5. Results and Analysis

Approximation with Uniform Distribution

Numerical Solution

Four finite difference schemes which are second-order central difference, forth-order central difference, pade compact scheme and sixth-order compact scheme, are performed in order to approximate the first derivative of typical function. The approximation is performed with computational domain size of 2π. The computational range is [-π, π]. The number of grid point is 8 points and uniform distribution is applied firstly. Then the approximation with non-uniform distribution will be performed with typical transformation function. The small number of grid points is applied in order to compare the numerical solution from each approximation scheme easily. The boundary condition is periodic boundary condition where, $f(0) = f(L)$

Since the typical function is $f(x) = \sin(x)$, it has been known that its first derivative can be calculated analytically. The exact first derivative of typical function is

$$f'(x) = \cos(x) \text{ ----- } (5.1)$$

By applying the second-order central difference scheme with uniform distribution (Equation (1)), the following graphs are obtained from MS EXCEL where the blue line indicates the analytic solution and the pink line indicate the approximating solution.

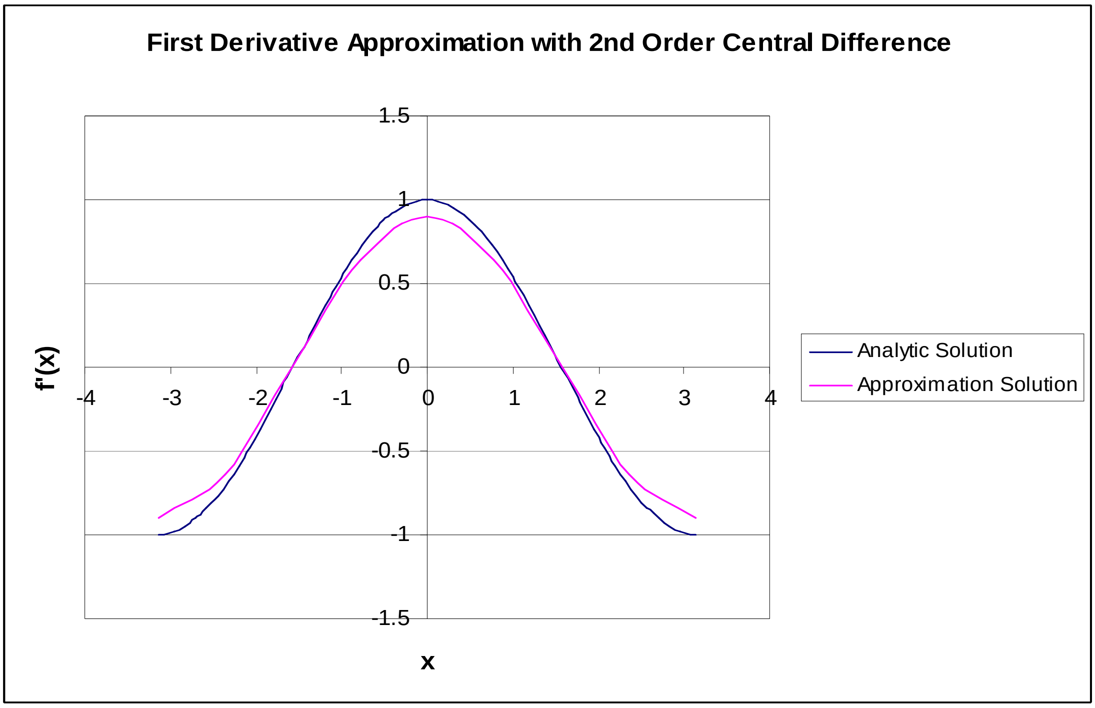


Figure 5.1 : First Derivative Approximation by Second-Order Central Difference

According from the figure (5.1), since, the grid spacing is uniform; the error is quite uniform from starting boundary to ending boundary as well. The error is large because very small number of grid point is applied. This error can be reduced when the number of grid point is increased.

Then, the higher order central difference which is forth-order central difference is now being applied with the uniform distribution of 8 grid points. The numerical solution is shown in figure 5.2

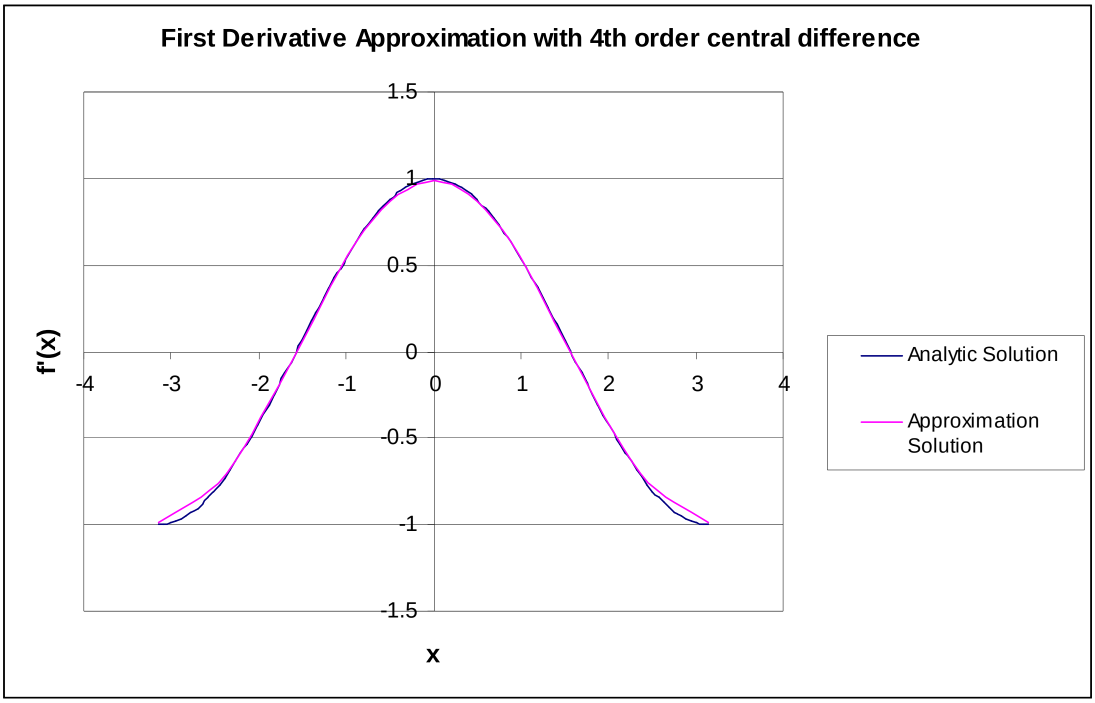


Figure 5.2 : First Derivative Approximation by Forth-Order Central Difference

When the higher order scheme is applied, the error is reduced automatically although the number of grid point is exactly the same. The approximation solution is developed, then, it is very close to the analytic solution. Both second-order and forth-order central difference are explicit method. Next, the implicit method will be used for the approximation. The first method to be applied is the pade compact scheme. Eight grid points with uniform distribution is used in the approximation. The boundary condition is also periodic boundary. Especially for the implicit method, the matrix algorithm is needed. As it has been explained in the *methodology* section, there are two types of matrix algorithms which are normal type and periodic type. The difference in the numerical solution between applying those two algorithms will be shown later. The algorithm that will be applied in this section is the periodic one. The following graph shows the numerical result obtained by applying the normal pade scheme where the coefficient matrix is solved by the periodic matrix algorithm.

