence holds while \( d\omega/d\Omega \) changes more than 10 times, that is doubtfully a coincidence.

One way to estimate the non-axisymmetric formation precession rate is to assume that the fluid composing it has the uniform vorticity, say, \( \tilde{\omega} \). For the formation to precess with the rate \( \omega \) without changing its shape, relation \( \omega = \tilde{\omega}/2 \) must be valid. The first term in relation (1.1.2), commonly used to model vorticity evolution, suggests that the ‘loss’ of vorticity is proportional to the area occupied by the fluid times \( \tilde{\omega} \). Similarly, vorticity ‘income’ is proportional to the area spanned by a propeller times \( \Omega \), which is the propeller rotation speed. Assuming that the area occupied by the precessing structure has characteristic size \((c + b)\), where \( c \) is the propeller radius and \( b \) is the gap radius, we suggest that \( d\omega/d\Omega \) is proportional to \( c^2/(c + b)^2 \) (see the scheme below Fig. 30). Preliminary sketching showed that it is worth considering somewhat ‘effective’ propeller
radius $4c$, so that
\[
\frac{d\omega}{d\Omega} \propto \frac{a^2}{(4c + b)^2} \tag{2.1.5}
\]

To find out if this relation holds, the value $d\omega/d\Omega$ is plotted versus $c^2/(4c + b)^2$ in logarithmic scale in Fig.30. Observe, that the graphs can be interpreted as having $45^\circ$ slope in certain range of parameters, so that in certain cases jet-like structures can be considered as formed by the fluid of uniform vorticity.

2.1.3 Jet precession rate and the amplitude of velocity perturbation

The amplitude of velocity variation between the area of air inflow and the jet for the single-jet flow regime in the air gap with the open outer edge (configuration as in Fig.8a) is studied. A primitive thermo-anemometer based on the thread of broken light bulb is placed to the gap edge (Fig.31a). Gap diameter is 30 cm, gap height is 2.4 cm. Four-blades propeller of 1.5 cm height and 13 cm span is in use. Signal from the anemometer is recorded for 25 s with the sampling rate of 100 Hz and then the Fast Fourier Transform is applied to the signal (digital oscilloscope Tektronix TDS420a is used). Typical signal and its spectrum are plotted in Fig.31b,c. Height of the first peak in the spectrum is measured and plotted versus propeller rotation speed together with position of the peak (that corresponds to the jet precession rate) in Fig.32. The jet precession rate linearly depends on propeller rotation speed in accordance with previous measurements. The amplitude of the first peak, that is associated with the squared amplitude of velocity variation, changes linearly with the propeller rotation speed as well. The value of propeller rotation speed, at which the linear fit crosses zero level (indicated with a dotted line)
can be associated with the point of the non-axisymmetric structure onset. Described measurement confirms that the non-axisymmetric structure has the non-zero precession rate in vicinity of the critical propeller rotation speed. Another point to note is that the linear fit in Fig. 32a suggests the square root dependence of amplitude of velocity variation on the propeller rotation speed, that is typical for the supercritical bifurcation.

2.1.4 Precession of the pattern in a differentially rotating gap

Precession rate of the non-axisymmetric structures is measured for the setup with rotating discs flash-mounted to the gap walls. The gap is partially filled with the water-glycerol mixture, so that the air bubble of diameter less than that of rotating discs remains (Fig. 8m). Flows, similar to the ones shown in Fig. 23 are video-recorded and then analyzed using the VCR jogging mode. Precession rate obtained is shown in Fig. 33 for the structures with angular orders from 2 to 6. Note, that dependence of the structures precession rate on the flash-mounted disks rotation speed is more or less linear for the structures of all angular orders. Preserving of the linear dependence $\omega(\Omega)$ throughout the measurements is surprising here, because the Reynolds number varies from around 100 to around 4000 for different points, and thus even the flow regime is expected to change its nature.

2.1.5 Setup with the jet precessing in negative direction

While performing measurements of velocity profile for the setup with the cylinder diameter equal to the gap height (setup as in Fig. 8ℓ), a regime with the jet-like structure precessing in the direction, opposite to direction of the cylinder rotation, is found. Two PIV images are shown in Fig. 34. In the case (a) the non-axisymmetric structure precesses
counterclockwise, i. e. in the same direction as the cylinder rotates. In the case (b) the structure precesses clockwise. Dependence of the flow pattern precession rate on the cylinder rotation speed for this setup is shown in Fig. 35.

2.2 PIV measurements for the stationary axisymmetric flow

To test the applicability of the asymptotic model presented in Section 3.4, a program of flow visualization and velocity measurement is conducted. Setup, similar to the one in Fig. 8, with the outer radius \( b = 12 \text{ cm} \) and the gap height \( 2H = 2 \text{ cm} \) is used. Angular velocity \( \Omega \) is changed in the interval from 0 to 500 rpm. The gap is filled with water-glycerol mixture of viscosity \( \nu = 0.43 \text{ cm}^2/\text{s} \) and of density \( \rho = 1.15 \text{ g/cm}^3 \).

To perform the flow visualization and velocity measurements, Particle Image Velocimetry is applied. We use the standard PIV system (manufactured by TSI) that consists of 5 W continuous Argon Ion laser, beam modulator, electronic synchronizer, CCD camera and software (Insight 3.0) for images acquisition and processing. Beam modulator forms pulses with duration from 1 to 100 ms and separation from 5 to 500 ms. Optical system transforms the laser beam into a 0.2 mm thickness light sheet. The images are captured by a CCD Kodak Megaplus 1.4 camera with the resolution 1316 by 1034 pixels and exposure time from 2 to 265 milliseconds. Silvered hollow glass particles with characteristic size 20 \( \mu \text{m} \) and density 1.2 g/cm\(^3\) are in use for the flow visualization and PIV measurements.

Sections of the flow by the lasersheet passing through the setup axis are presented in Fig. 36. For low enough cylinder rotation speed, two Taylor vortices resides in vicinity of the rotating cylinder (a). With the increase of the cylinder rotation speed, one of vortices
become dominating (b)-(g): the flow loses mirror symmetry with respect to the mid-gap plane, as it was earlier described in Pfister et al (1988). For high enough cylinder rotation speed, the flow loses rotational symmetry and the precessing structure occurs, so that the visualized velocity field becomes unstable (h),(i) (compare with Fig. 17).

Velocity measurements for the rotationally symmetric regimes, similar to the ones visualized in Fig. 36a–g, are conducted. As it is clear from visualization, the flow is concentrated in the Taylor vortices near the inner cylinder. Preliminary measurements showed that the angular component of velocity steeply decreases with the radius. Thus, to provide a sufficient accuracy of the PIV measurements, laser pulses with different time separation are used for different regions of the flow. In particular, pulses with duration 2 ms and separation down to 10 ms are in use for measurements in the vicinity of rotating cylinder. Maximum time separation 500 ms is used for the measurements of low velocities in the outer area. For the image processing either auto-correlation method or the two-frame cross-correlation method is applied. First is used for measurements in the vicinity of rotating cylinder where velocity is of the same order as the one of rotating cylinder and the second is applicable to measurements in the outer areas where velocity is low, so that the camera is able to acquire two images for two laser pulses.

We have carried out velocity measurements for the axisymmetric two-vortex regime (similar to the one shown in Fig. 36a) for several values of cylinder radius $a$ and its angular velocity $\Omega < 100 \text{ rpm}$. Obtained results are similar, and the case with $a = 25 \text{ mm}$ and $\Omega = 20.1 \text{ rpm}$ is chosen for presentation. Profiles of azimuthal velocity $v = v(r, z)$ for several values of $z$ are presented in linear scale in Fig. 37. The origin $z = 0$, $r = 0$ is placed to the centre of the gap middle cross-section, so that the levels $z = -h$ and $z = +h$
correspond to the gap walls. Continuous curves correspond to the theoretically obtained profiles (3.4.28). Exponential decrease of $v$ with the radius is clearly seen in Fig.38 where the same data is presented in logarithmic scale. The accuracy of PIV measurements for the angular component of velocity is around 5%. One can see that results of measurements are in a reasonable agreement with velocity profiles (3.4.28) predicted by asymptotic model both in vicinity of rotating cylinder and in the areas remote from the cylinder.

More difficult is to provide accurate PIV measurements for the radial velocity component $u = u(r, z)$. The reason is that the radial velocity is much smaller than the angular velocity while the absolute measurement errors for radial and azimuthal velocities are the same and are proportional to the error of measurement of reflective particles displacement. Therefore, relative error of the radial and axial velocities measurement is larger than that for the azimuthal velocity. In our case this error is around 25%. Profiles of radial velocity $u = u(r, z)$ for $\Omega = 60$ rpm for four values of $z$ are presented in Fig.39. Continuous curves correspond to the theoretically predicted profiles (3.4.33). Deviation of experimentally obtained points from theoretical curves is of the same order as the measurement error, however, the deviation is obviously biased. It can be caused by imprecision of the lasersheet positioning in axial direction.

Figs. 40, 41 show dependence of the angular velocity on radius in linear and logarithmic scales for $\Omega = 130$ rpm that corresponds to the axisymmetric flow after the bifurcation of breaking of $z \to -z$ symmetry, i.e. the flow with with one dominating Taylor vortex (Fig.36g). Now, velocity field $v(r, z)$ is not symmetric with respect to plane $z = 0$ and profiles of angular velocity significantly deviate from ones predicted by asymptotic model.

Asymptotic model, described in Section 3.4 is not applicable now, because the assumptions
on the order of radial and axial velocities (3.4.11), (3.4.13) are not valid and the advective terms affect equation for the azimuthal velocity $v$ (3.4.17). It reveals in the existence of the flat section in the interval $0.5h < r < 1.5h$, that is related to the Taylor vortex near the rotating cylinder. It is worth noticing that for $r > 1.5h$ velocity decreases exponentially in the way predicted by the asymptotic model (Fig.41). Some kind of boundary layer exists near the inner cylinder at $r < 0.5h$. The thickness of this boundary layer is of a scale different (smaller) than $h$.

Numerical solution of the Navier-Stokes equations in assumption of the angular symmetry of solution is performed using the finite-difference method in the way described by Cliffe (1983). The computer code was kindly granted by Prof. M.Yu.Zhukov. Comparison of experimental data with numerical simulations is presented in Fig.42. While showing good agreement, discrepancy of up to 30% is observed in certain areas, that can be caused by inaccuracy of the lasersheet positioning in axial direction.

### 2.3 Free surface flow with two precessing jets. PIV measurement #168

While constructing mathematical models of precessing structures and comparing them with experimental observations, a need for ‘sample’ flow regime, for which the most complete data set would be available, arose. Measurements of radial, azimuthal and axial velocities, critical cylinder rotation speed and a number of other parameters are performed for the flow with two precessing jets. For the sake of convenience, we call these series with the name used in our experimental journal, i.e. ‘Measurement #168’.

Setup used for Measurement #168 is similar to the one in Fig.8g: 2 cm depth free
surface layer of the water-glycerol mixture of viscosity of 0.127 cm²s⁻¹ in the vessel of 24 cm radius. Rotating cylinder radius is 6 cm. Cylinder rotation speed is 20.8 rpm that corresponds to velocity of the disk boundary of 13.1 cm/s and may cause the free surface deviation from horizontal not larger than $v^2(2g)^{-1} \approx 0.09$ cm. The jets precession rate is 1.08 rpm. PIV configuration, similar to the one described in previous section, is in use.

While image series is acquired, the flow pattern precesses, making its different parts exposed to the camera, so that observed time evolution corresponds to the flow angular structure. It seems now that the Laser Doppler Anemometry technique is more suitable to study velocity field in a steadily precessing structure. Though, it was not available at the time experiments were done. Radial profiles of azimuthal velocity $v$, measured with the help of PIV system, are plotted in Fig. 43. They are obtained from image series acquired with the camera placed above the flow and with the lasersheet oriented horizontally. Observe the radial interval of the changeable width with the high azimuthal velocity, that corresponds to the Taylor vortex in the vicinity of rotating cylinder: positive radial velocity advects the fluid with high $v$ from the cylinder, that is best shown by the velocity profiles at $z = 1.8$ cm (triangular markers). Angular dependence of the vortex size is better visualized in Fig. 44, where the contour plot based on the same data is shown.

An interesting study performed is the measurement of azimuthal velocity radial gradient in the very vicinity of the rotating cylinder. Due to the relatively low spatial resolution of PIV technique, error of velocity measurement near the cylinder boundary (where the azimuthal velocity gradient is steep) is too large to resolve velocity dependence on radius. To achieve better results, vertical lasersheet is set to be tangent to the cylinder boundary as it is sketched in Fig. 45a. In this geometry, distance from the axis of the camera lens
(say, \(x\)) is converted to the distance from the cylinder surface \((r - a)\), as \(r - a \approx \frac{x^2}{2a}\), so that the ratio \(dr/dx\) varies from 0 to about 0.4, increasing spatial resolution of the system. Resolution is now limited by lasersheet thickness, that is 0.2 mm. Velocity gradient, measured by the method described, is plotted versus \(z\) at different time instants (i.e. for different flow cross-sections \(\varphi\)) in Fig.46. The line plot (a) shows that \(dv/dr\) changes linearly in axial direction, that can be used when modelling the boundary layer near the rotating cylinder. The contour plot (b) shows that \(dv/dr\) changes linearly in azimuthal direction along the most of circumference (from 30 to 150 degrees, i.e. between the jets) and jumps afterwards, that can be useful when modelling separation of the Taylor vortex from the cylinder (Fig.75b). Another conclusion done is that \(dv/dr\) does not change more than 20% with \(\varphi\). It suggests that there is no point of the boundary layer separation near the cylinder edge, that can be associated with the jet appearance. It suggests the idea to consider the model, where the cylinder will be associated with a ‘source of angular momentum’ rather than with a rigid boundary. In other words, the condition \(dv/dr = \text{const}\) can be more appropriate than the condition \(v = \Omega a\) at the boundary of rotating cylinder located at \(r = a\).

Vertical profiles of azimuthal velocity \(v\) and axial velocity \(w\) are measured in the geometry similar to the one shown in Fig.45b with the lasersheet located at different distances (0.2 cm, 0.6 cm, 1 cm, 1.5 cm, 2 cm and 3 cm) from the cylinder. Vertical profiles of azimuthal velocity \(v(z)\) are shown in Fig.47. Contour plots of \(v(\varphi, z)\) at different radii are shown in Fig.48. Profiles of axial velocity are presented in Fig.49. Note, that the sign of \(w\) changes from positive in the vicinity of the cylinder to negative in the outer areas, that corresponds to existence of Taylor vortex. Comparing Fig.49 and Fig.47, we
highlight that the ratio of axial to azimuthal velocity components $w/v$ reaches up to 0.2 at $r - a = 2\text{ cm}$, that is different from what happens within Taylor vortices in the long Couette-Taylor setup, where the axial velocity is negligible in comparison with the azimuthal velocity.

It is interesting to find out if the principal flow structure looks much different being observed from the frame of reference, rotating with the jet. To do that, the value $\omega r$, where $\omega = 1.08 \text{ rpm}$, is to be subtracted from azimuthal velocity $v$. For the largest distance from cylinder we consider ($r - a = 3\text{ cm}$), this value is $1.02 \text{ cm/s}$, so that velocity profiles shown in Fig.47 are not significantly different from that observed from the frame of reference where the jet is steady.

To make the set of measurements devoted to this particular setup complete, the non-axisymmetric structure precession rate is measured versus cylinder rotation speed (Fig.50a). A slight decrease of the pattern precession rate is detected for the cylinder rotation speed slightly higher than the critical. A number of attempts have been undertaken to construct the asymptotic model, based on the layer depth as a parameter. So, measurements of the critical cylinder rotation speed and the non-axisymmetric structure precession rate on the layer depth are performed (Fig.50b,c). Since deviation of the flow from axisymmetric is small for the cylinder rotation speed, close enough to the critical one, precessing structures are hardly visible by naked eye and the studies on the critical disk rotation speed are made by measuring deviation of the laser beam, reflected by a tiny mirror, affixed to the thin thread submerged to the flow (Fig.51a). Squared amplitude of laser beam deviation is plotted versus cylinder rotation speed for different fluid layer depths (Fig.51b). Points where the amplitude can be extrapolated to be zero correspond
to the critical disc rotation speed and are plotted in Fig. 50b. Two side observations made are that the squared amplitude of the non-axisymmetric structure grows with the cylinder rotation speed in vicinity of its critical value and that the phenomenon has no observable hysteresis effect. It suggests that the jet appearance is the supercritical bifurcation of the axisymmetric flow.

One possible way to explain the observed jet is to associate it with separation of the Taylor vortex from the rotating cylinder. So, the vortex can be considered as a precessing structure (say, as a kind of boundary layer), that originates at certain point, then grows and separates from the rotating cylinder (Fig. 75b). In this case, condition of the structure $2\pi$-periodicity can somehow define precession rate of the vortex origin and the vortex separation point. No reasonable model to describe the vortex is constructed, though, measurement of distance between Taylor vortex origin and its point of separation is performed. The obstacle (a cylinder of 3.5 cm diameter) is placed adjacent to the rotating cylinder in the setup, used in the Measurement #168 (see sketch in Fig. 52). Thus, the point of vortex origin is fixed and the $2\pi$-periodicity of the flow is violated. Angle, at which Taylor vortex separates from the cylinder, forming a jet-like structure, is measured while varying the cylinder rotation speed (Fig. 52). An interesting fact is that the measured angle decreases with the increase of cylinder rotation speed almost as $\Omega^{-1}$. Another fact is that the distance between two jets in the structure, precessing with the rate about 2.5 rpm when the disc rotation speed is 31 rpm (Fig. 50c), is 180° while the linear extrapolation of data in Fig. 52 suggests more than 360° for the vortex with fixed origin. It can be explained by the fact that the Taylor vortex is not completely destroyed at the point where the jet separates from the cylinder.
2.4 Miscellaneous experimental observations

While playing around with the hysteresis effect in the free surface flows, similar to the ones in Fig.15, bare fingers are used to set the flow required regime. In transitional stages, we observe situations when the three jets are precessing more or less stably, while the fourth one is ‘running’ faster, overtaking those three and continuing moving ahead, completing up to three turns around setup axis before and then disappearing. Such behaviour resembles the one of solitons, colliding without interaction. Similar, though not as clearly visible structures are observed in the flow shown in Fig.23 in the form azimuthally propagating waves on the boundary of the air bubble.

When visualizing turbulent flows in the propeller-driven gap, it was noticed that the jet-like structure is not as spectacular when the propeller fills the whole gap height (Fig.8d) and no hole exists near the propeller shaft, so that no total radial air flux is present. It suggests that, when the propeller height is small (Fig.8a), strong boundary layers with the flow directed inwards are formed near the gap walls, so that the positive radial flux appears in the middle of the gap, that is responsible for the jet formation.

2.5 Setups, in which the jet-like structures do not appear

2.5.1 Propeller-driven air gap with the sealed outer edge

Experiments in the air gap with the sealed outer edge showed that when the outer radius is not large enough, air in the entire gap starts to rotate and no jets appear. To be precise, the following experiment is done: a primitive thermo-anemometer made of the broken light bulb is placed to the gap as in Fig.8b. The signal from the anemometer is
recorded, and the Fast Fourier Transform is applied to the signal, i.e. the same system as the one described in Section 2.1.3 (Fig.31) is in use. Existence of the well-defined peak in the signal spectrum is interpreted as existence of some coherent structure (precessing jet). For the four-blades cross-shaped propeller of 13 cm span and 1.5 cm height placed to the sealed gap of 2.4 cm height, coherent non-axisymmetric structures do not appear when the gap diameter is less than 50 cm. If the propeller height is changed to 0.55 cm, critical diameter become around 40 cm and another effect appears: the structure exists if the propeller rotation rate is less than 2000 rpm and do not exist for higher propeller rotation rates. When removing the seal from the outer edge of the gap, making it similar to one in Fig.8a, the jet-like structure appears immediately.

2.5.2 Setup with the cylinder filling the whole air gap height

The air flow in a gap with the open outer edge (Fig.8f) is studied. The gap height is 2.2 cm, gap diameter is 85 cm. The cylinder of 12 cm diameter and 2 cm height, rotating with the speed 3000 rpm is placed to the gap centre (note that almost no space left between the cylinder and the gap walls). The tiny oil fog source is introduced to the gap for the flow visualization. Irregular flow located near the cylinder with no significant outflow is observed (Fig.53a): the smoke remained near the cylinder without leaving the gap.

At the next stage, a bolt M8 with the hexagonal head of 1.3 cm width and 0.6 cm height is screwed into the cylinder’s side so that only the head remained above the surface. Precessing non-axisymmetric flow pattern is now observed (Fig.53b). When the bolt is unscrewed so that around 1.5 cm of its trunk is exposed, a jet-like structure, visually similar to the one observed with a propeller, appeared (Fig.53c).
Described observation leads to the following construction. The bolt moving in the air creates a wake behind. If it rotates fast enough, i.e. if its revolution time is much less than the characteristic time of the flow evolution, the bolt creates the average force imposed on the fluid. Thus, the 2-dimensional flow induced by a force, localized at certain radius and dependent on the fluid speed, is to be modelled (see Section 3.1). An experimental setup, aimed to fit this scheme, is constructed (Fig. 8c): a square plate of 1.6 cm by 1.6 cm is suspended on the rotating arm of 4.5 cm length in the gap of 1.7 cm height (construction, resembling a single-blade propeller is thus obtained). When the arm rotates fast enough, the jet-like structure is observed (Fig. 53d).

2.6 Miscellaneous experiments where precessing structures arise

2.6.1 Circular shear layer with the faster rotating fluid located further from the axis

One interesting case when precessing non-axisymmetric flow appears in the circular geometry is formation of the meso-vortices in the bath-tub-like setup, described by Vladimirov and Tarasov (1980) and then studied by Montgomery, Vladimirov and Denissenko (2002). This experimental study is undertaken to model formation of meso-vortices in a typhoon eye. The sink in the bottom of the layer of rotating fluid is considered (Fig. 54a). Fluid is pumped out from the lower vessel and then tangentially injected in vicinity of the outer edge of the upper vessel, forming rotating flow in the upper vessel. The sink is located on the vessel axis, so, in the absence of viscosity, azimuthal velocity profile is \( v \propto r^{-1} \) and the pressure radial gradient is, \( p_r \propto v^2 r^{-1} \propto r^{-3} \). Viscous boundary layer is formed
near the bottom of the vessel. In this boundary layer, azimuthal velocity is depressed and, thus, radial pressure gradient is less than the one mentioned above. So, the inflow is formed near the bottom of upper vessel (Fig. 54b). Reaching the sink orifice, boundary layer penetrates to the lower vessel and forces out the fluid from the lower vessel to the upper vessel, creating the upwelling near the setup axis. Since the fluid in lower vessel does not rotate, cylindrical shear layer is formed where it gets in touch with the rotating fluid in the upper vessel. This shear layer is unstable, and it turns into an impressively stable system of precessing meso-vortices spread at radius, slightly less than the one of the sink (spirals in Fig. 54b). Ink visualization of the system of meso-vortices is shown in Fig. 55. It is worth noticing, that the flow is essentially two-dimensional, i.e. velocity profiles are almost independent of the vertical coordinate (apart from existence of the boundary layer near the bottom). Stable structures, containing 2, 3 and 4 meso-vortices are observed for various layer depth, orifice radius and the total flux through the sink. Precessing structures, though not always stable, with larger numbers of meso-vortices are observed for thin enough fluid layer. The flow structure is robust in the sense that, being destroyed, the system of meso-vortices appears again. No hysteresis is detected in this setup.

The case described differs from the ones where jet-like structures are observed by the fact that the faster rotating fluid is located further from the setup axis. Unlike the case when jets are formed, unstable shear layer causes formation of the meso-vortices, which are localized near the line of separation of fast and slow rotating fluid and do not penetrate far in the radial direction.
2.6.2 Non-axisymmetric structure in the rarified granular flow

One funny case of the ordered non-axisymmetric behaviour in the axisymmetric setup is observed in the circular pan-like basin with the rotating bottom, filled with dried peas (Fig. 56a). Ordered patterns of the angular orders 1 and 2 are observed for the motion of around 100 dried peas (Fig. 56b). An interesting feature is that direction of the non-axisymmetric structures precession is opposite to that of the bottom rotation.

Idea to perform such an experiment appeared while observing flow in the setup, similar to the one in Fig. 8i. Impression is that in flows, similar to ones in Figs. 14, 15, fluid particles are thrown apart from rotating disk and then return back due to the gravity force because the averaged over circumference fluid depth increases with the radius, making the bottom of the layer effectively concave.

2.6.3 Source flow in a circular gap

While studying rotating flows, one reasonable question appeared. What if, instead of driving the flow with some rotating object, we simply inject gas to the centre of the gap with circular outer edge, opened to the atmosphere? Experiments are performed in the geometry shown in Fig. 8n with the air injected either radially or with a certain tangential velocity component through the holes in the perforated cylinder placed to the centre of the gap. The cylinder with holes drilled either radially or with the certain inclination is used. When the air is injected radially, three stationary radial jets appear (Fig. 57a). When the air is injected with the angle about 20° to the radial direction, 2 to 3 unstable radial jets, precessing in the azimuthal direction, appear (Fig. 57b). When the angle of air injection increases to about 45°, i.e. at the radius of injection air has similar radial and
azimuthal velocities, the jet-like structures disappear and the flow becomes disordered (Fig. 57c).
Chapter 3

Three models of precessing structures and one model of the axisymmetric flow

In this chapter, three approaches to modelling of the non-axisymmetric structures described in Section 1.2 are presented together with an asymptotic model for the stationary viscous axisymmetric flow in a narrow gap. The first model (Section 3.1) is aimed to describe the origination of the non-axisymmetric perturbations by considering the linear stability of the two-dimensional viscous flow driven by an imposed force in the form of a $\delta$-function of radius and a friction term. Parameters of the neutrally stable modes are studied. The second model (Section 3.2) is a long wave approximation, applied to the two-dimensional ideal flow with the three layers of constant vorticity, embedded into the circular gap. Conditions under which the system is hyperbolic, are studied and the properties of discontinuities, propagating in the azimuthal direction, are discussed. The third model (Section 3.3) deals with the evolution of a system of decaying vortices in the infinite two-dimensional domain, filled with ideal fluid, with the positively and negatively signed vortices generated at different radii. The last Section 3.4 is devoted to the asymp-
totic model for the viscous three-dimensional stationary axisymmetric flow in a narrow gap. Velocity profiles, obtained with the help of this model, are compared with the ones experimentally measured in Section 2.2 and can be used for further theoretical analysis of the three-dimensional flow in a narrow gap.

3.1 Flow in the air gap at large Reynolds numbers. Darcy friction

One obvious way to model the viscous flow induced by a propeller in a narrow gap is to consider the flow as being two-dimensional, imitating the propeller by an imposed axisymmetric azimuthally-directed body force dependent on radius and imitating the influence of the gap walls by an added friction force. In this work, we chose the driving force to be localized at a certain radius and the friction force to have the form $-\varepsilon \vec{V}$, where $\vec{V}$ is the fluid velocity, so that in the absence of any external forcing the flow will exponentially decay with time in the manner the plain Poiseuille flow does. The term $-\varepsilon \vec{V}$ is commonly known as the Darcy friction term. The model described is relevant to the flow regimes where the momentum transport parallel and perpendicular to the gap walls are of the different nature, so that it is possible to choose the friction coefficient $\varepsilon$, responsible for the momentum diffusion in axial direction (across the gap), to be independent of either Newtonian or eddy viscosity $\nu$, responsible for the momentum diffusion in radial and azimuthal directions (parallel to the gap walls). An attempt to derive the model by some kind of asymptotic procedure failed, however, the approach looks reasonable and thus it was developed.
3.1.1 Basic equations

The system of equations, describing the flow of viscous fluid in the two-dimensional annular domain $D$ with imposed external force and the Darcy friction term, is

\begin{align}
  u_r + \frac{1}{r} u + \frac{1}{r} v_\varphi &= 0 \quad (3.1.1) \\
  u_t + uu_r + \frac{1}{r} vu_\varphi - \frac{v^2}{r} &= -p_r + \nu (\Delta u - \frac{u}{r^2} - \frac{2v_\varphi}{r^2}) - \varepsilon u + F_{\text{radial}} \quad (3.1.2) \\
  v_t + vv_r + \frac{1}{r} vu_\varphi + \frac{u v}{r} &= -\frac{1}{r} p_\varphi + \nu (\Delta u - \frac{v}{r^2} + \frac{2u_\varphi}{r^2}) - \varepsilon v + F_{\text{azim}} \quad (3.1.3) \\
  D &= \{a < r < b, \ 0 < \varphi < 2\pi\}, \ \nu > 0, \ \varepsilon > 0. \quad (3.1.4)
\end{align}

Here $r$ and $\varphi$ are radial and azimuthal coordinates, $u(r, \varphi)$ and $v(r, \varphi)$ are radial and azimuthal velocities, $p(r, \varphi)$ is pressure, $\nu$ is Newtonian viscosity. The terms $-\varepsilon u$ and $-\varepsilon v$ depict the fluid deceleration due to its friction against the gap walls, and $\mathbf{F} = (F_{\text{radial}}, F_{\text{azim}})$ is the imposed force that corresponds to a propeller, disk or any solid rotating in the centre of the gap. The friction coefficient $\varepsilon$ is defined from some empirical considerations or from the estimates to the order of magnitude. For example, $\varepsilon$ can be associated with the deceleration time of the plain Poiseuille flow of the fluid of viscosity $\nu$ in the gap of height $H$:

$$
\varepsilon \sim \frac{8\nu}{H^2} \quad (3.1.5)
$$

The boundary conditions are the non-slip conditions at $r = a$ and $r = b$. Not to lose generality, we put the inner boundary to be rotating with the angular velocity $\Omega_{in}$, so that

$$
  u(a, \varphi) = 0, \ v(a, \varphi) = \Omega_{in}a \quad \text{and} \quad u(b, \varphi) = 0, \ v(b, \varphi) = 0. \quad (3.1.6)
$$
3.1.2 Axisymmetric stationary solution

If the imposed force $F$ does not depend on $\varphi$, the problem (3.1.1)–(3.1.6) has the stationary axisymmetric solution:

\[
u = 0, \quad v = V_0(r), \quad p = p_0(r).
\] (3.1.7)

Assuming that the radial component of external force $F_{radial}$ is zero, to determine $V_0(r)$ we have from (3.1.3):

\[
u \frac{1}{r} \frac{d}{dr} \left( r \frac{dV_0}{dr} \right) - \nu \frac{V_0}{r^2} - \varepsilon V_0 + F_{azim} = 0.
\] (3.1.8)

The boundary conditions (3.1.6) now turn to

\[
V_0(a) = \Omega_in a,
\] (3.1.9)

\[
V_0(b) = 0.
\] (3.1.10)

We take the external force, imposed on the fluid, as

\[
F_{azim} = \Phi \delta(r - c),
\] (3.1.11)

where $\Phi$ is a constant and $\delta$ is Dirac delta-function. The form of imposed force is chosen in order to obtain a simple analytical solution for the axisymmetric flow. Such force can be associated with a propeller with short blades, suspended on the thin arms (as in Fig.8c, 53d). The solution of (3.1.8) is solution of the homogenous equation

\[
\frac{d^2V_0}{dr^2} + \frac{1}{r} \frac{dV_0}{dr} + \left( -\frac{\varepsilon}{\nu} - \frac{1}{r^2} \right) V_0 = 0 \quad \text{for} \quad a < r < c \quad \text{and} \quad c < r < b,
\] (3.1.12)

with the discontinuous first derivative $\frac{dV_0}{dr}$ at $r = c$. Performing the substitution $\rho = r \sqrt{\varepsilon/\nu}$ in (3.1.12), we obtain Bessel equation for $V_0$:

\[
\frac{d^2V_0}{d\rho^2} + \frac{1}{\rho} \frac{dV_0}{d\rho} - \left( 1 + \frac{1}{\rho^2} \right) V_0 = 0,
\] (3.1.13)
that has a solution in the form

\[ V_0 = C_{1i} \cdot I_1(\rho) + C_{2i} \cdot K_1(\rho), \quad \tilde{a} \leq \rho < \tilde{c} \]  

\[ V_0 = C_{1o} \cdot I_1(\rho) + C_{2o} \cdot K_1(\rho), \quad \tilde{c} < \rho \leq \tilde{b}. \]  

Here \( \tilde{a} = a \sqrt{\varepsilon/\nu}, \tilde{b} = b \sqrt{\varepsilon/\nu}, \tilde{c} = c \sqrt{\varepsilon/\nu}, I_1(\rho), K_1(\rho) \) are first and second kind Bessel functions of pure imaginary argument (Watson, 1944). Indices \( i \) and \( o \) stand for inner and outer intervals of \( r \). To match solutions (3.1.14) and (3.1.15), we are to provide continuity of \( V_0 \) at \( \rho = \tilde{c} \):

\[ V_0 \bigg|_{\rho=\tilde{c}-0} = V_0 \bigg|_{\rho=\tilde{c}+0}. \]  

Another matching condition is obtained from (3.1.8), multiplied by \( r \), by integration from \( c - 0 \) to \( c + 0 \), with the account for (3.1.11):

\[ \frac{\partial V_0}{\partial \rho} \bigg|_{\rho=\tilde{c}+0} - \frac{\partial V_0}{\partial \rho} \bigg|_{\rho=\tilde{c}-0} = -\Phi/\sqrt{\nu \varepsilon} = -\bar{\Phi} \]  

Boundary conditions (3.1.9), (3.1.10) and the matching conditions (3.1.16), (3.1.17)) give:

\[ C_{1i} I_1(\tilde{a}) - C_{2i} K_1(\tilde{a}) = \Omega_{in} a \]  

\[ C_{1o} I_1(\tilde{b}) + C_{2o} K_1(\tilde{b}) = 0 \]  

\[ C_{1i} I_1(\tilde{c}) + C_{2i} K_1(\tilde{c}) - C_{1o} I_1(\tilde{c}) - C_{2o} K_1(\tilde{c}) = 0 \]  

\[ C_{1o} \left( I_0(\tilde{c}) - \frac{I_1(\tilde{c})}{\tilde{c}} \right) + C_{2o} \left( -K_0(\tilde{c}) - \frac{K_1(\tilde{c})}{\tilde{c}} \right) \]
\[ -C_{1i} \left( I_0(\tilde{c}) - \frac{I_1(\tilde{c})}{\tilde{c}} \right) - C_{2i} \left( -K_0(\tilde{c}) - \frac{K_1(\tilde{c})}{\tilde{c}} \right) = -\bar{\Phi} \]  

Dividing (3.1.20) by \( \tilde{c} \) and subtracting it from (3.1.21), we obtain:

\[ C_{1i} I_0(\tilde{c}) - C_{2i} K_0(\tilde{c}) - C_{1o} I_0(\tilde{c}) + C_{2o} K_0(\tilde{c}) = \bar{\Phi}. \]  

\[ \text{(3.1.21)} \]
Denoting $I_1(\tilde{a}) = I_1^a$, $K_1(\tilde{a}) = K_1^a$, $I_1(\tilde{b}) = I_1^b$ etc., we have

\[
\begin{align*}
&I_1^a \quad K_1^a \quad 0 \quad 0 \quad \text{etc.}
\end{align*}
\]

Then,

\[
\begin{align*}
&\begin{pmatrix}
I_1^a & K_1^a & 0 & 0 \\
0 & 0 & I_1^b & K_1^b \\
I_1^c K_1^b & K_1^c K_1^b & -I_1^b K_1^b + I_1^c K_1^c & 0 \\
I_1^c K_1^b & -K_0^c K_1^b & -I_0^c K_1^b - I_1^c K_1^c & 0
\end{pmatrix}
\begin{pmatrix}
C_{1i} \\
C_{2i} \\
C_{1o} \\
C_{2o}
\end{pmatrix}
= \begin{pmatrix}
\Omega_{in} a \\
0 \\
0 \\
\tilde{\Phi}
\end{pmatrix}
\]
where

\[
\begin{align*}
a_{11} &= (K^a_1 I^a_1 - K^a_0 I^a_0) K^b_1 \\
a_{12} &= (-I^c_1 K^b_1 + I^b_1 K^c_1) I^a_1 \\
a_{21} &= (-K^a_0 I^a_1 - K^a_1 I^a_0) K^b_1 \\
a_{22} &= (-I^b_0 K^b_1 - I^b_1 K^b_0) I^a_1 \\
b_1 &= -I^c_1 K^b_1 \Omega_{in} a \\
b_2 &= -I^c_0 K^b_1 \Omega_{in} a + K^b_1 I^a_1 \Phi \\
D &= (K^a_1 I^a_1 - K^a_0 I^a_0)(-I^c_0 K^b_1 - I^b_1 K^c_1) - (K^a_0 I^a_1 - K^a_1 I^a_0)(-I^c_0 K^b_1 + I^b_1 K^c_1) \\
&= -(K^a_1 I^a_0 + I^a_1 K^c_0)(K^a_1 I^b_1 + I^a_1 K^b_1) \quad (3.1.27)
\end{align*}
\]

Thus, the Green’s function of the boundary value problem (3.1.8)-(3.1.10) is obtained.

### 3.1.3 Linearized problem

Since the two-dimensional incompressible flow is considered, we introduce the stream function \( \psi(r)e^{im\varphi}e^{\sigma t} \), so that

\[
\begin{align*}
v &= V_0(r) + \psi'(r)e^{im\varphi}e^{\sigma t}, \
u &= -\frac{im}{r}\psi(r)e^{im\varphi}e^{\sigma t}, \
p &= p_0(r) + q(r, \varphi, t).
\end{align*}
\quad (3.1.28)
\]

Now we assume that the non-axisymmetric perturbation is small, i.e.

\[
\begin{align*}
v - V_0 &= o(1), \
u &= o(1), \
q &= o(1) \quad \text{and} \quad V_0 = O(1).
\end{align*}
\quad (3.1.29)
\]
Substituting (3.1.28) into (3.1.2) and (3.1.3), cancelling zeroth order terms due to (3.1.8) and omitting higher (than first) order terms, we obtain linearized equations:

\[
\psi'''' \nu r^4 + \psi'''(2r^2 V_0) + \\
+ \psi''(-m_\sigma r^2 + m^2 r V_0 - i m^3 - i \varepsilon m r^2) + \frac{r^3 \partial q}{\rho \partial r} e^{-im\varphi} e^{-\sigma t} = 0 \quad (3.1.30)
\]

\[
\psi'''' \nu r^3 + \psi''' \nu r^2 - \psi' (\sigma r^3 + imr^2 V_0 + nm^2 r + \nu r + \varepsilon r^3) - \\
- \psi(-imr^2 V_0r - 2\nu m^2 - imr V_0) - \frac{r^2 \partial q}{\rho \partial \varphi} e^{-im\varphi} e^{-\sigma t} = 0 \quad (3.1.31)
\]

Eliminating the pressure perturbation \( q \), we have

\[
\psi'''' \nu r^4 + \psi''' 2\nu r^3 + \\
+ \psi'' (-2\nu m^2 r^2 - \nu r^2 - \varepsilon r^4 - \sigma r^4 - im^3 V_0) + \\
+ \psi' (-\varepsilon r^3 + \nu r + 2\nu m^2 r - im^2 V_0 - \sigma r^3) + \\
+ \psi (-4\nu m^2 + \varepsilon m^2 r^2 + \nu m^4 + im^3 r V_0 - imr V_0 + im^2 V_0r + im^3 V_{0rr} + m^2 \sigma r^2) = 0 \quad (3.1.32)
\]

Taking into account expression (3.1.12) for \( V_0 \) and expression (3.1.11) for \( F_{azim} \), we obtain the following equation for \( \psi(r) \) (that is similar to the Orr-Sommerfeld equation):

\[
\psi'''' = \\
- \psi'' \left( \frac{2}{r} \right) + \\
+ \psi'' \left( \frac{2m^2}{r^2} + \frac{1}{r^2} + \frac{\varepsilon}{\nu} + \frac{\sigma}{\nu} + \frac{im}{r\nu} V_0 \right) + \\
+ \psi' \left( \frac{\varepsilon}{\nu r} - \frac{1}{r^3} + \frac{2m^2}{r^3} + \frac{im}{r^2 \nu} V_0 + \frac{\sigma}{\nu r} \right) + \\
+ \psi \left( \frac{4m^2}{r^4} - \frac{\varepsilon m^2}{\nu r^2} - \frac{m^4}{r^4} - \frac{im^3}{\nu r^3} V_0 - \frac{im\varepsilon}{\nu^2 r} V_0 - \frac{m^2 \sigma}{\nu r^2} \right) + \psi \frac{im}{\nu^2 r^2} \Phi \delta(r - c). \quad (3.1.33)
\]
We solve this equation using the shooting procedure. Two intervals are considered: inner \((a \leq r \leq c)\) and outer \((c \leq r \leq b)\). Boundary conditions for inner and outer solutions follow from the non-slip conditions \((3.1.6)\):

\[
\psi(a) = 0, \quad \psi'(a) = 0, \tag{3.1.34}
\]

\[
\psi(b) = 0, \quad \psi'(b) = 0. \tag{3.1.35}
\]

Matching of the inner and the outer solutions at \(r = c\) requires

\[
\psi\bigg|_{r=c-0} = \psi\bigg|_{r=c+0}, \quad \psi'\bigg|_{r=c-0} = \psi'\bigg|_{r=c+0}, \quad \psi''\bigg|_{r=c-0} = \psi''\bigg|_{r=c+0}. \tag{3.1.36}
\]

Integration of \((3.1.33)\) from \(c - 0\) to \(c + 0\) gives the discontinuity of \(\psi''(r)\) at \(r = c\):

\[
\psi''\bigg|_{r=c+0} - \psi''\bigg|_{r=c-0} = \frac{im}{\nu^2 c} \Phi \psi\bigg|_{r=c} = i\gamma \psi\bigg|_{r=c}, \quad \gamma = \frac{m\Phi}{\nu^2 c}. \tag{3.1.37}
\]

Here \(\gamma\) is a real-valued constant.

### 3.1.4 Linear shooting procedure

To find the spectrum (the values of \(\sigma\)) of the linear problem \((3.1.33)-(3.1.35)\), we use the shooting procedure. Denote solutions of \((3.1.33)\), corresponding to initial conditions

\[
\begin{pmatrix}
\psi(a) \\
\psi'(a) \\
\psi''(a) \\
\psi'''(a)
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
\psi(b) \\
\psi'(b) \\
\psi''(b) \\
\psi'''(b)
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \tag{3.1.38}
\]

as \(\psi_1(r), \psi_2(r)\) and \(\psi_1'(r), \psi_2'(r)\) respectively. Solutions with index ‘-’ are valid for \(a \leq r \leq c\), and solutions with index ‘+’ are valid for \(c \leq r \leq b\). We look for solution of \((3.1.33)-(3.1.37)\) in the form

\[
\psi = \alpha_1 \psi_1 - \alpha_2 \psi_2, \quad a \leq r \leq c,
\]

\[
\psi = \beta_1 \psi_1 + \beta_2 \psi_2, \quad c \leq r \leq b. \tag{3.1.39}
\]
The function $\psi(r)$ (3.1.39) satisfies (3.1.34) and (3.1.35) automatically. Now we are to choose coefficients $\alpha_1, \alpha_2, \beta_1, \beta_2$ to satisfy matching conditions (3.1.36) and (3.1.37).

Denote

$$
\begin{pmatrix}
\Psi_1 - \Psi_2 - \Psi_1' - \Psi_2' - \Psi_1'' - \Psi_2'' - \Psi_1''' - \Psi_2''' \\
\Psi_1' - \Psi_2' - \Psi_1'' - \Psi_2'' - \Psi_1''' - \Psi_2''' \\
\Psi_1'' - \Psi_2'' - \Psi_1''' - \Psi_2''' \\
\Psi_1''' + i\gamma \Psi_1 - \Psi_2''' + i\gamma \Psi_2 - \Psi_1''' - \Psi_2'''
\end{pmatrix} =
\begin{pmatrix}
\psi - (c) \psi' - (c) \\
\psi' - (c) \\
\psi'' - (c) \\
\psi''' - (c)
\end{pmatrix},
$$

$$
\begin{pmatrix}
\Psi_1 + \Psi_2 + \Psi_1' + \Psi_2' + \Psi_1'' + \Psi_2'' + \Psi_1''' + \Psi_2'''
\Psi_1' + \Psi_2' + \Psi_1'' + \Psi_2'' + \Psi_1''' + \Psi_2'''
\Psi_1'' + \Psi_2'' + \Psi_1''' + \Psi_2'''
\Psi_1''' + \Psi_2''' + \Psi_1''' + \Psi_2'''
\end{pmatrix} =
\begin{pmatrix}
\psi + (c) \psi' + (c) \\
\psi' + (c) \\
\psi'' + (c) \\
\psi''' + (c)
\end{pmatrix}.
$$

Conditions (3.1.36) and (3.1.37) yield the complex-valued system of the linear algebraic equations for $\alpha_1, \alpha_2, \beta_1, \beta_2$:

$$
\begin{pmatrix}
\Psi_1 - \Psi_2 - \Psi_1' - \Psi_2' - \Psi_1'' - \Psi_2'' - \Psi_1''' - \Psi_2''' \\
\Psi_1' - \Psi_2' - \Psi_1'' - \Psi_2'' - \Psi_1''' - \Psi_2''' \\
\Psi_1'' - \Psi_2'' - \Psi_1''' - \Psi_2''' \\
\Psi_1''' + i\gamma \Psi_1 - \Psi_2''' + i\gamma \Psi_2 - \Psi_1''' - \Psi_2'''
\end{pmatrix} \cdot
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\beta_1 \\
\beta_2
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}.
$$

The system (3.1.41) has a non-zero solution if and only if the complex-valued determinant of the matrix in the left hand side of (3.1.41) vanishes:

$$
\det \Lambda = 0, \quad \text{where} \quad \Lambda =
\begin{pmatrix}
\Psi_1 - \Psi_2 - \Psi_1' - \Psi_2' - \Psi_1'' - \Psi_2'' - \Psi_1''' - \Psi_2''' \\
\Psi_1' - \Psi_2' - \Psi_1'' - \Psi_2'' - \Psi_1''' - \Psi_2''' \\
\Psi_1'' - \Psi_2'' - \Psi_1''' - \Psi_2''' \\
\Psi_1''' + i\gamma \Psi_1 - \Psi_2''' + i\gamma \Psi_2 - \Psi_1''' - \Psi_2'''
\end{pmatrix}.
$$

Vanishing of real and imaginary parts of $\Lambda$ gives two conditions on real and imaginary parts of $\sigma$ in (3.1.28).

### 3.1.5 Numerical realization

To solve the problem (3.1.33), the Runge-Kutta-Fehlberg method is applied to solve the initial value problem for the vector $(Z_1, Z_2, Z_3, Z_4)$, correspondent to $(\psi, \psi', \psi'', \psi''')$ for
the intervals $a \leq r \leq c$ and $c \leq r \leq b$ and obeying the following system of differential equations:

$$
\frac{dZ_1}{dr} = Z_2 \\
\frac{dZ_2}{dr} = Z_3 \\
\frac{dZ_3}{dr} = Z_4 \\
\frac{dZ_4}{dr} = -Z_4 \left( \frac{2}{r} \right)
$$

(3.1.43)

The term with $\delta$-function is responsible for the discontinuity of $Z_4$ at $r = c$. To optimize the equation (3.1.41) for numerical investigation in the case when the ratio $b/a$ is significantly greater than one, two changes of variables are tested.

First, we write the linear problem (3.1.33) in the form

$$
(\sigma + \varepsilon)\Delta_m \psi - \nu \Delta_m^2 \psi + \frac{im}{\nu r} \left\{ V_0 \Delta_m \psi - \psi \frac{1}{r} \frac{d}{dr} \frac{1}{r} \frac{d}{dr} rV_0 \right\} = 0,
$$

(3.1.44)

where the stream function $\psi$ is introduced in accordance with (3.1.28) and

$$
\Delta_m = \frac{1}{r} \frac{d}{dr} r \frac{d}{dr} - \frac{m^2}{r^2}.
$$

(3.1.45)

Recalling the equation for $V_0$ (3.1.8) and the particular form of the forcing term $F$ (3.1.11), we have

$$
(\sigma + \varepsilon)\Delta_m \psi - \nu \Delta_m^2 \psi + \frac{im}{\nu r} \left\{ V_0 \Delta_m \psi - \frac{\varepsilon V_0}{\nu} \psi \right\} + \frac{im}{\nu r} \Phi \delta(r - c) = 0.
$$

(3.1.46)
The equation (3.1.33) contains terms with the high negative powers (down to \( r^{-4} \)), that will affect the calculation precision when \( b/a \geq 1 \). So, we introduce the vector \((Z_1, Z_2, Z_3, Z_4)\), that looks to be more natural for the solution of (3.1.46):

\[
\begin{align*}
\psi &= Z_1, \\
rd\psi/dr &= Z_2, \\
\frac{1}{r}d\frac{1}{r}d\psi/dr &= Z_3, \\
r \frac{1}{dr} \frac{d}{dr}^{3} \frac{d\psi}{dr} &= Z_4.
\end{align*}
\] (3.1.47)

Now, equations for \((Z_1, Z_2, Z_3, Z_4)\) are

\[
\begin{align*}
\frac{dZ_1}{dr} &= \frac{1}{r}Z_2 \\
\frac{dZ_2}{dr} &= rZ_3 \\
\frac{dZ_3}{dr} &= \frac{1}{r}Z_4 \\
\frac{dZ_4}{dr} &= \left( \frac{imV_0}{\nu} + \frac{\sigma r}{\nu} + \frac{\varepsilon r}{\nu} + \frac{2m}{r} \right) Z_3 - \left( \frac{4m^2}{r^3} \right) Z_2 \\
&\quad + \left( \frac{4m^2}{r^3} - \frac{m^4}{r^3} - \frac{im^3V_0}{\nu r^2} - \frac{\sigma m^2}{\nu r} - \frac{im\varepsilon V_0}{\nu^2} \right) Z_1 + \frac{im}{\nu^2} \Phi \delta(r - c) Z_1.
\end{align*}
\] (3.1.48)

The second way to transform the equations (3.1.33) is to introduce the variable \( x = \ln r \). Now \( \frac{d}{dx} = r \frac{d}{dr} \), and for the vector \((Z_1(x), Z_2(x), Z_3(x), Z_4(x)) = (\psi, \psi', \psi'', \psi''')\),
where the differentiation is performed by \( x \), we have

\[
\frac{dZ_1}{dx} = Z_2
\]
\[
\frac{dZ_2}{dx} = Z_3
\]
\[
\frac{dZ_3}{dx} = Z_4
\]
\[
\frac{dZ_4}{dx} = 4Z_4 + \left(2m^2 - 4 + \frac{1}{\nu} e^{2x}(\sigma + \varepsilon) - \frac{imV_0}{\nu} e^x\right) Z_3 - 4m^2Z_2
\]
\[+ \left(4m^2 - m^4 - \frac{1}{\nu} e^{2x}(\sigma + \varepsilon) + \frac{im^3}{\nu} e^x - \frac{im\varepsilon V_0}{\nu^2} e^{3x}\right) Z_1\]
\[+ \frac{im}{\nu^2} e^{2x} \Phi \frac{1}{c} \delta(x - \ln(c)) Z_1.\]

While solving the initial value problem (3.1.33), (3.1.38) with the Runge-Kutta-Fehlberg method, values of \( \psi \) and its derivatives change in order of magnitude significantly (tens of orders). Arising numerical errors spoil the calculation. To avoid the growth of numerical error, Gramm-Schmidt orthogonalization procedure is applied during the shooting procedure to the pair of vectors \((\psi_1, \psi'_1, \psi''_1, \psi'''_1)\) and \((\psi_2, \psi'_2, \psi''_2, \psi'''_2)\), corresponding to the orthogonal initial values (3.1.38) at every step of the Runge-Kutta-Fehlberg procedure. In particular, vector \((\psi_1, \psi'_1, \psi''_1, \psi'''_1)\) is replaced with itself, divided by its norm, and the vector \((\psi_2, \psi'_2, \psi''_2, \psi'''_2)\) is replaced with its normalized component, perpendicular to \((\psi_1, \psi'_1, \psi''_1, \psi'''_1)\). While changing the absolute value of \( \det \Lambda \) in (3.1.42), described orthogonalization procedure does not change the sign of real and imaginary parts of the determinant. First, one column of the matrix (3.1.42) is divided by its norm (positive real number), that does not change the sign of the determinant real or imaginary part, then one column is subtracted from another with the certain weight, that does not change the determinant, then the second column is divided by its norm, that again does not change the signs of real and imaginary parts. Application of the orthogonalization procedure
makes dependence of $\Lambda$ in (3.1.42) on $\sigma$ non-smooth and thus it makes impossible the usage of Newton method to solve the two-dimensional equation (3.1.42). So, a simple algorithm is elaborated to find the values $\text{Re}(\sigma)$ and $\text{Im}(\sigma)$, where the lines, along which $\text{Re}(|\det \Lambda|)$ and $\text{Im}(|\det \Lambda|)$ change the sign, intersects. The typical pattern of the such lines is shown in Fig. 58a. To locate the point of intersection more exactly, the following iteration is repeated several times. Signs of $\text{Re}(|\det \Lambda|)$ and $\text{Im}(|\det \Lambda|)$ are calculated in the set of (say, 17) points uniformly distributed at certain distance from some point $\sigma_0$, located near the lines intersection (Fig. 58b). Then the middle of the segment, connecting neighboring points where $\text{Re}(|\det \Lambda|)$ has different signs is connected with another such point. The same procedure is performed with $\text{Im}(|\det \Lambda|)$. Intersection of the two segments constructed is assigned to be $\sigma_1$. Then the procedure is repeated with circle of smaller radius and the centre at $\sigma_1$ to obtain better approximation.

Similar procedure is used to find the eigenvalues by varying other pairs of parameters, different from $(\text{Re}(\sigma), \text{Im}(\sigma))$. In particular, pairs $(\nu, \text{Im}(\sigma))$, $(\nu, m)$, $(m, \text{Im}(\sigma))$, $(\Phi, \nu)$ and $(\Phi, \text{Im}(\sigma))$ are used. Ellipses with the appropriately chosen axis are used instead of the circles where characteristic scales of parameters differs.

As it follows from Fig. 58, search for the eigenvalues is rather time-consuming procedure if no prior information on the parameters is available. So, for a certain parameter set, the point of intersection has been found by extensive plotting of patterns similar to the one shown in the figure, and to find the eigenvalues for the various parameters sets, their evolution in terms of two chosen variables is traced while gradually changing other parameters. Say, to find the set of parameters, for which the neutral mode $\text{Re}(\sigma) = 0$ exists, knowing the pair shown in Fig. 58, we change $\text{Re}(\sigma)$ from 0.3 to 0, observing
evolution of $m$ and $\text{Im}(\sigma)$ while other parameters fixed. A number of tricks is applied to save the computation time. Denote the pair of parameters, being adjusted to satisfy $\det \Lambda = 0$ (3.1.42), as $\vec{q}$ and the set of other parameters, describing the system as $\vec{P} = (a, c, b, \ldots)$. When changing $\vec{P}$ from $\vec{P}_{\text{begin}}$ to $\vec{P}_{\text{end}}$, parametrization $\vec{P}(\alpha)$ is in use, so that

$$\vec{P}_i = (1 - \alpha)\vec{P}_{\text{begin}} + \alpha\vec{P}_{\text{end}}. \quad (3.1.50)$$

While changing $\alpha$ from 0 to 1, we look for the value of $\vec{q}_{i+1}$ in vicinity of the point

$$\vec{q}_{i+1} \approx \vec{q}_i + \frac{\alpha_{i+1} - \alpha_i}{\alpha_i - \alpha_{i-1}}(\vec{q}_i - \vec{q}_{i-1}). \quad (3.1.51)$$

In other words, we linearly extrapolate the change of $\vec{q}$ with the change of $\alpha$. The use of higher order approximations did not appear to be fruitful. The step $(\alpha_{i+1} - \alpha_i)$ is a variable: it is increased ($\sqrt{2}$ times) after two successful attempts to find the eigenvalue $q_{i+1}$ in certain vicinity $\delta \vec{q}$ of (3.1.51), and it is decreased ($\sqrt{2}$ times) after one unsuccessful attempt.

Varying the flow parameters, while keeping $\text{Re}(\sigma) = 0$, a set of neutral curves is constructed.

### 3.1.6 Neutral curves and parameters of some neutral modes

Consider a system resembling our experimental setup, similar to the one in Fig.8c, i.e. $a = 1 \text{ cm}$, $c = 5 \text{ cm}$, $b = 20 \text{ cm}$. Fortunately, dimensional values are of order of 1, so we solve the problem keeping the dimensional values. To start with, some typical values of viscosity $\nu$ and friction coefficient $\varepsilon$ are to be defined from some empiric observations. We choose $\varepsilon \sim 1 \text{ s}^{-1}$, that corresponds to the decay time of 1 s of the non-forced flow. It can be associated with the decay time of the air (viscosity $0.15 \text{ cm}^2\text{s}^{-1}$) flow in the gap
of approximately 1 cm height (compare with (3.1.5). The amplitude of driving force $\Phi$ is chosen to be $500 \, \text{cm} \, \text{s}^{-2}$, that corresponds to rotation of the single blade (as in Fig.8c) with the radial width of 1 cm with the speed about 360 rpm. The neutral curve (the line correspondent to $\text{Re}(\sigma) = 0$) is plotted in Fig.59 for $a = 1$, $c = 5$, $b = 20$, $\Phi = 500$, $\varepsilon = 1$. The Reynolds number is chosen as

$$\text{Re} = \frac{V_{\text{max}} c}{\nu}.$$  

(3.1.52)

In spite of restriction for $m$ to be an integer not less than one, wider range is used. Physically, it corresponds to the flow without azimuthal periodicity and corresponds to some kind of spiral domain (as in Fig.75c), where $2\pi$-periodicity is not required and perturbations with any azimuthal wave-numbers are allowed to exist. Consideration of $m$ as the continuous parameter makes it possible to compare the general features of the neutral curves with the ones obtained by other authors for the flow in a straight channel (Lin (1955), Betchov and Criminale (1967), Drazin and Reid (1981)).

Four different types of neutrally stable perturbation modes are observed for the Reynolds numbers higher than $\text{Re} \sim 1400$.

To compare results obtained with the experiments in the air gap, ratio of the non-axisymmetric structure precession rate to the maximum angular velocity of the fluid in the domain is introduced:

$$\frac{\omega}{\Omega} = \frac{\text{Im}(\sigma)/m}{V_{\text{max}}/c}.$$  

(3.1.53)

The ratio $\omega/\Omega$ is shown in the Fig.60 by the vertical bars, starting at the points on the neutral curve. The segment, corresponding to $\omega = \Omega$, is plotted for reference. In the experimentally observed flows $\omega/\Omega$ varies in the range from 0.001 to 0.1.
Stream function perturbations $\psi(r)$ from (3.1.28), corresponding to the points, marked by numbers in Fig. 59, are presented in Fig. 61. The absolute value of the complex function $\psi(r)$ from (3.1.28) is plotted. The perturbations are somehow divided into ones localized near the radius where the force is applied ($r = c$) and the ones spanning the whole gap. The first type of perturbations corresponds to the bearing-shaped formations, similar to the ones observed by Rabaud and Couder (Fig. 4b). The latter type of perturbations can be responsible for the formation of the jet-like structures, which extend to the outer area of the domain, as in Fig. 13b-f.

To visualize the neutral modes obtained, typical contour plots of $\text{Re}(\psi(r)e^{im\varphi})$ from the equation (3.1.28) are shown in Fig. 62a, b. To plot resulting flow patterns, these perturbations are imposed on a basic flow $v = V_0(r)$ with some amplitude $\alpha$. The flow is observed from the frame of reference, rotating with the rate of the non-axisymmetric pattern precession, so that perceived pattern corresponds to

$$
\psi_{\text{visual}}(r, \varphi) = \int_a^r V_0(\xi)d\xi + \alpha\text{Re} \left( \psi(r)e^{im\varphi} \right) + \frac{1}{2} \frac{\text{Im}(\sigma)}{m} r^2.
$$

(3.1.54)

Here the first term gives the stream function of the basic flow, the second term is the perturbation, the third term corresponds to the rigid body rotation with the angular speed $\text{Im}(\sigma)/m$. The contour lines of $\psi_{\text{visual}}$ are plotted in Fig. 62c, d. Patterns, resembling the ones experimentally observed, can be spotted. Even the intuitive difference between the forwards (a) and backwards (b) precessing structures exists (compare with the forwards precessing structure in Fig. 20f and the backwards precessing structure in Fig. 15a or the forwards and backwards precessing structures in Fig. 34). However, the visualization procedure performed is not quite legal, because the amplitude $\alpha$ is chosen to be large enough for the velocity perturbation to be comparable to the velocity in the basic flow, that
violates the condition (3.1.29).

Up to this point, to plot the neutral curves, viscosity $\nu$ is varied to achieve different values of Reynolds number (3.1.52). However, variation of $\nu$ affects maximum azimuthal velocity in the basic flow $V_{0_{\text{max}}}$, and the choice of Reynolds number as a coordinate for plotting the neutral curves does not seem to be natural when $\nu$ is not a constant.

Now, we plot the neutral curves by varying the amplitude of imposed force $\Phi$ in (3.1.11) for different friction coefficients $\varepsilon$. It looks to be more appropriate because the maximum velocity $V_{0_{\text{max}}}$ and, thus, the Reynolds number, are proportional to $\Phi$. With increase of the friction coefficient $\varepsilon$, increase of the minimum forcing amplitude $\Phi$ and the minimum Reynolds number $Re$, at which the neutral mode occurs, is observed (Fig. 63). It might not be clearly visible, but $Re_{\text{min}}(\varepsilon = 0.001) < Re_{\text{min}}(\varepsilon = 0.01)$ and the minimum Reynolds number, at which the non-axisymmetric pattern appears, is around 26. The linear fits imposed on the logarithmic-scaled graph $Re_{\text{min}}(\varepsilon)$ and $\Phi_{\text{min}}(\varepsilon)$ (Fig. 64) show that, when the friction is not too small, minimum Reynolds number is proportional to the square root of the friction coefficient $\varepsilon$ and the minimum forcing amplitude is proportional to $\varepsilon$. To compare this result with the experiment, we notice that the friction $\varepsilon$ can be associated with the gap height $H$ (3.1.5). Values of $H$ for the gap filled with air ($\nu = 0.15 \text{ cm}^2\text{s}^{-1}$) are plotted above the graph for reference. For the turbulent flow, $\Phi \propto \rho \Omega^2$ and thus $\Omega_{\text{min}} \propto H$, that can be experimentally tested by plotting several sets of graphs similar to Fig. 26, 27 and following evolution of the point of lines intersection while changing $H$. Precession rate of the non-axisymmetric structures appearing at minimum $Re$ is also plotted versus $\varepsilon$ in Fig. 64. Discussing dependence $\frac{\omega}{\Omega}$ ($\varepsilon$), we note that the non-axisymmetric structure precession rate increases with $\varepsilon$, that is somehow in agreement with the experiment: both
\( \frac{\omega}{\Omega} \) given by the model (diamonds in Fig.64) and \( \frac{d\omega}{d\Omega} \) obtained experimentally (Fig.28) increases with the decrease of \( H \).

A reasonable test to perform is to study dependence of the neutral mode precession rate as a function of geometric parameters, i.e. the inner domain radius \( a \), radius where the force is imposed \( c \) and the outer domain radius \( b \). Graphs of the perturbation precession rate \( \omega = \text{Im}(\sigma)/m \) divided by the maximum angular velocity of the fluid in the basic flow \( \Omega = V_{\text{max}}/c \) versus geometric parameters is presented in Fig.65. We observe that, as it should appear from general reasoning, the ratio \( \omega/\Omega \) does not have a singularity when the inner radius \( a \) goes to zero and is almost independent of the outer radius \( b \) when it is large enough. Negative \( w \) appearing for certain values of \( c \) can be associated with the structures mentioned by Rabaud and Couder (Fig.4d).

To compare results obtained for the non-axisymmetric structure precession speed with the ones obtained experimentally, we plot its dependence on the maximum angular velocity of the fluid in a basic flow \( \omega(\Omega) \) for different values of the outer radius \( b \) in Fig.66. To plot the neutral curve, radius \( c \), at which the external force is applied, is changed. This method looks reasonable, because \( c \) is not defined from the physical reasoning in the turbulent flow, driven by a propeller. Vaguely linear dependence \( \omega(\Omega) \) is observed. Similarly to what is observed experimentally, non-axisymmetric structure precession rate decreases when the outer radius increases (see Fig.66b and Fig.29). No explicit comparison with the experimental data can be done, because in experiments outer edge of the gap is open to atmosphere, while in the model the non-slip conditions are imposed at the outer radius \( b \).

Summarizing said in this section, we see that some properties of the non-axisymmetric
perturbations arising in the two-dimensional model with friction term and external forcing are similar to the ones of the experimentally observed jet-like structures.

3.2 Layers of constant vorticity in the long wave approximation

The shallow water model is developed for the evolution of the layers of constant vorticity embedded between two cylinders, similarly to the model constructed by Lyapidevskii in 1994 and Ovsiannikov et al (1985). The model is based on the Euler equations for the two-dimensional fluid flow. Non-permeability conditions are imposed at the inner and outer boundaries. The axial component of vorticity is assumed to be piecewise-constant function of radius (Fig.67a). In the long wave approximation, vorticity depends on the azimuthal velocity only and thus the azimuthal velocity is a piecewise-differentiable function of radius. Under these assumptions, velocity profiles are $\gamma/r$ in the areas of zero vorticity and $\alpha r + \beta/r$ in the areas of constant non-zero vorticity, where $\alpha$, $\beta$, $\gamma$ are constants.

The system of quasilinear hyperbolic equations is obtained to describe behaviour of the boundaries of the layer of constant vorticity. The model allows to define velocity of the discontinuities propagation in the azimuthal direction, that can be associated with the rate of jets precession.
3.2.1 Basic equations

The two-dimensional Euler equations and the continuity equation have the following non-dimensional form:

\[
U_t + \vec{V} \cdot \nabla U - \frac{V^2}{r} = -P_r
\]

\[
V_t + \vec{V} \cdot \nabla V + \frac{UV}{r} = -\frac{1}{r}P_\varphi
\]  

(3.2.1)

\[
\frac{1}{r}(rU)_r + \frac{1}{r}V_\varphi = 0
\]  

(3.2.2)

Here \(\vec{V} = (U, V)\) is the fluid velocity, \(P\) is the pressure and the operators \(\vec{V} \cdot \nabla\) and \(\Delta\) are defined as

\[
\vec{V} \cdot \nabla = U \frac{\partial}{\partial r} + \frac{V}{r} \frac{\partial}{\partial \varphi}
\]

\[
\Delta = \frac{1}{r} \frac{\partial}{\partial r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2}.
\]  

(3.2.3)

The flow domain \(D\) and the boundary conditions are defined as

\[
D = \{a < r < b\}
\]

\[
U = 0 \text{ at } r = a, \ r = b.
\]  

(3.2.4)

Note that no \(2\pi\)-periodicity condition is implied.

\(, a, b\) are the radii of inner and outer cylinders. Dimensional (marked with the asterisk) and dimensionless parameters are related as follows:

\[
t = \Omega_* t_s, \ r_* = R_* r,
\]

\[
(U^*, V^*) = \Omega_* R_* (U, V), \ P_* = \rho_* \Omega_*^2 R_*^2 P
\]

\[
\nu = \frac{\nu_*}{R_*^2 \Omega_*}, \ a = \frac{R^\text{in}_*}{R_*}, \ b = \frac{R^\text{out}_*}{R_*},
\]  

(3.2.5)

Here \(\Omega_*, R_*, R^\text{in}_*, R^\text{out}_*, \rho_*, \nu_*\) are dimensional values corresponding to angular velocity, radial dimension, inner radius, outer radius, fluid density and kinematic viscosity.
3.2.2 Asymptotic model

To construct the asymptotic model, the multi-scale expansion method is applied. Introduce the parameter $h \to 0$. Introduce ‘slow’ variables $\varphi$ and $\tau$ as

$$\tau = ht, \; \phi = h\varphi. \quad (3.2.6)$$

The essence of substitution (3.2.6) is compression of the azimuthal angle. It leads to the model, describing local (soliton-like) azimuthally propagating disturbances. We look for the solution of (3.2.1), (3.2.2) in the form

$$U = hu(r, \phi, \tau) + O(h^2),$$

$$V = v(r, \phi, \tau) + O(h),$$

$$P = p(r, \phi, \tau) + O(h) \quad (3.2.7)$$

Substituting (3.2.7) into (3.2.1), (3.2.2) and keeping the main terms only, we get the following asymptotic model:

$$-\frac{v^2}{r} = -p_r,$$

$$v_r + uv_r + \frac{1}{r}vv_\phi + \frac{uv}{r} = -\frac{1}{r^2}p_\phi,$$

$$(ru)_r + v_\phi = 0 \quad (3.2.8)$$

In the approximation (3.2.6), (3.2.7), vorticity has the form

$$\text{curl} \, \vec{v} = \frac{1}{r} (rv)_r \, \vec{e}_z + O(h), \quad (3.2.9)$$

so that after eliminating the pressure and omitting the terms $O(h)$, equations (3.2.8) can be written as

$$\frac{d}{d\tau} |\text{curl} \, \vec{v}| = 0, \; \text{where} \; \frac{d}{dt} (ru) + u( )_r + \frac{1}{r}v( )_\phi = 0 \quad (3.2.10)$$

$$(ru)_r + v_\phi = 0. \quad (3.2.11)$$
This is the two-dimensional model of the time-dependent flow in the \((r, \phi)\) plane.

### 3.2.3 Main assumptions and relations

As it follows from Euler equations, in a two-dimensional flow the vorticity remains constant for the liquid particle moving with the fluid, i.e. vorticity is constant in Lagrangian coordinates. This is the main idea of further consideration. Following Liapidevskii, consider three non-mixing areas as in Fig. 67a. We require

\[
|\text{curl} \mathbf{v}(r, \phi, \tau)| = \begin{cases} 
0 & a < r < A(\phi, \tau) \\
\Omega_0 & A(\phi, \tau) < r < B(\phi, \tau) \\
0 & B(\phi, \tau) < r < b 
\end{cases} \tag{3.2.12}
\]

In other words, vorticity is a non-zero constant for \(A(\phi, \tau) < r < B(\phi, \tau)\) and the flow is irrotational for \(a < r < A(\phi, \tau)\), and \(B(\phi, \tau) < r < b\). Boundaries of the vortex layer \(r = A(\phi, \tau), r = B(\phi, \tau)\) are assumed to be unknown, vorticity \(\Omega_0\) is assumed to be given. Recalling the expression for vorticity (3.2.9), we have the following expressions for \(v(r, \phi, \tau)\):

\[
v(r, \phi, \tau) = \begin{cases} 
c_1(\phi, \tau) r^{-1} & a < r < A(\phi, \tau) \\
\frac{1}{2}\Omega_0 r + c_0(\phi, \tau) r^{-1} & A(\phi, \tau) < r < B(\phi, \tau) \\
c_2(\phi, \tau) r^{-1} & B(\phi, \tau) < r < b 
\end{cases} \tag{3.2.13}
\]

Here \(c_0, c_1, c_2\) are functions of \(\phi\) and \(t\). Now we require \(v(r, \phi, t)\) to be continuous at \(r = A(\phi, \tau)\) to obtain

\[
v(r, \phi, \tau) = \begin{cases} 
c_1 r^{-1} & a < r < A(\phi, \tau) \\
\frac{1}{2}\Omega_0 r + \left(c_1 - \frac{1}{2}\Omega_0 A^2\right) r^{-1} & A(\phi, \tau) < r < B(\phi, \tau) \\
c_2 r^{-1} & B(\phi, \tau) < r < b 
\end{cases} \tag{3.2.14}
\]

The structure of the azimuthal velocity (3.2.14) can be associated with the velocity profiles measured experimentally near the gap middle-plane either in the axisymmetric case (Figs. 40, 41, 42) or in the regime with jets (Figs. 43, 44). It is worth noticing, that in both vortex and irrotational area the profile (3.2.14) satisfies Navier-Stokes equations with
the non-zero viscosity. Demanding continuity of \( v(r, \phi, \tau) \) at \( r = B(\phi, \tau) \), we have the following relation for \( c_1 \) and \( c_2 \):

\[
c_1 - c_2 = \frac{1}{2} \Omega_0 \left( A^2 - B^2 \right) \tag{3.2.15}
\]

Another relation between \( c_1 \) and \( c_2 \) we get by introducing the total fluid flux in the azimuthal direction \( Q(\tau) \), that is independent of \( \phi \) due to the continuity. We assume that \( Q \) is independent of \( \tau \) as well. Note that in all further calculations \( Q \) is a parameter, so that no crucial changes would be introduced if we assume dependence \( Q(\tau) \). Taking into account (3.2.14), we have

\[
Q = \int_a^b v \, dr = \int_a^A v \, dr + \int_A^B v \, dr + \int_B^b v \, dr =
\]

\[
= c_1 \ln \frac{b}{a} + c_2 \ln \frac{b}{B} + \frac{1}{4} \Omega_0 (B^2 - A^2) - \frac{1}{2} \Omega_0 A^2 \ln \frac{B}{A} \tag{3.2.16}
\]

Using (3.2.15) and (3.2.16), we express \( c_1 \) and \( c_2 \) via \( A \) and \( B \):

\[
c_1 \ln \frac{b}{a} = \frac{1}{4} \Omega_0 (A^2 - B^2)(1 + 2 \ln b) + \frac{1}{2} \Omega_0 (B^2 \ln B - A^2 \ln A) + Q
\]

\[
c_2 \ln \frac{b}{a} = \frac{1}{4} \Omega_0 (A^2 - B^2)(1 + 2 \ln a) + \frac{1}{2} \Omega_0 (B^2 \ln B - A^2 \ln A) + Q \tag{3.2.17}
\]

### 3.2.4 Shallow water equations

Boundaries \( r = A(\phi, \tau), \ r = B(\phi, \tau) \) separate areas of potential and vortex flows, so the kinematic boundary conditions are to be set at these radii. Further, it will lead to the shallow water equations for the areas with potential flow. At the boundary \( r = B(\phi, \tau) \), kinematic boundary condition has the form:

\[
\frac{dr}{dt} = u = B_t + \frac{1}{r} v B_\phi \quad \text{at} \quad r = B(\phi, \tau) \tag{3.2.18}
\]
The non-permeability boundary condition at the outer gap edge (3.2.4) gives:

\[ u = 0 \quad \text{at} \quad r = b. \]  
(3.2.19)

Togethe with the continuity equation (3.2.2) it gives

\[ r u = - \int_{b}^{r} v_{\phi}(x, \phi, \tau) \, dx \]

\[ u(B(\phi, \tau), \phi, \tau) = -\frac{1}{B} \int_{b}^{B} v_{\phi}(x, \phi, \tau) \, dx. \]  
(3.2.20)

Combining (3.2.18) and (3.2.20), we have

\[ B_{\tau} + \frac{1}{B} \frac{\partial}{\partial \phi} \left\{ c_{2}(A, B) \ln \frac{B}{b} \right\} = 0. \]  
(3.2.21)

Recalling the profile of azimuthal velocity (3.2.14), we obtain the partial differential equation, typical for the shallow water theory, for the outer area \( B(\phi, \tau) < r < b \) with the potential flow:

\[ B_{\tau} + \frac{1}{B} \frac{\partial}{\partial \phi} \left\{ c_{2}(A, B) \ln \frac{B}{b} \right\} = 0 \]  
(3.2.22)

Similar equation for the inner area with potential flow \( a < r < A(\phi, \tau) \) has the form

\[ A_{\tau} + \frac{1}{A} \frac{\partial}{\partial \phi} \left\{ c_{1}(A, B) \ln \frac{A}{a} \right\} = 0. \]  
(3.2.23)

The system of equations (3.2.22), (3.2.23) with \( c_{1}, c_{2} \), defined by (3.2.17), is a system of quasi-linear partial differential equations, describing behaviour of boundaries \( r = A(\phi, \tau) \) and \( r = B(\phi, \tau) \). This system is the analog of Liapidevskii model in the circular geometry.

Now we can rewrite equations (3.2.22), (3.2.23) in conservative form, which allows to set the Rankine-Hugoniot conditions across the line of discontinuity of \( A \) and \( B \). Introduce the following notations:

\[ \alpha = A^{2}, \quad \beta = B^{2}, \quad q = \frac{4Q}{\Omega_{0}}. \]  
(3.2.24)
Then
\[
\begin{align*}
\alpha_t + \Omega \{ f_1(\alpha, \beta) \}_\phi &= 0 \\
\beta_t + \Omega \{ f_2(\alpha, \beta) \}_\phi &= 0
\end{align*}
\]  
(3.2.25)

where (compare with (3.2.17))
\[
\begin{align*}
f_1(\alpha, \beta) &= \left[ (\alpha - \beta)(1 + 2 \ln b) + \beta \ln \beta - \alpha \ln \alpha + q \right] (\ln \alpha - 2 \ln a) \\
f_2(\alpha, \beta) &= \left[ (\alpha - \beta)(1 + 2 \ln a) + \beta \ln \beta - \alpha \ln \alpha + q \right] (\ln \beta - 2 \ln b) \\
\Omega &= \frac{\Omega_0}{4 \ln(b/a)} \\
a^2 \leq \alpha \leq \beta \leq b^2
\end{align*}
\]  
(3.2.26)

Rewriting the system in matrix form, we have
\[
\begin{pmatrix} \alpha \\ \beta \end{pmatrix}_t + \Omega \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}_\phi = 0.
\]  
(3.2.28)

Here
\[
\begin{align*}
k_{11} &= \frac{\partial f_1}{\partial \alpha} = -(\ln \alpha - k_1)(\ln \alpha - k_2) + \alpha^{-1}[(\alpha - \beta)(1 + k_2) + \beta \ln \beta - \alpha \ln \alpha] + \frac{q}{\alpha}, \\
k_{12} &= \frac{\partial f_1}{\partial \beta} = (\ln \beta - k_2)(\ln \alpha - k_1), \\
k_{21} &= \frac{\partial f_2}{\partial \alpha} = -(\ln \beta - k_2)(\ln \alpha - k_1), \\
k_{22} &= \frac{\partial f_2}{\partial \beta} = (\ln \beta - k_1)(\ln \beta - k_2) + \beta^{-1}[(\alpha - \beta)(1 + k_1) + \beta \ln \beta - \alpha \ln \alpha] + \frac{q}{\beta},
\end{align*}
\]  
(3.2.29)

where $k_1 = \ln(a)$ and $k_2 = \ln(b)$. The system (3.2.28) is hyperbolic if the eigenvalues $\lambda_i$, $i = 1, 2$ of the matrix $(k_{ij})$, $i, j = 1, 2$, are real. Equation for $\lambda$ is
\[
\begin{align*}
\lambda_i^2 - (k_{11} + k_{22})\lambda_i + k_{11}k_{22} - k_{21}k_{12} &= 0 \\
\lambda_{1,2} &= \frac{1}{2}((k_{11} + k_{22}) \pm \sqrt{D})
\end{align*}
\]  
(3.2.30)

Thus, condition for the system (3.2.28) to be hyperbolic is
\[
D \equiv (k_{11} - k_{22})^2 + 4k_{12}k_{21} \geq 0
\]  
(3.2.32)
The condition $D > 0$ corresponds to the case when characteristics of (3.2.28) are different. Equality $D = 0$ corresponds to degenerate case when characteristics coincide. As it follows from (3.2.32) and (3.2.29), condition for the system (3.2.28) to be hyperbolic does not depend on $Q$ or $\Omega_0$ separately, but depends only on $q$, i.e. on the ratio of $Q$ and $\Omega_0$. The area (3.2.27) is divided into two parts, in one of which the system (3.2.28) is not hyperbolic. In the range where (3.2.32) is valid, the values $\Omega\lambda_1(\alpha, \beta)$ and $\Omega\lambda_2(\alpha, \beta)$ are the characteristic velocities and can be associated with the angular velocities of the disturbances propagation. The conservation law for jumps of the quantities $\alpha$ and $\beta$ across the discontinuity is

$$
\omega[\alpha]_{\phi = \phi_s} = \Omega[f_1(\alpha, \beta)]_{\phi = \phi_s},
$$

$$
\omega[\beta]_{\phi = \phi_s} = \Omega[f_2(\alpha, \beta)]_{\phi = \phi_s},
$$

where $\omega = \frac{d\phi_s(\tau)}{dt}$ is the speed of the discontinuity propagation.

Here the notation $[f]$ means the difference between values of $f$ at different sides of discontinuity line $\phi = \phi_s(\tau)$. These are the Rankine-Hugoniot conditions at the line where $\alpha(\phi)$ and $\beta(\phi)$ are discontinuous.

The possible development now is to try to find the periodic piecewise-continuous solution, dependent on the combination $(\phi - \omega\tau)$, that is analogous to so-called Roll-Waves, described, for example, in Stoker (1957). In other words, it would be interesting to construct the system of the stable discontinuities and the rarefaction waves, propagating with the speed $\omega$ in the azimuthal direction without changing the shape. In this work, we did not consider rarefaction waves but analyzed the stable discontinuities appearing in the problem (3.2.28). In the next section we study parameters, for which the stable discontinuities are suitable for description of the experimentally observed azimuthally
3.2.5 Azimuthally propagating discontinuities

The simple interactive program is designed for visual study of the discontinuities, arising in the system (3.2.28). The program allows to change parameters $a$, $b$, $\Omega_0$, $Q$, $\omega$ and to find pairs $(A_0, B_0)$ and $(A_1, B_1)$, such that the discontinuity $(A_0, B_0) \Rightarrow (A_1, B_1)$, propagating in the azimuthal direction with the speed $\omega$, is allowed in accordance to the Rankine-Hugoniot conditions (3.2.33). Fixing the point $(A_0, B_0)$, we plot two contour levels, so that one of conditions (3.2.33) is satisfied along each of them. Recalling relation between $\alpha$, $\beta$ and $A$, $B$ (3.2.24), we plot

\[
\text{Line 1 : } \omega(A_1^2 - A_0^2) - \Omega(f_1(A_1^2, B_1^2) - f_1(A_0^2, B_0^2)) = 0,
\]

\[
\text{Line 2 : } \omega(B_1^2 - B_0^2) - \Omega(f_2(A_1^2, B_1^2) - f_2(A_0^2, B_0^2)) = 0.
\]

(3.2.34)

One intersection of Line 1 and Line 2 is, obviously, $(A_0, B_0)$. The other intersection point of these lines corresponds to the point $(A_1, B_1)$ at the other side of the discontinuity. The typical screen shot, made while looking for the allowed discontinuities, is shown in Fig.67b. Numerical algorithm of drawing the lines is simple: a square grid is spanned over the space $(A, B)$ and the pair of nodes between which the expressions in the left hand side of (3.2.34) or (3.2.34) change the sign is highlighted as belonging to the corresponding line.

The problem described contains five parameters: $a$, $b$, $\Omega_0$, $Q$, $\omega$. To avoid an extensive study on their influence, we use the set, that can be associated with the middle-plane of the two-jet flow, observed in the water-glycerol mixture in the Measurement #168 (Section 2.3). Setup with $R_{in}^* = 6\text{ cm}$, $R_{out}^* = 24\text{ cm}$ is in use. Parameters, observed
near the layer mid-plane when the jet appears, are based on the data from Fig.43,44: 
\( \Omega_0 \approx -2 \, \text{s}^{-1} \), that corresponds to the azimuthal velocity decrease by 2 cm/s per 1 cm, 
and \( Q \approx 7.2 \, \text{cm}^2\text{s}^{-1} \) that is approximate area under the curve \( V(r) \). The value of \( Q \) is defined with precision around 50% and is chosen to be 7.2 for convenience. Recalling relation between dimensional and dimensionless values (3.2.5), and assigning \( R = 6 \, \text{cm} \), 
\( \Omega = 2 \, \text{s}^{-1} \), we have \( a = 1, \, b = 4, \, \Omega_0 = -1, \, Q = 0.1 \). The area in the \((A, B)\) plane, where the system (3.2.28) is hyperbolic, is shown in Fig.68. Contour levels of the characteristic velocities \( \lambda_1 \) and \( \lambda_2 \) (3.2.31), correspondent to the velocities of disturbances propagation, are shown. Eigenvalues (3.2.31) coincide at the edge of the area where the system (3.2.28) is hyperbolic. Bold lines indicate zero levels of \( \lambda_1 \) and \( \lambda_2 \), that corresponds to the change of direction of disturbances’ propagation.

Changing \( \Omega_0, \, Q, \, \omega \) and moving the point \((A_0, B_0)\), we look for the stable discontinuities. In accordance with Lax conditions for the two-dimensional hyperbolic system, discontinuity, propagating with (say, positive) speed \( \omega \) is stable, if one of the characteristic values exceeds \( \omega \) on the left from the discontinuity and is less than \( \omega \) on the right of discontinuity, while the other characteristic value is either less or larger than \( \omega \) everywhere. Assume that \( \phi \) axis is directed to the right. Enumerate the characteristic values so, that \( \lambda_2 \geq \lambda_1 \) everywhere. Mark the characteristic value on the left of discontinuity with the
index ‘-’ and the value on the right with the index ‘+’. The Lax conditions are now

\[
\Omega \lambda_1^- > \omega > \Omega \lambda_1^+ \\
\Omega \lambda_2^- > \omega \quad \text{and} \quad \Omega \lambda_2^+ > \omega \tag{3.2.35}
\]

or

\[
\Omega \lambda_2^- > \omega > \Omega \lambda_2^+ \\
\Omega \lambda_1^- < \omega \quad \text{and} \quad \Omega \lambda_1^+ < \omega. \tag{3.2.36}
\]

Since the structures we observe in experiments precess with the rate much lower than \(\Omega_0\), to start with, we put \(|\omega| \leq 0.1\). In Fig.67b we present a typical screen-shot, made while looking for pairs \((A_0, B_0)\) and \((A_1, B_1)\), satisfying conditions (3.2.33) and one of conditions (3.2.35), (3.2.36). Being hardly comprehensive, the screen-shots are useful to the one who will develop or modify the model. For the chosen set of parameters, stable discontinuities (discontinuities, satisfying both Rankine-Hugoniot and Lax conditions) are naturally separated into ones for which the absolute value of jump of outer radius \([B] = B_1 - B_0\) across the discontinuity is significantly larger than the absolute value of inner radius jump \([A] = A_1 - A_0\) and the ones where the jump of \(A\) dominates. The first class of discontinuities corresponds to the case (3.2.35) and the latter one to (3.2.36).

Four types of the stable azimuthally propagating with velocity \(\omega = \pm 0.1\) shock-like structures are sketched in Fig.69. None of these discontinuities can be associated with the flow, similar to the one shown in Fig.16. In other words, no

\[\text{anticlockwise-propagating sudden increase of } B\] \tag{3.2.37}

is found. However, zooming in the area \((A, B) \rightarrow a\), discontinuity, similar to (3.2.37), where \((A, B)\) changes from \((1.0007, 1.032)\) to \((1.0004, 1.037)\), can be observed (Fig.70).
Here, the vortex layer is too close to the inner cylinder to be associated with the structure appearing in the experiments. As the discontinuity propagation speed $\omega$ increases, the pair discovered near the inner cylinder moves outwards, and at $\omega = 0.27$ we observe the pair $(A, B) = (1.23, 1.32) \rightarrow (1.27, 1.97)$. Jump of the outer radius of vortex layer from about 1.32 to 1.97 in the non-dimensional scale corresponds to the jump from about 8 cm to 12 cm in the setup, used for the Measurement #168. The speed of the discontinuity propagation $\omega = 0.27$ corresponds to the jet precession rate of 0.54 rad/s, that is 5 rpm. Experimentally measured value is about 1 rpm. Change of the value $\Omega_0$ in (3.2.12) from $-1$ to $-0.5$ gives the discontinuity propagation rate of order of 3 rpm.

Dependence of the discontinuity (3.2.37) propagation rate on several parameters is presented in (Fig. 72). Flows with the inner radius of the vortex layer close to $a$ are considered. Numeric experiments show that the system is not too sensitive to the value of $A$ when $A \rightarrow a$ (compare with the screen-shot in Fig. 70, 71). The first two graphs are related to the small jump of the vortex layer outer radius. Unlike the shallow water equations, in the described system the infinitely small discontinuities propagate with the finite velocity. Dependence of this velocity on the outer radius of the vortex layer is shown in Fig. 72a. When the outer radius $B$ exceeds the certain value, discontinuity (3.2.37) disappears. As it follows from Fig. 72b, discontinuity propagation rate is proportional to the vorticity $\Omega_0$ when the latter is large enough. Graphs Fig. 72c,d show that the discontinuity propagation rate is almost independent of its amplitude.

As it was mentioned above, the initial idea of using the long-wave approximation was to construct the system of azimuthally propagating discontinuities (shock-waves) and the rarefaction waves, analogous to the so-called Roll-Waves. This attempt failed, because
for the rarefaction wave to exist, the values of \((A_1, B_1)\) and \((A_2, B_2)\) at its ends should be the eigenvectors, corresponding to the characteristic values \(\lambda_1(A_1, B_1)\) and \(\lambda_1(A_2, B_2)\) or \(\lambda_2(A_1, B_1)\) and \(\lambda_2(A_2, B_2)\). It turned out, that the points where this condition holds lie in the part of \((A, B)\) plane, where the stable discontinuities of the type (3.2.37) do not exist.

3.3 Flow of ideal fluid, initiated by a bulk of decaying vortices

The vortex layer (Fig.67a) appearing in the model considered in Section 3.2, can be associated with the vorticity sheet, separating from the tips of propeller blades in the setup, similar to the one in Fig.8a. In 2002, the similar consideration for the flow as in Fig.8h (Shnilerman, A., private communication) lead to the model, in which the volume (axisymmetric) force, imposed on the two-dimensional flow of ideal fluid, is represented as a source of vorticity of different signs at certain radii. Since the total vorticity, generated by the spatially localized force is zero, Shnilerman suggested to throw the vortices of one sign evenly spread around circumference at the radius, correspondent to the radius of the rotating disk and to throw the same amount of vortices of another sign, evenly spread by area covered by the disk.

One consequence of the enforced generation of the same amount of vorticity of different signs at different radii (say, positive vorticity at \(r_1\) and negative vorticity at \(r_2\), \(r_1 < r_2\)) is that, in spite of the non-axisymmetric structures formation, one can expect the average velocity circulation at \(r \in (r_1, r_2)\) to be positive, that can naturally cause precession of the flow pattern in positive direction. Described conditions can be realized experimentally by
using a propeller with the blades, suspended on the thin arms (Fig. 8c). Following this idea, the evolution of the bulk of decaying vortices is modelled numerically in the infinite two-dimensional space filled with ideal fluid.

A population of decaying vortices, similar to the point vortices, is studied. The velocity field, generated by an individual vortex, is
\[
\vec{V} = \frac{\vec{\Gamma} \times \vec{r}}{r^2 + \delta^2},
\]
where \( \Gamma = |\vec{\Gamma}| \) is the strength of the vortex. The value \( \delta \) is introduced to limit the maximum velocity, generated by a vortex, to avoid too small time step in the Runge-Kutta-Fehlberg procedure while solving the system of ordinary differential equations. It is believed, that if \( \delta \) is small enough, it does not affect the overall behaviour of the vortex population. So, we solve the system of the ordinary differential equations for \( N \) vortices with the coordinates \((x_i, y_i)\) and strength \( \Gamma_i \):
\[
\begin{align*}
\frac{dx_i}{dt} &= + \sum_{j=1}^{N} \frac{\Gamma_j (y_i - y_k)}{(x_i - x_k)^2 + (y_i - y_k)^2 + \delta^2} \\
\frac{dy_i}{dt} &= - \sum_{j=1}^{N} \frac{\Gamma_j (x_i - x_k)}{(x_i - x_k)^2 + (y_i - y_k)^2 + \delta^2} \\
\frac{d\Gamma_i}{dt} &= -\Gamma_i / \tau,
\end{align*}
\]
The last equation describes the vortices exponential decay with the characteristic time constant \( \tau \) and can be interpreted as the influence of Rayleigh friction, similar to models (1.1.2).

Initially, vortices with positive \( \Gamma \) are evenly distributed at radii \( c + d \) and equal amount of vortices with the same negative \( \Gamma \) is evenly distributed at radii \( c - d \). When the strength of the vortex \( k \) decays below certain level \( \Gamma_k < \Gamma_{\text{min}} \), or the vortex distance from the origin
become greater than the certain value $x_k^2 + y_k^2 > b^2$ (that corresponds to the fluid leaving the gap with the outer edge open to the atmosphere), the vortex $k$ is replaced with the new one in accordance with the following procedure. Two random numbers $R_1$ and $R_2$ in the interval $(-1, 1)$ are generated, and the vortex with the following parameters appears:

$$\begin{align*}
\Gamma_k &= \Gamma_0 R_1 \\
x_k &= (c + d \cdot \text{sign}(R_1)) \cos(\pi R_2) \\
y_k &= (c + d \cdot \text{sign}(R_1)) \sin(\pi R_2),
\end{align*}$$

(3.3.3)

where $\Gamma_0$ is a constant. In words, the vortex of a random strength is generated and placed at random angular position to the radius $(c + d)$ if its strength is positive and to $(c - d)$ is its strength is negative. Instantaneous velocity field is recorded at certain points to simulate probes for the velocity measurement, and the passive markers are introduced to the flow to simulate the oil fog used in experiments for the flow visualization.

Precessing structures, visually similar to the ones formed in experiments with either disk or propeller rotating in a narrow gap (Figs.12,11,10), are observed (Fig.73). No extensive study on the flow behaviour is performed, however, it is worth noting that more or less stable non-axisymmetric structures appear for wide range of parameters, so development of this model will be continued.

3.4 Axisymmetric flow in a narrow gap at moderate Reynolds numbers

In 1997 the asymptotic model to describe the axisymmetric flow in the geometry as in Fig.8 was developed. The model is similar to the one suggested by Rabaud and Couder (1983).
The flow domain $D$ is an annulus enclosed between two vertical concentric cylinders and two parallel horizontal planes

$$D = \{ a < r < b, \quad 0 < \varphi < 2\pi, \quad -H < z < H \}. \quad (3.4.1)$$

Here $a$ is radius of the cylinder, rotating with frequency $\Omega$, $b$ is the outer radius, $2H$ is the gap height. Let $(\hat{u}, \hat{v}, \hat{w})$ be the fluid velocity, $\hat{p}$ be the pressure. The boundary conditions are:

$$\hat{u} = 0, \quad \hat{v} = 2\pi \Omega a, \quad \hat{w} = 0 \quad \text{at} \quad r = a \quad (3.4.2)$$

$$\hat{u} = \hat{v} = \hat{w} = 0 \quad \text{at} \quad z = \pm H \quad (3.4.3)$$

$$\hat{u} = \hat{v} = \hat{w} = 0 \quad \text{at} \quad r = b \quad (3.4.4)$$

Solutions, possessing the mirror symmetry with respect to the gap middle-plane $z = 0$ are considered:

$$\hat{u}(r, \varphi, z, t) = \hat{u}(r, \varphi, -z, t), \quad \hat{v}(r, \varphi, z, t) = \hat{v}(r, \varphi, -z, t), \quad \hat{w}(r, \varphi, z, t) = -\hat{w}(r, \varphi, -z, t), \quad \hat{p}(r, \varphi, z, t) = \hat{p}(r, \varphi, -z, t). \quad (3.4.5)$$

So that the stress tensor vanishes at $z = 0$:

$$\sigma_{\varphi z} = 0, \quad \sigma_{rz} = 0 \quad \text{at} \quad z = 0, \quad (3.4.6)$$

where tangential components of the stress tensor are given as

$$\sigma_{\varphi z} = \nu \left( \frac{\partial \hat{v}}{\partial z} + \frac{r}{r} \frac{\partial \hat{w}}{\partial \varphi} \right), \quad \sigma_{rz} = \nu \left( \frac{\partial \hat{w}}{\partial r} + \frac{\partial \hat{u}}{\partial z} \right).$$

If we take (3.4.6) as a boundary condition, we can consider only a ‘half’ $-H < z < 0$ of the original flow domain (3.4.1) and to solve Navier-Stokes equations in the semi-domain $D$:

$$D = \{ a < r < b, \quad 0 < \varphi < 2\pi, \quad -H < z < 0 \} \quad (3.4.7)$$
The correspondent boundary conditions are:

\begin{align*}
\hat{u} &= 0, \quad \hat{v} = 2\pi \Omega a, \quad \hat{w} = 0 \quad \text{at} \quad r = a \\
\hat{u} = \hat{v} = \hat{w} &= 0 \quad \text{at} \quad z = -H \\
\hat{u} = \hat{v} = \hat{w} &= 0 \quad \text{at} \quad r = b \\
\sigma_{\varphi z} = 0, \quad \sigma_{rz} &= 0, \quad v_z = 0 \quad \text{at} \quad z = 0
\end{align*}

(3.4.8)

Problem (3.4.6), (3.4.7), (3.4.8) corresponds to so called ‘flow with the non-deformable free boundary’ located at \( z = 0 \).

Domain with the small aspect ratio is considered:

\[ \frac{b - a}{a} \sim 1, \quad H \ll 1, \quad (3.4.9) \]

and the gap half-height \( H \) is chosen as a small parameter in the asymptotic model. The new variables are introduced:

\[ \rho = \frac{r - a}{H}, \quad \zeta = \frac{z}{H}, \quad \tau = \frac{t}{H}, \quad (3.4.10) \]

The main terms of the fluid velocity and pressure are

\begin{align*}
\hat{u} &= H u(\rho, \zeta, \tau, \varphi) + \ldots \quad (3.4.11) \\
\hat{v} &= v(\rho, \zeta, \tau, \varphi) + \ldots \quad (3.4.12) \\
\hat{w} &= H w(\rho, \zeta, \tau, \varphi) + \ldots \quad (3.4.13) \\
\hat{p} &= H p(\rho, \zeta, \tau, \varphi) + \ldots \quad (3.4.14)
\end{align*}

Also it is assumed that

\[ \nu = H \nu_0, \quad \nu_0 \sim 1 \quad (3.4.15) \]
Substitution of (3.4.10)-(3.4.15) into the Navier-Stokes equations and retention the main terms yields:

\[ u_\tau - \frac{v^2}{a} = -p_\rho + \nu_0 \Delta_2 u \]  
\[ v_\tau = \nu_0 \Delta_2 v \]  
\[ w_\tau = -p_\zeta + \nu_0 \Delta_2 w \]  
\[ u_\rho + \frac{1}{a} v_\varphi + w_\zeta = 0 \]  
\[ \Delta_2 = \frac{\partial^2}{\partial \rho^2} + \frac{\partial^2}{\partial \zeta^2} \]

The boundary conditions (3.4.2)–(3.4.4) take the form

\[ u(0, \zeta, \tau, \varphi) = 0, \quad v(0, \zeta, \tau, \varphi) = 2\pi \Omega a, \quad w(0, \zeta, \tau, \varphi) = 0; \]  
\[ u(\rho, \pm 1, \tau, \varphi) = 0, \quad v(\rho, \pm 1, \tau, \varphi) = 0, \quad w(\rho, \pm 1, \tau, \varphi) = 0 \]

There is one more important assumption related to (3.4.4). The condition of the mirror symmetry (3.4.6) (if any) is:

\[ v_\zeta(\rho, 0, \tau, \varphi) = 0, \quad u_\zeta(\rho, 0, \tau, \varphi) = 0, \quad w(\rho, 0, \tau, \varphi) = 0 \]

The problem (3.4.16)–(3.4.23) splits into two separate problems: one for \( v \) and one for \( u \) and \( w \).

We look for the time-independent axisymmetric flow regime. For the azimuthal velocity we have \( v(\rho, \zeta, \tau, \varphi) = V(\rho, \zeta) \) and

\[ \Delta_2 V(\rho, \zeta) = 0 \]

in the half-strip

\[ 0 \leq \rho < \infty, \quad -1 \leq \zeta \leq 1 \]
with the boundary conditions

\[ V(0, \zeta) = 2\pi \Omega a, \quad V(\rho, \pm 1) = 0, \quad V(+\infty, \zeta) = 0. \]  (3.4.26)

Solution of this problem can be found in the form of Fourier series

\[ V(\bar{\rho}, \bar{\zeta}) = 8\Omega a \sum_{k=0}^{\infty} \frac{(-1)^k}{2k + 1} \exp \{-(2k + 1)\bar{\rho}\} \cos(2k + 1)\bar{\zeta}. \]  (3.4.27)

The series converges and finally we obtain

\[ V(r, z) = 4a\Omega \arctan \left\{ \frac{\cos \left( \frac{\pi z}{2H} \right)}{\sinh \left( \frac{\pi (r-a)}{2H} \right)} \right\}. \]  (3.4.28)

In the article by Rabaud and Couder (1983) this formula is written as

\[ V(r, z) = 4a\Omega \left( \frac{\pi}{4} - \frac{1}{2} \arctan \left\{ \frac{\sinh^2 \left( \frac{\pi (r-a)}{2H} \right) - \cos^2 \frac{\pi z}{2H}}{2 \sinh \left( \frac{\pi (r-a)}{2H} \right) \cos \frac{\pi z}{2H}} \right\} \right). \]  (3.4.29)

To find the radial and axial velocities, the stream function \( \psi(\rho, \zeta) \) is introduced:

\[ U = \psi_\zeta, \quad W = -\psi_\rho \]  (3.4.30)

Eliminating the pressure \( p \) from (3.4.16), (3.4.18), we have

\[ \Delta^2 \psi = -\frac{1}{\alpha \nu_0} (V^2)_\zeta \]  (3.4.31)

with the boundary conditions

\[ \psi(0, \zeta) = 0, \quad \psi(\rho, \pm 1) = 0, \quad \psi_\zeta(+\infty, \zeta) = 0 \]
\[ \psi_\rho(0, \zeta) = 0, \quad \psi_\zeta(\rho, \pm 1) = 0, \quad \psi_\rho(+\infty, \zeta) = 0, \]  (3.4.32)

Solution of the second problem gives

\[ \psi(\rho, \zeta) = \frac{4\Omega^2 a}{\nu_0 \pi^2} \Psi \left( \frac{\pi}{2}, \frac{\pi}{2} \rho, \zeta \right). \]  (3.4.33)
where the function $\Psi(\rho, \zeta)$ is universal and does not depend on any parameters and can be represented as the series, where only four first terms are kept to achieve the reasonable precision:

$$
\psi(\bar{\rho}, \bar{\zeta}) = \sum_{k=1}^{4} \psi_k(\bar{\rho}) \varphi_k(\bar{\zeta}),
$$

where

$$
\varphi_k(\bar{\zeta}) = B_k^{-1} \left\{ \sinh \left( \frac{\pi}{2} \ell_k \right) \sin(\ell_k \bar{\zeta}) - \sin \left( \frac{\pi}{2} \ell_k \right) \sinh(\ell_k \bar{\zeta}) \right\}
$$

$$
\psi_k(\bar{\rho}) = 8 \sum_{\alpha=1}^{8} C_{\alpha} \Psi^{(\alpha)}_k \exp(\lambda_{\alpha} \bar{\rho}) + \sum_{i=0}^{15} A_{ki} \exp \left\{ -2(i + 1)\bar{\rho} \right\}
$$

Values $l$, $B$, $\lambda$, $C$, $\Psi$, $A$ are presented in the tables in Appendix A. Contour plot of the function $\Psi \left( \frac{\pi}{2} \rho, \frac{\pi}{2} \zeta \right)$ in (3.4.33) is presented in Fig. 74.

Comparison of velocity profiles obtained by asymptotic modelling with the experimentally measured ones is performed in Section 2.2.
Chapter 4

Conclusions

4.1 Two-dimensional perturbations of the unstable velocity profile

The zoo of experimental setups, in which the essentially non-axisymmetric (lacking rotational symmetry) patterns have been observed, is sketched below. In the sketch below, we conventionally divided all cases into laminar and non-laminar, separating the case when the radial fluid flux is generated and no rotating solid is present. The most interesting properties of the flows in particular setups are stated to the right from the sketch. We suggest that the reason for the essentially non-axisymmetric patterns to appear is similar for all these setups.

An extensive experimental study of the laminar and turbulent precessing jet-like structures appearing either in a gap with the non-deformable boundaries or in a layer with a free surface is performed (Chapter 2). An accent is done on the measurements of the jet-like structures precession rate, which is easy to measure and that is believed to be a relevant flow characteristics. A series of PIV measurements of velocity field is performed for the axisymmetric flow in a short Couette-Taylor setup whose outer radius several
times exceeds its inner radius and its axial size. Another series of PIV measurements is performed for the flow regime with the two precessing jets in the layer with a free surface. A number of experimental variations of axisymmetric setups with the source of angular momentum near the axis is carried out.

Comparison of results of experimental and theoretical studies showed that the appearance and behaviour of the non-axisymmetric structures in axisymmetric setup can be explained, to certain extent, by the two-dimensional equations for the incompressible fluid with the extra-terms accounting for the influence of the gap walls.

Linear stability analysis of the two-dimensional Navier-Stokes equations with the friction term (Section 3.1) showed the existence of neutrally stable and unstable two-dimensional non-axisymmetric perturbations, visually resembling the ones experimentally observed (compare Fig. 62c,d with visualizations in Figs. 9-22). The precession rate of the
neutrally stable perturbations turned out to possess the similar properties to that of the experimentally observed structures. In particular, for certain set of parameters the pattern precession rate \( \omega \) linearly depends on the speed of the driving solid rotation \( \Omega \) (compare Fig. 66 and Fig. 29), \( \omega \) decreases with the increase of the domain outer radius \( b \) (compare Fig. 65 and Fig. 29), \( \omega \) increases with the decrease of friction coefficient \( \varepsilon \) (3.1.5) that is associated with the gap height \( H \) (compare Fig. 64 and Fig. 28).

Study of the evolution of the two-dimensional fluid layer of constant vorticity embedded between two irrotational fluid layers in a circular gap filled with the ideal fluid (Section 3.2) uncovered azimuthally propagating discontinuities, which can be associated with the jet-like structures observed in experiments. Similarly to the experimentally observed jet-like structures, these formations are localized in azimuthal direction, and have the propagation rate linearly dependent on the magnitude of vorticity in the vortex layer (Fig. 72b,c,d).

The said above leads to conclusion that the appearance and properties of the essentially non-axisymmetric structures described in Section 1.2 can be studied by considering the two-dimensional equations for the incompressible fluid with an unstable radial profile of azimuthal velocity \( v(r) \) in the basic axisymmetric flow. For such an unstable velocity profile to appear in the two-dimensional flow, an external reason is required. It can be (r.1) a kind of axisymmetric body force, e.g. the force imposed by a rotating propeller, suspended in the gap;

(r.2) some specific boundary conditions, e.g. existence of the rotating area of the gap walls (Fig. 23);

(r.3) the 3-dimensional structure of the basic axisymmetric flow, causing advection of
the azimuthal momentum by radial and axial velocities, e.g. advection of the vorticity generated in the boundary layer near the rotating cylinder to the outer area with formation of the radius with the maximum vorticity in the middle-gap plane (Figs. 42, 43);

(r.4) the 3-dimensional structure of the basic flow, providing formation of the circular shear layer far from rigid boundaries (Fig. 54).

Hence, reasons for the non-axisymmetric flow pattern to appear are more or less clear. Now, we attempt to explain the four features of the developed non-axisymmetric flows, listed in Section 1.2:

(ff.1) Concentration of radial outflow in the narrow areas compare to the areas of inflow occurs due to the mass conservation. Indeed, the fluid velocity in the outflow areas (jets) is an order higher than that in the inflow areas since the fluid is getting into the jets after being accelerated by a rotating solid while returning to the central area after losing its momentum. The product of characteristic velocity of the flow formation by its characteristic width should be similar for the inflow and the outflow. Thus, the jets width should be less than the width of the inflow areas.

(ff.2) Essential condition for the jets length to exceed the other characteristic dimensions of the system, in particular, to be larger than the radius of the shear layer causing instability, is positioning of the fluid with higher azimuthal velocity at the smaller radii. In this case, the fluid with high azimuthal velocity, being disposed to the outer areas, where azimuthal velocity is low, continues to move from the centre due to the so-called centrifugal force. This effect is similar to the floating-up of hot fluid plumes when the thermal convection is induced by heating of a fluid layer from below.

(ff.3) It is not surprising that the non-axisymmetric structures are advected azimuthally
by somehow averaged velocity field. The fact that the jet precession rate is an order less then the angular speed of the driving solid become clear if we notice that in our geometries the fluid velocity is similar to that of the rotating solid only in the small vicinity of the boundary of this solid and is significantly less in the rest of the volume. Unexpectedly, cases when the jet-like structure precesses in the direction, opposite to the direction of the cylinder rotation, are observed (Fig.15a, Fig.34b and, not so obviously relevant, Fig.56). No proper explanation for this effect is found, however, models described in Section 3.2 and Section 3.1 allow the negative precession rates and the model with the point vortices (Section 3.3) does not prohibit negative precession rate explicitly.

The fact that the flow is robust and indestructible by severe changes in experimental conditions suggests that formation of the circular shear layer far from the rigid boundaries often happens in axisymmetric geometry.

As the last touch, we recall measurements of the amplitude of the non-axisymmetric velocity perturbation versus the solid rotation speed, performed for the propeller-driven flow in the air gap (Fig.32) and for the cylinder-driven free surface flow (Fig.51). Both experiments can be interpreted as discovery of the supercritical bifurcation of the basic axisymmetric flow.

To summarize said above, we present a table where the common features of the essentially non-axisymmetric structures arising in the axisymmetric setup are listed and the occurrence of these feature in each of the four experimental arrangements and each of the four mathematical descriptions is indicated. The numbers refer to the relevant figures, the .. sign means that no information is available, N/A is the abbreviation for 'Not Applicable', 'No' means that the feature is not observed in the particular system.
### CHAPTER 4. CONCLUSIONS

<table>
<thead>
<tr>
<th>hysteretic behaviour</th>
<th>supercritical behaviour</th>
<th>greater $\omega \rightarrow \text{lower } m$</th>
<th>$\omega \times \Omega \rightarrow$</th>
<th>$\omega \ll \Omega$</th>
<th>$\omega$ decreases with $H$</th>
<th>$\omega$ increases with $b$</th>
<th>$\omega$ can be negative</th>
<th>jet length $\gg H$</th>
<th>jet length $\gg a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>gap + propeller (turbulent)</td>
<td>No</td>
<td>32</td>
<td>26, 27, 29</td>
<td>26-29</td>
<td>28</td>
<td>27-29</td>
<td>26, 28</td>
<td>No</td>
<td>9-13</td>
</tr>
<tr>
<td>gap + cylinder (laminar)</td>
<td>..</td>
<td>..</td>
<td>22</td>
<td>..</td>
<td>35</td>
<td>..</td>
<td>..</td>
<td>35</td>
<td>No</td>
</tr>
<tr>
<td>free surface layer + cylinder (lam)</td>
<td>15</td>
<td>51</td>
<td>16</td>
<td>50a, 50a,c</td>
<td>50c</td>
<td>..</td>
<td>..</td>
<td>15a</td>
<td>14-16</td>
</tr>
<tr>
<td>gap + circular shear (laminar)</td>
<td>..</td>
<td>..</td>
<td>23</td>
<td>33</td>
<td>33</td>
<td>..</td>
<td>..</td>
<td>No</td>
<td>23</td>
</tr>
<tr>
<td>2-d NS + friction, linearized</td>
<td>N/A</td>
<td>N/A</td>
<td>63</td>
<td>66</td>
<td>60, 64-66</td>
<td>64</td>
<td>65c</td>
<td>65a</td>
<td>60, 65a</td>
</tr>
<tr>
<td>2-d inviscid, vortex layers</td>
<td>N/A</td>
<td>N/A</td>
<td>..</td>
<td>3.2.28, 72a</td>
<td>72</td>
<td>..</td>
<td>3.2.26</td>
<td>3.2.26</td>
<td>69b</td>
</tr>
<tr>
<td>2-d low mode Galerkin</td>
<td>..</td>
<td>[53]</td>
<td>..</td>
<td>3.2.28</td>
<td>N/A</td>
<td>..</td>
<td>..</td>
<td>N/A</td>
<td>7d</td>
</tr>
<tr>
<td>3-d finite difference*</td>
<td>+</td>
<td>+</td>
<td>..</td>
<td>+</td>
<td>..</td>
<td>..</td>
<td>..</td>
<td>..</td>
<td>..</td>
</tr>
<tr>
<td>intuitively clear</td>
<td>..</td>
<td>..</td>
<td>..</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>..</td>
<td>+</td>
</tr>
</tbody>
</table>

* The data on the finite-difference DNS is obtained with the code kindly provided by Prof. M. Yu. Zhukov and is not discussed in this work.
4.2 Not yet answered questions

While vaguely explaining certain features of the jet-like flow formations and intuitively understanding the others, the author has several blank spots in comprehending the phenomenon.

(q.1) Measurements of the turbulent and laminar non-axisymmetric structures precession rate versus propeller or disk rotation speed (Figs. 26–33) uncovered a number of surprisingly regular relations, that should have some simple explanation author failed to provide.

(q.2) The jet-like structures precession in the direction opposite to that of the driving body rotation (Fig. 15, Fig. 34, Fig. 34b) looks mysterious and is not explained in spite of existence of the negatively precessing neutrally stable modes, shown in zoomed insert in Fig. 60.

(q.3) The positive radial pressure gradient, caused by so-called centrifugal force, results in appearance of the inflow within boundary layers located near the gap walls (a similar effect is shown in Fig. 54). Now, the mass conservation requires existence of the positive radial outflow in the fluid outside the mentioned boundary layers. No setup avoiding this effect is available and the influence of the positive radial flux on the jet-like structures behaviour (and even on their existence) is unclear. Here, further progress will be done by introducing a fluid sink of the known strength to the centre of the gap.

4.3 Prospective theoretical approaches

The study on the essentially non-axisymmetric structures in axisymmetric geometries is worth continuing. Here we list several theoretical approaches that are to be developed in future.
(t.1) It looks possible to construct an asymptotic model for the viscous three-dimensional axisymmetric flow in a layer at large enough Reynolds numbers by considering the system of three boundary layers: one boundary layer forming near the cylinder (some experimental measurements devoted to this layer are shown in Fig. 46), another layer near the free surface (or the rigid boundary if the flow in a gap is considered) and the third layer near the bottom or near the other rigid boundary (Fig. 75a). When the velocity field for the basic axisymmetric flow is known, its stability with respect to non-axisymmetric perturbations can be studied.

(t.2) Another way to use the boundary layer approach is to model of Taylor vortex with parameters slowly changing in azimuthal direction (Fig. 75b), in the approximation of high Reynolds number. Then, the system of ordinary differential equations for the vortex parameters evolution (say, evolution of its radial size and the radial velocity amplitude) may be obtained and studied in the way it is done for the vortex layer parameters (3.2.28).

(t.3) Various friction terms can be substituted to the linear model described in Section 3.1 or to the code for Galerkin simulation described by Zhukov and Petrovskaya (2001). The structures obtained by simulation of the two-dimensional viscous flow in the annular domain visually resemble the ones detected experimentally (compare Fig. 7 with Fig. 34b), and the addition of a friction term to the model introduces parameter related to the layer thickness in the axial dimension, that can lead to significant progress.

(t.4) Models for the two-dimensional flow of inviscid fluid, built similarly to the model of a system of vortex layers, described in Section 3.2, can be constructed. Various azimuthal velocity profiles can be substituted instead of (3.2.13) to obtain the system of differential equations, similar to (3.2.28). Hints on choosing the appropriate velocity profiles can be
given by the asymptotic models (T1) and (T2).

(t.5) An approach, based on the evolution of vortex layers in the inviscid two-dimensional flow (Section 3.2) can be applied to the \((r, z, t)\) space instead of \((r, \varphi, t)\) space. In other words, dynamics of the radial vorticity component and the structures visualized in Fig.18 can be studied.

(t.6) Being simple in terms of numerical realization, point vortex model, similar to the one described in Section 3.3, is to be studied for different distributions of the vorticity source.

(t.7) The two-dimensional shallow water equations with somehow accounted viscosity and somehow modelled source of motion imitating the rotating cylinder, can be applied to the free surface flow such as in Fig.14, where variation of the layer depth is significant.

(t.8) An attempt to construct the soliton-like solution of the two-dimensional shallow water equations for the fluid with the non-zero axial vorticity component can be done.

(t.9) A study using the three-dimensional finite-difference numerical simulation of the viscous flow is to be performed. The code is written by M.Yu.Zhukov and E.V.Shiryaeva in 2003 and the preliminary numerical experiments are done.

4.4 Prospective experimental arrangements

To understand the role of particular features of the experimental setup in formation of the jet-like structures, a number of modifications of the experiments described in this work can be useful.

(e.1) A setup with violated angular periodicity condition, e.g. a cylinder-driven spiral domain as shown in Fig.75c, can be constructed. Localized in angular direction soliton-like
or shock-like formations are expected to appear.

(e.2) The flow velocity field and, in particular, the amplitude of the non-axisymmetric velocity perturbation versus driving solid rotation speed is to be investigated using the Laser Doppler Velocimetry. No LDV diagnostics was available during this study. It led to some sophisticated and unreliable experimental arrangements like the-mirror-on-the-thread to measure the critical cylinder rotation speed (Fig. 51). Experimentally obtained phase portraits of the system are to be studied in order to plot bifurcation curves if any. Measurements of the average velocity fields in the turbulent propeller-induced flow is to be performed in order to understand the role of boundary layers forming near the gap walls (Fig. 9b).

(e.3) The setups already studied can be placed to the rotating table, making the jet-like structures stable in the laboratory system of reference. Experiments in the rotating frame of reference can lead to the understanding the role of centrifugal and coriolis terms.

(e.4) Following Zhukovskij (1914) experiments, vortex sources of various types can be studied, i.e. the vortex source in either a free surface layer or in a gap with the vorticity generated by either propeller or the stationary blades placed into the feeding pipe (Fig. 75d). These arrangements can fit the conditions for which the model by Goldshtik, Shtern and Yavorski (1989) and Goldshtik, Hussan and Shtern (1991), revealing azimuthally propagating localized structures, is developed. One simple experiment to perform is to introduce a sink created by, say, a hoover to the centre of the gap at one of existing experimental setups to study the role of radial flux in our experiments (see unanswered question q.2 above).

(e.5) The granular flows, similar to the one shown in Fig. 56, are to be studied for various
shape of the concave surface. It is believed that patterns of the shape similar to that in Fig. 14b can be obtained for not too steep bowl walls when using the regularly shaped grains, say, puck-like pills or bearing balls. Experiments with the granular flow can lead to interesting conclusions related to the role of medium continuity.
Bibliography


Figures

Figure 1: A rotating solid located in the centre of a thin axisymmetric layer generates the stable precessing jet-like formation(s). When the solid rotates fast enough the jet(s), which are the spatially concentrated area(s) with high positive radial velocity, form on the background of low negative radial velocity, so that the total radial flux is zero. The flow can be either laminar or turbulent. A system of a few jets can appear. In further consideration, the Reynolds number is defined as the maximum speed of rotating solid boundary times layer depth over the fluid kinematic viscosity.
Figure 2: A sketch of the setup used by Zhukovskij (1914). A cross-shaped propeller is placed to the centre of a narrow circular gap between two rigid plates and the air is injected along the propeller axis. Paper flags are suspended around circumference to visualize the flow. Zhukovskij detected the precessing jet-like formation and suggested a simple analytical model with the vortex sheet separated from the propeller blades.
Figure 3: Experiments by Nezlin, M.V., Snezhkin, E.N. (1990): (a), (b) flow domains, the free surface layers with the parabolic-shaped differentially rotating bottom; (c) Rossby soliton, observed in the lower setup in (a), (d) a system of Rossby waves, observed in the setup (b). White circle corresponds to the azimuthal velocity jump between the counter-rotating rings; (e) one side experiment: schematic of transition of the triangular vortex to the three vortex pairs and a remainder.
Figure 4: Experiments on the instabilities in the differentially rotating fluid layer by Rabaud, M. and Couder, Y. (1983): (a) experimental setup, a gap with the differentially rotating walls; (b), (c) interference patterns formed by a soap film spanned in the mid-gap; (d) schematic of the flow, similar to (c), where the fluid located near the axis rotates in the direction, opposite to that of the inner part of the gap walls rotation.
Figure 5: (a) Gledzer, E.B., Dolzhanskij, F.V., Obukhov, A.M. (1981): experimentally observed system of stationary vortices arising in conductive fluid with the radial electric current, differentially rotating in axial magnetic field; (b) the low-dimensional Galerkin simulation of the flow (a) by Ponomarev, V.M. (1980).
Figure 6: (a), (b) Orlandi, P. and van Heijst, G.F. (1992): experimentally observed tripolar vortex and the vorticity contour levels for the simulated flow; (c), (d) Carnevale, G.F. and Kloosterziel, R.C. (1994): experimentally observed tripolar vortex and the vorticity contour levels for the simulated flow; (e), (f) Xavier Carton and Bernard Legras (1994): vorticity contour levels for the simulated flow at two time instants. Compare these three cases with the visualization in Fig. 20, where the rotating cylinder acts as a core vortex.
Figure 7: Zhukov and Petrovskaya (2001): vortex-like (a) and jet-like (b, c, d) precessing structures appearing in the low-mode Galerkin simulation of the viscous flow in the annular domain. Velocity of the rotating cylinder edge is 1. Velocity field is plotted in the points where the fluid velocity lies in the interval from 0.62 to 0.75.
Figure 8: Experimental setups in which the non-axisymmetric structures are observed. Narrow gaps or fluid layers with rotating propellers or cylinders: (a) a propeller suspended in a gap with the outer edge open; (b) a propeller suspended in a gap with the outer edge sealed; (c) a blade on a thin arm suspended in a gap; (d) a propeller filling the whole gap height; (e) a disk suspended in a gap; (f) a cylinder filling the whole gap height; (g) a cylinder in a free surface fluid layer which depth can be considered constant; (h) a cylinder, flash-mounted to the gap wall; (i) a cylinder in a free surface layer of variable depth; (j) types of non-rotationally-symmetric rotating solids: propellers with the 1, 2, 3, 4, 16 rectangular blades, a propeller with twisted blades (driving the fluid in axial direction), a blade suspended on a thin arm; (k) a cylinder suspended above the layer bottom; (ℓ) a cylinder in a gap with the sealed outer edge; (m) a gap with rotating central area (two flash-mounted rotating discs) partially filled with the fluid; (n) an air gap with the source.
Figure 9: A single-jet regime in a propeller-driven air gap (as in Fig. 8a): the planform view and (b) the side view from the direction indicated by a pair of arrows in (a). Gap: diameter 100 cm, height 2 cm, outer edge open. Propeller: span 6.5 cm, height 1 cm, rotation speed 1000 rpm (anticlockwise). Propeller in its real size is marked by a cross. Visualization: the oil fog is introduced to the centre of the gap through the hole in the upper plate. It is believed that the hot oil fog does not affect the flow significantly. The jet precesses anticlockwise.
Figure 10: A double-jet regime in a propeller-driven air gap (as in Fig. 8a). Gap: diameter 50 cm, height 2 cm, outer edge open. Propeller: span 20 cm, height 1 cm, rotation speed 1000 rpm (anticlockwise). Propeller in its real size is marked by a cross. Visualization: the oil fog is introduced to the centre of the gap through the hole in the upper plate. The jets precess anticlockwise.
Figure 11: A single-jet and a double-jet regimes in a disk-driven air gap (as in Fig. 8e). Gap: diameter 100 cm, height 2 cm, outer edge open. Disk: diameter 19 cm, height 1 cm, rotation speed (anticlockwise) (a) 500 rpm, (b) 200 rpm, (c) 1000 rpm. Visualization: oil fog is introduced near the disk axis through the hole in the lower plate. The non-axisymmetric structures precess anticlockwise.
Figure 12: A double-jet, a single-jet and a triple-jet regimes in the air gap with a disk flash-mounted to the wall (as in Fig. 8h). Gap: diameter 100 cm, height 2 cm, outer edge open. Disk: diameter 20 cm, rotation speed (anticlockwise) (a) 220 rpm, (b) 2000 rpm, (c) quickly changed from 0 to 200 rpm (transitive regime). Visualization: the oil fog is introduced to the centre of the gap through the hole in the upper plate. Asymmetric structures precess anticlockwise.
Figure 13: A jet-like structure generated by a propeller suspended in the gap (same as in Fig. 8b) filled with water (viscosity 0.01 cm$^2$s$^{-1}$). Gap: height 0.63 cm, diameter 46 cm. Propeller: height 0.5 cm, span 6 cm, rotation speed (anticlockwise) (a) 100 rpm, (b) 250 rpm, (c) 500 rpm, (d) 800 rpm, (e) 1200 rpm, (f) 1200 rpm. Propeller in its real size is marked by a cross. Visualization: ink is injected through the hole in the upper plate. It is believed that the ink injection does not affect the flow significantly. The jet-like structure precesses anticlockwise. The dark object on the bottom-left is a tachometer. Two black lines on the top left are segments of the rubber belt driving propeller shaft (located behind the vessel).
Figure 14: The triple-jet regime in a disk-driven fluid layer (as in Fig.8k). Layer: diameter 50 cm, depth 0.5 cm. Liquid: water-glycerol mixture with the rheoscopic concentrate added for visualization, viscosity 0.27 cm²s⁻¹. Disk: diameter 14 cm, rotation speed (clockwise) (a) 451 rpm, (b) 405 rpm. Disk is suspended 0.3 cm above the bottom. Visualization: rheoscopic concentrate and reflection from the layer free surface. The jets precess clockwise. The flow pattern in (a) precesses very slowly and the tips of the jets in (a) are oscillating.
Figure 15: Hysteresis phenomenon in a disk-driven fluid layer (as in Fig. 8k). Layer: diameter 50 cm, depth 0.5 cm. Liquid: water-glycerol mixture with the rheoscopic concentrate added for visualization, viscosity $0.27 \text{ cm}^2\text{s}^{-1}$. Disk: diameter 14 cm, rotation speed 201 rpm (clockwise). Disk is suspended 0.3 cm above the bottom. Regimes with (a) two, (b) three, (c) four arms are observed for identical parameters of the setup. Visualization: rheoscopic concentrate and light reflection from the layer surface. Asymmetric structures precess (a) anticlockwise; (b),(c) clockwise. A schematic of hysteresis diagram is shown in (d).
Figure 16: A triple-jet and a double-jet structures in a cylinder-driven fluid layer (as in Fig.8g). Layer: diameter 23 cm, depth 1.5 cm. Liquid: water-glycerol mixture with the rheoscopic concentrate added for visualization, viscosity $0.73 \text{cm}^2\text{s}^{-1}$. Cylinder: diameter 8 cm, rotation speed (anticlockwise) (a) 87 rpm, (b) 166 rpm. The non-axisymmetric structures precess anticlockwise.
Figure 17: Sequence of the cross-sections of the flow with precessing double-jet structure in a vessel with the free surface (as in Fig. 8g). Layer: diameter 50 cm, depth 2 cm. Liquid: water-glycerol mixture, viscosity 0.127 cm²s⁻¹. Cylinder: diameter 12 cm, rotation speed 20.8 rpm (parameters as in Measurement #168). Direction of fluid motion in the vortex is indicated by an arrow in the first frame. Visualization: lasersheet and silvered hollow glass particles. Lasersheet is oriented vertically, so that the cylinder axis lays in its plane. Bright vertical bar on the right is the cylinder edge. Images are acquired while the jet-like structure precesses, exposing its different cross-sections (numbered rectangles in the sketch) to the camera.
Figure 18: A set of cross-sections of the flow with a precessing single-jet structure (as in Fig.19b), by a vertical lasersheet, parallel to the cylinder axis and spaced from the cylinder axis by 2.3 cm. Layer: diameter 50 cm, depth 1.9 cm. Liquid: water-glycerol mixture, viscosity 0.43 cm²s⁻¹. Cylinder: diameter 2 cm, rotation speed 676 rpm. Position of the jet with respect to the lasersheet is roughly sketched on the right of each frame (top view). The arrows show direction of the vortex curling, not indicating the ratio of the fluid velocity components parallel and perpendicular to the vortex core. Bright vertical bars correspond to the light reflection from the metal cylinder axis. The perspex cylinder itself is visible. A pair of areas with the oppositely directed radial vorticity can be spotted.
Figure 19: A double-jet and a single-jet regimes in a cylinder-driven fluid layer with a free surface (as in Fig. 8g). Layer: diameter 23 cm, depth 2 cm. Liquid: water-glycerol mixture with the rheoscopic concentrate added for visualization, viscosity 0.35 cm$^2$s$^{-1}$. Cylinder: diameter (a) 1.6 cm, (b) 2 cm, rotation speed (clockwise) (a) 520 rpm, (b) 408 rpm. Asymmetric structures precess clockwise. Some ink that is slightly lighter than the water-glycerol mixture is dropped on the free surface in (b).
Figure 20: Development of the non-axisymmetric structure in the setup, similar to the one in Fig.8g. Layer: diameter 50 cm, depth 2 cm. Liquid: water-glycerol mixture. Cylinder: diameter 5 cm, rotating clockwise. Initially, an axisymmetric Taylor vortex is visualized by ink (a). Next, the cylinder rotation speed is increased, so that the Taylor vortex expands (b) and then turns to the non-axisymmetric structure, precessing clockwise (d)-(f). Dark lines on the background are segments of the rubber belt driving the cylinder shaft (located behind the vessel).
Figure 21: A pair of shoe-like vortices consisting of the fluid that is never entrained by a Taylor vortex in the setup similar to the one in Fig. 8g. Layer: diameter 50 cm, depth 2 cm. Liquid: water-glycerol mixture. Cylinder: diameter 5 cm, rotating clockwise. Visualization: ink that is lighter than the fluid. Dark lines on the background are segments of the rubber belt driving the cylinder shaft (located behind the vessel). The flow direction is indicated by dotted arrows. The non-axisymmetric structure precesses clockwise.
Figure 22: A single-jet and a double-jet regimes in a cylinder-driven gap (as in Fig.8c). Gap: diameter 23 cm, height 1.7 cm. Liquid: water-glycerol mixture with the rheoscopic concentrate added for visualization, viscosity $1.3 \text{ cm}^2\text{s}^{-1}$. Cylinder: diameter 3.5 cm, rotation speed (clockwise) (a) 1396 rpm, (b) 1200 rpm. The non-axisymmetric structures precess clockwise. Regimes shown are ones with the single Taylor vortex (as in Fig.36h,i). Taylor vortex is oriented so that the flow on the side, facing the camera, is (a) outwards, (b) inwards.
Figure 23: The asymmetric flow in a gap with the rotating disks flash-mounted to the gap walls (as in Fig.8m). Gap: radius 8 cm, height 0.5 cm. Fluid: water-glycerol mixture with rheoscopic concentrate added for visualization, viscosity 0.17 cm²s⁻¹. Disks: radius 5 cm, rotating speed (clockwise) (a) 48 rpm, (b) 38 rpm, (c) 30 rpm. Asymmetric structures precess clockwise. Polygonal air bubble is seen in the gap centre.
Figure 24: A structure (a set of trajectories of the fluid particles) of the laminar flow in a free surface layer (domain as in Fig. 8g) with the three-jet precessing structure as viewed from the frame of reference rotating with the structure precession rate, i.e. the frame of reference in which the flow is stationary. The core of the Taylor vortex is shown with the bold line. At certain angle, fluid in the Taylor vortex splits into two parts. One part remains in the vortex, and the other part leaves to the outer area of the flow domain to be again entrained by the Taylor vortex. The fluid separates from the surface along dotted lines and leaves to the depth of the layer. The shoe-like patterns visualized in Fig. 21 are indicated by grey colour. Spirals indicate vortices extending from the shoe-shaped areas to the corner between the rotating cylinder and the bottom. These vortices are feebly visible in Fig. 21. Since the frame of reference rotates with the non-axisymmetric structure, the outer edge of the domain rotates in the direction opposite to that of the cylinder rotation (indicated by the arrow).
Figure 25: The structure of the essentially two-dimensional flow with the turbulent two-jet precessing structure. A high Reynolds number flow in a gap with the outer edge open to atmosphere is sketched. The flow domain is naturally divided into more or less laminar inflow area and the area, affected by the rotating solid. Fluid from the outside of the gap enters the flow domain through the outer edge and moves towards the gap centre. Reaching the area, affected by a rotating solid (propeller), fluid is entrained in the azimuthal direction and then is thrown away radially in the form of precessing jets. Structures with one or two jets can be observed for various sets of parameters. Certain amount of fluid within the jets is not reaching the gap edge, being entrained back to the area where the solid rotates. The solid rotates clockwise and the whole structure precesses clockwise.
Figure 26: The flow in the air gap, similar to the one visualized in Fig.9. Dependence of the jet precession rate on the propeller rotation speed for different propeller span. Gap: diameter 100 cm, height 2.22 cm. Propeller: cross-shaped four-blade, height 1 cm. Data markers indicate the propeller span. Note that linear fits to data, correspondent to propellers with different spans, intersect in the point different from the origin.
Figure 27: The flow in the air gap, similar to the one visualized in Fig. 9. Dependence of the jet precession rate on the propeller rotation speed for different gap diameter. Propeller: cross-shaped four-blade, height 0.55 cm, diameter 14 cm. Gap: height 1.2 cm. Data markers indicate the gap diameter. Note that linear fits to data, correspondent to gaps of different diameters, intersect in the point different from the origin.
Figure 28: Scatter plot of ratio of the jet precession rate increment to the propeller rotation speed increment $d\omega/d\Omega$ versus combination (2.1.3). Here $b$ is the gap radius, $2c$ is the propeller span, $H$ and $h$ are the gap height and propeller height. Setup as in Fig.8a with a cross-shaped four-blade propeller, flow as in Fig.9. Annular hole of about 2 cm$^2$ exists in the gap wall around the propeller shaft. Plot is based on 77 measurements of dependence $\omega$ on $\Omega$ for the latter varying from 500 rpm to 3000 rpm (Table 4.5).
Figure 29: Dependence of the jet precession rate on propeller rotation speed for the setup as in Fig. 8d. Gap: height 1.7 cm, outer edge open. Propeller: height 1.5 cm, span 13 cm; (a) dependence $\omega(\Omega)$ for different gap diameter $D$ with the linear fits; (b) dependence of the slope $d\omega/d\Omega$ on the logarithm of ratio of gap diameter to doubled propeller span (2.1.4) while other parameters fixed.
Figure 30: Dependence of $d\omega/d\Omega$ on $c^2/(4c+b)^2$ for the experimental series with the fixed gap height and propeller height. Relation (2.1.5) is tested. Values of $d\omega/d\Omega$ for the different series are shifted vertically for convenience. For reference, thin lines with 45° slope are traced, indicating linear dependence between $d\omega/d\Omega$ and $c^2/(4c+b)^2$. Bold solid lines correspond to the series in which the gap diameter is changed while the propeller diameter fixed. A dotted line corresponds to a series in which the propeller diameter is changed while the gap diameter is fixed. In the mentioned experiments, the gap outer edge is open to the atmosphere. A thin line corresponds to a series with the blade on the thin arm in a gap with the outer edge sealed.
Figure 31: Typical signal (b) and its spectrum (c) (acquired by digital oscilloscope Tektronix TDS420A), used for the measurement of the velocity perturbation versus propeller rotation speed. Signal varies with the ohmic resistance of the thread of the broken 6.5 V light bulb, connected as shown in the schematic (a) and cooled by the air flow in the gap. Air gap: diameter 30 cm, height 2.4 cm. Propeller: four blades, height 1.5 cm, span 13 cm, rotation speed 600 rpm.
Figure 32: Dependence of the signal amplitude (height of the first, not zeroth, peak in Fig. 31c) on the propeller rotation speed. Setup as in Fig. 8b. Air gap: diameter 30 cm, height 2.4 cm, outer edge open. Propeller: four blades, height 1.5 cm, span 13 cm, rotation speed 600 rpm. Linear dependence of the signal amplitude on the propeller rotation speed corresponds to the square root dependence of the flow velocity variation on the propeller rotation speed, that is typical for the super-critical bifurcation.
Figure 33: The non-axisymmetric structure precession rate $\omega$ versus disks rotating speed $\Omega$ in a gap with the flash-mounted rotating disks (as in Fig.8m). Flows similar to the ones shown in Fig.23 are studied. Gap: radius 8 cm, height 0.5 cm. Fluid: water-glycerol mixture of viscosity 0.3 cm$^2$s$^{-1}$ (circles) and 0.07 cm$^2$s$^{-1}$ (diamonds). Disks radius is 5 cm. Labels show the angular order of the structures.
Figure 34: PIV images of the flow in the domain as in Fig. 8f. Time separation between laser pulses is 0.016 s. Gap: height 2 cm, outer radius 12 cm. Fluid: water-glycerol mixture with the silvered hollow glass particles, viscosity 0.23 cm²s⁻¹. Cylinder: radius 1 cm, rotation speed (counterclockwise) (a) 393 rpm; (b) 411 rpm. The lasersheet is parallel to the gap walls and is located 0.8 cm from the bottom. Taylor vortex is oriented so that the inflow area is located near the top. Asymmetric structure precession rate is (a) 4.6 rpm, counterclockwise; (b) 5.4 rpm, clockwise.
Figure 35: Dependence of the jet-like structure precession rate on the cylinder rotation speed for the setup as in Fig. 8ℓ. Gap: height 2 cm, outer radius 12 cm. Fluid: water-glycerol mixture with the silvered hollow glass particles, viscosity 0.23 cm$^2$/s. Cylinder: radius 1 cm. Note that the jet-like structure precession rate changes the sign with the increase of the cylinder rotation speed. Visualization of the positively and negatively precessing flow patterns are shown in Fig. 34.
Figure 36: Side view of the flow in a vessel, similar to the one in Fig. 8[6]. Gap: radius 11.5 cm, height 1.7 cm. Fluid: water-glycerol mixture of viscosity of 0.43 cm$^2$s$^{-1}$. Cylinder: radius 2.5 cm, height 1.7 cm, rotation speed $\Omega$ is indicated in revolutions per minute. For low enough cylinder rotation speed, the flow represents two Taylor vortices (a). When the cylinder rotation speed increases, one vortex starts to dominate and finally occupies the whole gap (b)-(g). Last two snapshots ($\Omega = 500$ rpm) correspond to the non-axisymmetric regime with a precessing jet. Visualization: a lasersheet, positioned so that the cylinder axis lays in its plane, and reflective silvered hollow glass particles, suspended in the fluid.
Figure 37: Radial profiles of azimuthal velocity $v(r)$ for the flow as in Fig. 36a in a cylinder-driven gap as in Fig. 8. Cylinder rotation speed is $\Omega = 20.1 \text{ rpm} = 0.15\Omega_1$, that is far below the bifurcation of two Taylor vortices to the one. Gap height $2H = 2\text{ cm}$, vessel diameter $2b = 46\text{ cm}$, cylinder diameter $2a = 5\text{ cm}$, water-glycerol mixture viscosity $\nu = 0.25\text{ cm}^2s^{-1}$, density $\rho = 1.17\text{ gcm}^{-3}$. Solid lines depict profiles (3.4.28) obtained from the asymptotic model, described in Section 3.4. Velocity field is measured with PIV technique. Positions of the lasersheet, corresponding to different values of $z$, are indicated by dotted lines in the insert.
Figure 38: Azimuthal velocity profiles same as in Fig. 37, plotted in logarithmic scale.
Figure 39: Radial profiles of radial velocity $u$ for the same setup as in Figs. 37, 38. The cylinder rotation speed is 60 rpm. Note that the radial velocity reaches about 10% of (azimuthal) velocity of the cylinder boundary, that is higher than in conventional Taylor vortices in the Couette-Taylor setup that is long in the axial direction.
Figure 40: Radial profiles of azimuthal velocity $v(r)$ for the same setup as in Figs.37-39. Flow with one dominating Taylor vortex (Fig.36g) is studied. The cylinder rotation speed is $\Omega = 130 \text{ rpm} \approx \Omega_1$, that is right after the bifurcation of the two Taylor vortices to the one. Solid lines depict profiles, predicted by the asymptotic model (3.4.28).
Figure 41: Azimuthal velocity profiles same as in Fig. 40, plotted in the logarithmic scale. Note that for large enough distance from the rotating cylinder, where the radial and axial velocity components do not affect the transport of azimuthal momentum, $v(r)$ decreases logarithmically in accordance with the asymptotic model.
Figure 42: Radial profiles of azimuthal velocity $v$ same as in Fig. 40 compared with the ones obtained by the finite difference simulation for the stationary axisymmetric flow (solid lines). The code was kindly granted by Prof. M.Yu.Zhukov.
Figure 43: Measurement #168. The free surface flow (as in Fig. 8g). Layer: depth 2 cm, outer radius 24 cm. Fluid: viscosity 0.127 cm$^2$s$^{-1}$. Cylinder: radius 6 cm, rotation speed 20.8 rpm (velocity of the disk boundary 13.1 cm/s). Radial profiles of azimuthal velocity $v$, normalized by the speed of cylinder boundary at different heights $z$ at different time instants $t$ (or, equivalently, for different azimuthal cross-sections). Obviously, the plotted value is equal to unit at the $r = a$. Numbers on the right show the angular coordinate of the section in degrees. Observe the interval of $r - a$ with the high azimuthal velocity, that corresponds to the Taylor vortex in the vicinity of rotating cylinder.
Figure 44: Measurement #168. Contour plot of $v$, based on the same data as the Fig. 43. Coordinates are labelled in centimeters and the contour levels are labelled in cm/s. Velocity of the cylinder boundary is 13.1 cm/s.
Figure 45: PIV configuration, used to study the azimuthal velocity gradient near the rotating cylinder (left) and the axial profiles of azimuthal and axial velocities (right).
Figure 46: Measurement #168. Radial gradient of azimuthal velocity in vicinity of the rotating cylinder, derived from the measurements in the configuration Fig.45a: (a) line plot: each line corresponds to the certain time instant or, that is the same, to the certain angular coordinate with respect to the precessing non-axisymmetric formation; (b) contour plot of the same data: lines are the levels of constant $dv/dr$. Free surface flow as in Fig.8g. Layer: depth 2 cm, outer radius 24 cm. Fluid: viscosity 0.127 cm$^2$s$^{-1}$. Cylinder: radius 6 cm, rotation speed 20.8 rpm=2.18 s$^{-1}$ (velocity of the disk boundary is 13.1 cm/s).
Figure 47: Measurement #168. Vertical profiles of azimuthal velocity $v$ at different time instants (or, alternatively, different azimuthal coordinate $\varphi$) at different radii $r$ (or distances from the rotating cylinder edge ($r - a$)). Lines correspondent to different $\varphi$ are shifted horizontally. Scale above shows the angle to which the point $z = -2$ ($v = 0$ on the layer bottom) corresponds. Free surface flow as in Fig.8g. Layer: depth 2 cm, outer radius 24 cm. Fluid: viscosity $0.127 \text{cm}^2\text{s}^{-1}$. Cylinder: radius 6 cm, rotation speed 20.8 rpm (velocity of the disk boundary 13.1 cm/s).
Figure 48: Measurement #168. Contour plot of $v(\varphi, z)$ at different radii $r$, based on the same data as Fig.47: levels of constant azimuthal velocity $v$ in $(\varphi, z)$ plane at different radii $r$ (or distances from the rotating cylinder edge $(r-a)$). Labels show $v$ in cm/s. Free surface flow as in Fig.8g. Layer: depth 2 cm, outer radius 24 cm. Fluid: viscosity $0.127 \text{ cm}^2\text{s}^{-1}$. Cylinder: radius 6 cm, rotation speed 20.8 rpm (velocity of the disk boundary 13.1 cm/s).
Figure 49: Measurement #168. Vertical profiles of axial velocity $w$ at different time instants (or, alternatively, different azimuthal coordinate $\varphi$) at different radii $r$ (or distances from the rotating cylinder edge $(r - a)$). Lines corresponded to different $\varphi$ are shifted horizontally. Scale above shows the angle to which the point $z = -2$ ($w = 0$ on the layer bottom) corresponds. Free surface flow as in Fig. 8g. Layer: depth 2 cm, outer radius 24 cm. Fluid: viscosity 0.127 cm$^2$s$^{-1}$. Cylinder: radius 6 cm, rotation speed 20.8 rpm (velocity of the disk boundary 13.1 cm/s).
Figure 50: Measurement #168. Dependence of (a) jets precession rate on the cylinder rotation speed; (b) critical cylinder rotation speed on the fluid layer depth (derived from Fig. 51b); (c) jets precession rate on the fluid layer depth. The dotted line in (b) shows values of $\Omega$ for which the Reynolds number, based on the cylinder edge speed and the layer depth, is equal to 100. Free surface flow (as in Fig. 8g). Layer: outer radius 24 cm, depth 2 cm (in (a)). Fluid: viscosity $0.127 \text{cm}^2\text{s}^{-1}$. Cylinder: radius 6 cm, rotation speed 31 rpm (in (c)). The measurements have been performed using the arrangement, shown in Fig. 51a.
Figure 51: Measurement #168: (a) experimental arrangement, used to study the jet parameters for the near-critical cylinder rotation speeds. Laser beam is deviated by a mirror affixed to the thread submerged to the flow; (b) dependence of the squared jet amplitude characteristic (squared laser beam deviation) on the cylinder rotation speed for various fluid layer depth.
Figure 52: Measurement #168. Dependence of the angle at which Taylor vortex separates (the jet occurs) from the rotating cylinder on the cylinder rotation speed. The free surface flow in the layer of 1 cm depth (other parameters as in Measurement #168) is studied. The straight line corresponds to the $\alpha \propto \Omega^{-1}$ dependence.
Figure 53: Experiments with setup, similar to the one in Fig. 8f, side and planform views: (a) plain cylinder, the flow is disordered and is concentrated near the cylinder; (b) a bolt head juts out from the cylinder surface, the precessing flow pattern appears; (c) a bolt is attached to the cylinder, a well-developed single-jet structure exists; (d) a square plate is attached to a thin arm, a well-developed single-jet structure exists. Top view and the side view are shown in each case.
Figure 54: Setup, used to create the cylindrical shear layer (a); schematic of the flow pattern, arising near the sink (b). Fluid is pumped from the lower vessel and injected tangentially to the upper vessel. Boundary layer with the negative radial velocity is formed near the bottom of the upper vessel. Precessing meso-vortices (Fig. 55) are formed at a radius of approximately 1.5 times less than the radius of the sink orifice.
Figure 55: Ink visualization of meso-vortices appearing in the setup shown in Fig. 54 with the sink diameter 8.1 cm: (a) 4-vortex regime, layer depth 6.9 cm, water flux through the sink 150 ml/s; (b) 3-vortex regime, layer depth 8 cm, water flux 150 ml/s; (c) 4-vortex regime, layer depth 6.6 cm, water flux 166 ml/s.
Figure 56: Ordering of dried peas in a basin with rotating bottom. Central part of the basin bottom rotates anticlockwise. The peas pattern precesses clockwise.
Figure 57: Schematic of the flow in the air gap with a source in the centre. Gap: height 2 cm, diameter 100 cm, outer edge open. The air is injected through the holes in the cylinder of 10 cm diameter, located in the gap centre: (a) the air is injected radially, a steady three-jet structure appears; (b) the air is injected about 20° to the radial direction, a not-quite-stable two-jet precessing pattern appears; (c) the air is injected about 45° to the radial direction, the flow is disordered.
Figure 58: Shooting procedure. Search for the eigenvalues $\sigma$ of the linearized problem \((3.1.33)-(3.1.35)\): (a) a set of lines where Re(det $\Lambda$) (bold lines) and Im(det $\Lambda$) (thin lines) from \((3.1.42)\) change the sign. Intersections of these lines correspond to the eigenvalues. The only $\sigma$ with the positive real part, that corresponds to the growing perturbation, is observed. The value $\max|V_0/c| = 50$ corresponds to maximum angular velocity of the fluid in the basic axisymmetric flow, and the eigenvalue correspondent to this velocity is seen at $\sigma \approx -5 - 44i$; (b) sketch of one iteration of the algorithm used to find the point of the lines intersection.
Figure 59: Neutral curves ($\text{Re}(\sigma) = 0$ in (3.1.28)) for the two-dimensional flow with external forcing and Darcy friction for $a = 1$, $c = 5$, $b = 20$, $\Phi = 500$, $\varepsilon = 1$. The Reynolds number is chosen as $\text{Re} = \frac{V_{0\text{max}} c}{\nu}$. Solutions of the equation (3.1.33) for $\psi(r)$, correspondent to the numbered points, are illustrated in Fig. 61. Linear approximations are fitted to the tips of neutral curves. The Reynolds number is varied by changing viscosity $\nu$, whose values are shown above for reference.
Figure 60: Neutral curves, same as in Fig. 59. Vertical bars correspond to the ratio \( \text{Im}(\sigma)/m \), that is the non-axisymmetric perturbation precession rate. In the points located at Reynolds numbers around 150-200 (zoomed in area), the non-axisymmetric perturbation precesses in the direction, opposite to the one of velocity circulation in the basic flow.
Figure 61: Absolute value of the stream function perturbation $\psi(r)$ (3.1.28) for the modes, marked by numbers in Fig. 59. Thin lines correspond to the basic flow velocity profiles $V_0(r)$ (3.1.14), (3.1.15). The ratio of the perturbation pattern phase velocity $\omega = \text{Im}(\sigma)/m$ to the maximum angular velocity of the fluid in the basic flow $\Omega = V_{0\text{max}}/c$ is indicated. Note that the perturbation #6 precesses in the direction, opposite to the direction of velocity circulation in the basic flow.
Figure 62: Contour plots of the stream function perturbations $\text{Re}(\psi(r, \varphi))$ from (3.1.28) for (a) $m = 2$; (b) $m = 1$ (labelled points in Fig. 59). The contour plots (c), (d), shows the flow perturbations (a), (b) imposed on the basic flow (3.1.7) with some arbitrary amplitude as being observed from the frame of reference rotating with the rate of the non-axisymmetric perturbation precession (3.1.54). Parameters of the flow are the same as in Fig. 59. Radius $r = c$ is indicated with a dotted line. The structure (a),(c) precesses counterclockwise (in the direction of the velocity circulation in the basic flow), while the structure (b),(d) precesses in the opposite direction. Structures, visually similar to the ones experimentally observed, can be spotted in (c) and (d). Compare with the forwards precessing structure in Fig.20f and the backwards precessing structure in Fig.15a or the forwards and backwards precessing structures in Fig.34. Note, that the amplitudes of disturbances are large enough to violate the conditions (3.1.29), necessary for the problem linearization, that makes the visualization (c), (d) doubtfully lawful.
Figure 63: Neutral curves (Re(σ) = 0 in (3.1.28)) for the two-dimensional flow with the external forcing and Darcy friction for $a = 1$, $c = 5$, $b = 20$, $\nu = 1$ for different friction coefficients $\varepsilon$. Reynolds number is chosen as $V_{0\text{max}}c/\nu$. The Reynolds number is varied by changing the driving force magnitude $\Phi$ in (3.1.11).
Figure 64: Ratio of precession rate of the neutral modes to the maximum angular speed of the fluid particles in the basic flow (diamonds) for the modes appearing at the lowest possible values of driving force (left endings of the neutral curves in Fig.63); the Reynolds number (3.1.52) (empty circles), correspondent to the value of driving force $\Phi$ (triangles) in (3.1.11), for which these modes appear. To relate obtained results to the experimental observations, the height $H$ of the gap filled with air ($\nu = 0.15 \text{cm}^2\text{s}^{-1}$), related to $\varepsilon$ as (3.1.5) is shown above the graph.
Figure 65: Neutral curves. Ratio of precession rate of the neutral mode $\omega = \text{Im}(\sigma)/m$ to the maximum angular speed of the fluid in the basic flow $\Omega = V_{\text{max}}/c$ versus geometric parameters $a, c, b$. Values $m = 1$, $\nu = 1$ and $\varepsilon = 0.1$ are fixed.
Figure 66: Neutral curves. Dependence of the non-axisymmetric perturbation precession rate $\omega = \text{Im}(\sigma)/m$ on the maximum angular speed of the fluid in the basic flow $\Omega = V_{0\text{max}}/c$ for various outer radii $b$ (compare with Fig. 29). In the case of propeller-driven turbulent flow, which is associated with our model, $c$ is not explicitly defined by dimensions of the experimental setup, so it is chosen to be a variable, parameterizing the neutral curves. The inner radius $a$ is (a) 1, (b) 0.1. Values $m = 1$, $\nu = 1$ and $\varepsilon = 0.1$ are fixed. To illustrate that the value of $c$ does not change crucially, solely defining the pattern precession rate $\omega$, the same neutral curves are plotted to the right from the graphs in $(c, \omega)$ plane.
Figure 67: (a) A vortex layer, embedded between the two irrotational layers in a circular domain without condition of $2\pi$-periodicity (3.2.12); (b) a typical screen-shot made while looking for pairs $(A_0, B_0)$, $(A_1, B_1)$, satisfying Rankine-Hugoniot conditions (3.2.33) and Lax conditions (3.2.35) or (3.2.36). The point $(A_0, B_0)$ is moved around by pressing the arrows. It is indicated on the plane $(A, B)$. Line 1 and Line 2 are plotted in accordance with (3.2.34). Intersection of these lines is the point $(A_1, B_1)$. Values of $\Omega \lambda_{1,2}$ at points $(A_0, B_0)$ and $(A_1, B_1)$ are shown in square boxes on the right. Contour levels $\lambda_{1,2} = 0$ and $\lambda_{1,2} = \omega$ are plotted. If points $(A_0, B_0)$ and $(A_1, B_1)$ are separated by either line $\lambda_1 = \omega$ or the line $\lambda_2 = \omega$, but not both, discontinuous jump $(A_0, B_0) \rightarrow (A_1, B_1)$, propagating with the velocity $\omega$ (or $-\omega$, depending on the points enumeration), is stable by Lax conditions.
Figure 68: The set where the system (3.2.28) is hyperbolic and the contour levels of $\lambda_1$, $\lambda_2$ from (3.2.31) for $a = 1$, $b = 4$, $\Omega_0 = -1$, $Q = 0.1$. 
Figure 69: The four types of discontinuities with \((A - a)\) and \((B - a)\) of order of one (Fig.67)\( b \), allowed by the Rankine-Hugoniot conditions (3.2.34) and satisfying Lax conditions (3.2.35) or (3.2.35) for \( a = 1, b = 4, \Omega_0 = -1, Q = 0.1 \). Generic configuration of the discontinuities and the sketch of characteristics, correspondent to the eigenvalues \( \lambda_1, \lambda_2 \) compared with the discontinuity trajectory in \((t, \varphi)\) space are shown. No counterclockwise-propagating increase of \( B \), that can be associated with a jet in the flow from Fig.16, is observed among these types of discontinuities.
Figure 70: A screen-shot made while looking for the discontinuities \((A_0, B_0) \rightarrow (A_1, B_1)\) propagating with the angular velocity \(\omega = 0.1\), satisfying Rankine-Hugoniot conditions (3.2.33) and Lax conditions (3.2.35). The point \((A_0, B_0)\) is placed close to the origin \((A = B = a)\) and is visible on the insert. The two discontinuities satisfy Rankine-Hugoniot conditions: \((A_0, B_0) \rightarrow (A'_1, B'_1)\) and \((A_0, B_0) \rightarrow (A_1, B_1)\). The first discontinuity does not satisfy Lax conditions. The latter one is the discontinuity of the type we are interested in (3.2.37). The point \((A_1, B_1)\) is shown in the insert.
Figure 71: A screen-shot made while looking for the discontinuities $(A_0, B_0) \rightarrow (A_1, B_1)$ propagating with the angular velocity $\omega = 0.27$, satisfying Rankine-Hugoniot conditions (3.2.33) and Lax conditions (3.2.35). The point $(A_0, B_0)$ is placed close to the origin $(A = B = a)$ and is visible in the insert. The two discontinuities satisfy Rankine-Hugoniot conditions: $(A_0, B_0) \rightarrow (A'_1, B'_1)$ and $(A_0, B_0) \rightarrow (A_1, B_1)$. The first discontinuity does not satisfy Lax conditions. The latter one is the discontinuity of the type we are interested in: the anticlockwise-propagating sudden increase of $B$ (3.2.37). Points $(A_1, B_1)$ and $(A'_1, B'_1)$ are shown in the insert.
Figure 72: Evolution of a vortex layer embedded between the two irrotational layers in a circular domain: dependence of the discontinuity propagation rate $\omega$ from (3.2.33) on (a) the outer radius of the vortex layer $B$; (b) on the vorticity $\Omega_0$; (c) on the difference ($B_0 - B_1$) while the outer radius of the vortex layer to the right from the discontinuity ($B_1$) is fixed, (d) on the difference ($B_0 - B_1$) while the outer radius of the vortex layer to the left from the discontinuity ($B_0$) is fixed. The discontinuity with $|B_1 - B_0| \rightarrow 0$ is used to plot graphs (a), (c). Inner radius of the flow domain $a = 1$, outer radius $b = 4$. Inner radius $A$ of the vortex layer stays almost unchanged ($|A_1 - A_0| \ll a$) and is chosen to be close to $a$ for all four graphs. A straight line in (b) corresponds to linear dependence $\omega(\Omega_0)$. 
Figure 73: Evolution of a system of decaying vortices in ideal fluid. A pattern, formed for $c = 1.5$, $d = 0.5$, $b = 4$, $\delta = 0.01$, $\tau = 0.05$, $\Gamma_0 = 5$ by a bunch of vortices evolving in accordance with $(3.3.2)$, $(3.3.3)$: a scatter plot of (a) vortex coordinates and (b) traces left by moving vortices. Straight lines show the amplitude of the radial velocity near the outer edge of the domain. They imitate paper flags, used by Zhukovskij in his experiments (Section 1.1.1, Fig. 2). Observe the alternating areas of positive and negative radial velocities.
Figure 74: Visualization of the axisymmetric Taylor vortices: the horizontally mirrored $r - z$ cross-section from Fig. 36a and the contour plot of the stream function (3.4.33), correspondent to the radial and axial velocities $(u, w)$. 
Figure 75: Prospective theoretical and experimental developments: (a) for large enough Reynolds numbers, the three-dimensional flow in a gap can be modelled by considering the system of boundary layers. Available measurements of the azimuthal velocity gradient in the boundary layer near the rotating cylinder (Fig. 46) may be helpful; (b) the evolution of the Taylor vortex in azimuthal direction and its destruction can be modelled. Measurements on the angle between the vortex origin and separation from the cylinder (Fig. 52) may be helpful; (c) a circular gap without $2\pi$-periodicity condition, i.e. the spiral flow domain, can be constructed to study the azimuthally propagating azimuthally localized flow perturbations; (d) a laminar vortex source flow in a gap or in a free surface layer can be studied experimentally to uncover the influence of the non-zero total radial flux.
Tables

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\ell_k$</th>
<th>$B_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$2.4997526700739E+00$</td>
<td>$3.1768309864764E+01$</td>
</tr>
<tr>
<td>2</td>
<td>$4.4999995384835E+00$</td>
<td>$7.3599657693616E+02$</td>
</tr>
<tr>
<td>3</td>
<td>$6.4999999991381E+00$</td>
<td>$1.7031507537346E+04$</td>
</tr>
<tr>
<td>4</td>
<td>$8.499999999983E+00$</td>
<td>$3.9412088259229E+05$</td>
</tr>
</tbody>
</table>

Table 4.1: $\ell_k$, $B_k$ from (3.4.34)

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Re$\lambda_\alpha$</th>
<th>Im$\lambda_\alpha$</th>
<th>Re$C_\alpha$</th>
<th>Im$C_\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-2.38620E+00$</td>
<td>$+8.81854E-01$</td>
<td>$+2.44776E-03$</td>
<td>$-7.94096E-03$</td>
</tr>
<tr>
<td>3</td>
<td>$-8.13075E+00$</td>
<td>$+1.32579E+00$</td>
<td>$-8.79035E-02$</td>
<td>$+1.20160E-02$</td>
</tr>
<tr>
<td>5</td>
<td>$-6.39704E+00$</td>
<td>$+1.25031E+00$</td>
<td>$-4.08086E-02$</td>
<td>$-5.13259E-02$</td>
</tr>
<tr>
<td>7</td>
<td>$-4.42012E+00$</td>
<td>$+1.07817E+00$</td>
<td>$+4.93005E-03$</td>
<td>$-1.90021E-02$</td>
</tr>
</tbody>
</table>

Table 4.2: $\lambda_\alpha$, $C_\alpha$ from (3.4.34)
<table>
<thead>
<tr>
<th>$\Psi_1$</th>
<th>Re</th>
<th>Im</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Psi_1$</td>
<td>$+2.44484E + 02$</td>
<td>$+1.66890E + 02$</td>
</tr>
<tr>
<td>$\Psi_2$</td>
<td>$+2.14428E + 01$</td>
<td>$-1.15319E + 01$</td>
</tr>
<tr>
<td>$\Psi_3$</td>
<td>$-3.75241E + 00$</td>
<td>$+5.08832E - 01$</td>
</tr>
<tr>
<td>$\Psi_4$</td>
<td>$+1.00000E + 00$</td>
<td>$+0.00000E + 00$</td>
</tr>
<tr>
<td>$\Psi_5$</td>
<td>$+2.64650E - 02$</td>
<td>$-7.71234E - 03$</td>
</tr>
<tr>
<td>$\Psi_6$</td>
<td>$-1.12782E - 01$</td>
<td>$-6.93118E - 03$</td>
</tr>
<tr>
<td>$\Psi_7$</td>
<td>$+3.13373E - 01$</td>
<td>$+3.15100E - 01$</td>
</tr>
<tr>
<td>$\Psi_8$</td>
<td>$+1.00000E + 00$</td>
<td>$+0.00000E + 00$</td>
</tr>
<tr>
<td>$\Psi_9$</td>
<td>$-1.37102E - 02$</td>
<td>$-2.96513E - 01$</td>
</tr>
<tr>
<td>$\Psi_{10}$</td>
<td>$-7.90962E - 01$</td>
<td>$+1.18984E + 00$</td>
</tr>
<tr>
<td>$\Psi_{11}$</td>
<td>$+1.61754E + 00$</td>
<td>$+4.24524E + 00$</td>
</tr>
<tr>
<td>$\Psi_{12}$</td>
<td>$+1.00000E + 00$</td>
<td>$+0.00000E + 00$</td>
</tr>
<tr>
<td>$\Psi_{13}$</td>
<td>$-8.21747E - 01$</td>
<td>$-6.79718E + 00$</td>
</tr>
<tr>
<td>$\Psi_{14}$</td>
<td>$-2.39742E + 01$</td>
<td>$-1.71078E + 01$</td>
</tr>
<tr>
<td>$\Psi_{15}$</td>
<td>$-4.17784E + 00$</td>
<td>$+2.06247E + 00$</td>
</tr>
<tr>
<td>$\Psi_{16}$</td>
<td>$+1.00000E + 00$</td>
<td>$+0.00000E + 00$</td>
</tr>
</tbody>
</table>

*Table 4.3: $\Psi_\kappa$ from (3.4.34)*
<table>
<thead>
<tr>
<th>$A_{ki}$</th>
<th>$k = 1$</th>
<th>$k = 2$</th>
<th>$k = 3$</th>
<th>$k = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i = 0$</td>
<td>-4.57603E + 00</td>
<td>-1.66285E - 01</td>
<td>+3.00373E - 02</td>
<td>-8.40331E - 03</td>
</tr>
<tr>
<td>1</td>
<td>+1.02869E + 00</td>
<td>+1.46158E + 00</td>
<td>+9.42526E - 02</td>
<td>-2.36436E - 02</td>
</tr>
<tr>
<td>2</td>
<td>+9.01636E - 04</td>
<td>-5.33644E - 01</td>
<td>-7.19779E - 01</td>
<td>-5.73377E - 02</td>
</tr>
<tr>
<td>3</td>
<td>+3.21249E - 03</td>
<td>-9.10223E - 03</td>
<td>+3.26458E - 01</td>
<td>+3.99046E - 01</td>
</tr>
<tr>
<td>4</td>
<td>-1.68507E - 03</td>
<td>+2.40988E - 03</td>
<td>-1.43735E - 02</td>
<td>-6.56419E - 02</td>
</tr>
<tr>
<td>5</td>
<td>+1.68179E - 04</td>
<td>+2.62488E - 04</td>
<td>+1.58039E - 03</td>
<td>+4.18061E - 04</td>
</tr>
<tr>
<td>6</td>
<td>+4.96232E - 05</td>
<td>+1.83169E - 04</td>
<td>+4.40222E - 05</td>
<td>+4.16312E - 04</td>
</tr>
<tr>
<td>7</td>
<td>-9.41107E - 04</td>
<td>-5.78893E - 05</td>
<td>-4.51109E - 04</td>
<td>-2.22699E - 05</td>
</tr>
<tr>
<td>8</td>
<td>+3.16795E - 05</td>
<td>+8.80067E - 05</td>
<td>+3.01324E - 05</td>
<td>+1.37078E - 04</td>
</tr>
<tr>
<td>9</td>
<td>-5.99114E - 04</td>
<td>+2.76862E - 03</td>
<td>+1.42908E - 02</td>
<td>-2.38764E - 03</td>
</tr>
<tr>
<td>10</td>
<td>+1.98130E - 05</td>
<td>+4.51102E - 05</td>
<td>+1.83629E - 05</td>
<td>+5.67798E - 05</td>
</tr>
<tr>
<td>12</td>
<td>+5.00258E - 05</td>
<td>+5.14624E - 05</td>
<td>+1.58870E - 05</td>
<td>+3.67953E - 05</td>
</tr>
<tr>
<td>13</td>
<td>-5.43771E - 05</td>
<td>-4.99964E - 04</td>
<td>+2.77527E - 05</td>
<td>-4.97487E - 05</td>
</tr>
<tr>
<td>14</td>
<td>+1.04018E - 05</td>
<td>+1.55685E - 04</td>
<td>-5.14583E - 06</td>
<td>+3.38223E - 05</td>
</tr>
<tr>
<td>15</td>
<td>-4.60272E - 05</td>
<td>-1.04859E - 05</td>
<td>-1.74632E - 05</td>
<td>-5.75634E - 06</td>
</tr>
</tbody>
</table>

Table 4.4: $A_{ki}$ from (3.4.34)
| $\omega|_{\Omega=0}$ | $d\omega/d\Omega$ | $H$ | $2c$ | $h$ | $2b$ | date, 2000  |
|-----------------|-----------------|-----|------|-----|-----|------------|
| 0.341           | 0.00226         | 34.2| 60   | 10  | 100 | 05 June    |
| 1.012           | 0.00339         | 29.1| 60   | 10  | 100 | 05 June    |
| 6.375           | 0.00683         | 18.9| 60   | 10  | 100 | 05 June    |
| 5.423           | 0.0204          | 12  | 60   | 10  | 100 | 05 June    |
| 0               | 0.06137         | 6.9 | 140  | 5   | 60.5| 30 June    |
| 1.821           | 0.02825         | 12  | 140  | 5   | 60.5| 30 Jun     |
| 5.356           | 0.00967         | 18.9| 140  | 5   | 60.5| 30 Jun     |
| 2.272           | 0.00394         | 24  | 140  | 5   | 60.5| 30 Jun     |
| -0.265          | 0.00223         | 29.1| 140  | 5   | 60.5| 30 Jun     |
| 0               | 0.00098         | 34.2| 140  | 5   | 60.5| 30 Jun     |
| 5.143           | 0.02908         | 22.2| 140  | 20  | 100 | 16 June    |
| 2.977           | 0.02784         | 22.2| 140  | 15  | 100 | 16 June    |
| 0.84            | 0.0195          | 22.2| 140  | 10  | 100 | 16 June    |
| -0.201          | 0.00919         | 22.2| 140  | 5   | 100 | 16 June    |
| 3.516           | 0.00416         | 22.2| 60   | 10  | 100 | 07 June    |
| 2.808           | 0.00733         | 22.2| 75   | 10  | 100 | 07 June    |
| 3.186           | 0.0093          | 22.2| 90   | 10  | 100 | 13 June    |
| 1.436           | 0.0172          | 22.2| 130  | 10  | 100 | 13 June    |
| -0.611          | 0.02873         | 22.2| 180  | 10  | 100 | 13 June    |
| -2.681          | 0.0391          | 22.2| 220  | 10  | 100 | 13 June    |
| 8.414           | 0.00865         | 12  | 90   | 5.5 | 100 | 22 June    |
| 7.569           | 0.0102          | 12  | 90   | 5.5 | 82  | 21 June    |
| 4.391           | 0.01549         | 12  | 90   | 5.5 | 60.5| 21 June    |
| 0.318           | 0.02563         | 12  | 90   | 5.5 | 40.5| 21 June    |
| 0.814           | 0.02998         | 12  | 90   | 5.5 | 31  | 21 June    |
| 1.919           | 0.00972         | 22.2| 90   | 10  | 100 | 21 June    |
| 0.917           | 0.01315         | 22.2| 90   | 10  | 82  | 21 June    |
| 0.165           | 0.01597         | 22.2| 90   | 10  | 70  | 21 June    |
| -0.714          | 0.01821         | 22.2| 90   | 10  | 60.5| 21 June    |
| -0.927          | 0.02299         | 22.2| 90   | 10  | 50  | 21 June    |
| -1.471          | 0.02811         | 22.2| 90   | 10  | 40.5| 21 June    |
| -2.155          | 0.03183         | 22.2| 90   | 10  | 31  | 21 June    |
| 8.029           | 0.00488         | 12  | 45   | 5.5 | 100 | 23 June    |
| 7.999           | 0.00485         | 12  | 45   | 5.5 | 82  | 23 June    |
| 8.152           | 0.0049         | 12  | 45   | 5.5 | 70  | 23 June    |
| 8.34            | 0.00528         | 12  | 45   | 5.5 | 60.5| 23 June    |
| $\omega|_{\Omega=0}$ | $d\omega/d\Omega$ | $H$ | $2c$ | $h$ | $2b$ | date, 2000 |
|-----------------|-----------------|----|-----|---|-----|----------|
| 6.085           | 0.00682         | 12 | 45  | 5.5| 50  | 23 June  |
| 4.777           | 0.00869         | 12 | 45  | 5.5| 40.5| 23 June  |
| 1.157           | 0.0132          | 12 | 45  | 5.5| 31  | 23 June  |
| -2.787          | 0.01993         | 12 | 45  | 5.5| 22  | 23 June  |
| -10.925         | 0.02989         | 12 | 45  | 5.5| 16  | 23 June  |
| 6.703           | 0.01477         | 12 | 140 | 5.5| 100 | 27 June  |
| 4.394           | 0.02097         | 12 | 140 | 5.5| 82  | 27 June  |
| 2.225           | 0.02523         | 12 | 140 | 5.5| 70  | 27 June  |
| 0.057           | 0.0317          | 12 | 140 | 5.5| 60.5| 27 June  |
| -0.807          | 0.03544         | 12 | 140 | 5.5| 50  | 27 June  |
| -6.565          | 0.05625         | 12 | 140 | 5.5| 40.5| 27 June  |
| 3.422           | 0.00314         | 22.2| 45  | 10 | 100 | 29 June  |
| 3.408           | 0.00369         | 22.2| 45  | 10 | 82  | 29 June  |
| 2.583           | 0.0042          | 22.2| 45  | 10 | 70  | 29 June  |
| 2.524           | 0.00481         | 22.2| 45  | 10 | 60.5| 29 June  |
| 0.708           | 0.00726         | 22.2| 45  | 10 | 50  | 29 June  |
| 0.116           | 0.00826         | 22.2| 45  | 10 | 40.5| 29 June  |
| -0.608          | 0.01096         | 22.2| 45  | 10 | 31  | 29 June  |
| -1.19           | 0.01508         | 22.2| 45  | 10 | 22  | 29 June  |
| 1.482           | 0.0141          | 22.2| 75  | 10 | 60.5| 02 July  |
| -0.55           | 0.01704         | 22.2| 75  | 10 | 50  | 12 June  |
| 0.459           | 0.0171          | 24  | 90  | 10 | 60.5| 02 July  |
| 1.876           | 0.00876         | 24  | 90  | 10 | 100 | 02 July  |
| 1.735           | 0.02233         | 24  | 90  | 10 | 40.5| 02 July  |
| -0.196          | 0.02539         | 24  | 90  | 15 | 60.5| 02 July  |
| 2.068           | 0.00482         | 24  | 90  | 5.5| 60.5| 02 July  |
| -0.106          | 0.01516         | 24  | 60.2| 10 | 40.5| 02 July  |
| 0.538           | 0.01907         | 24  | 149 | 10 | 100 | 02 July  |
| 0.685           | 0.01169         | 24  | 90  | 7.5| 60.5| 04 July  |
| 0              | 0.00126         | 24  | 90  | 2  | 60.5| 04 July  |
| 1.159           | 0.02418         | 24  | 90  | 20 | 60.5| 04 July  |

Table 4.5: Precession rate of the jet in the air gap. Coefficients of the linear fits $\omega = \omega|_{\Omega=0} + \alpha \Omega$ to the data sets $\omega(\Omega)$. Outer edge of the gap is open to the atmosphere, 4-blades propeller is mid-gap positioned, axis of the propeller coincide with the axis of the gap. Measurements are performed with stop-watch. Summer 2000, Hong Kong University of Science and Technology.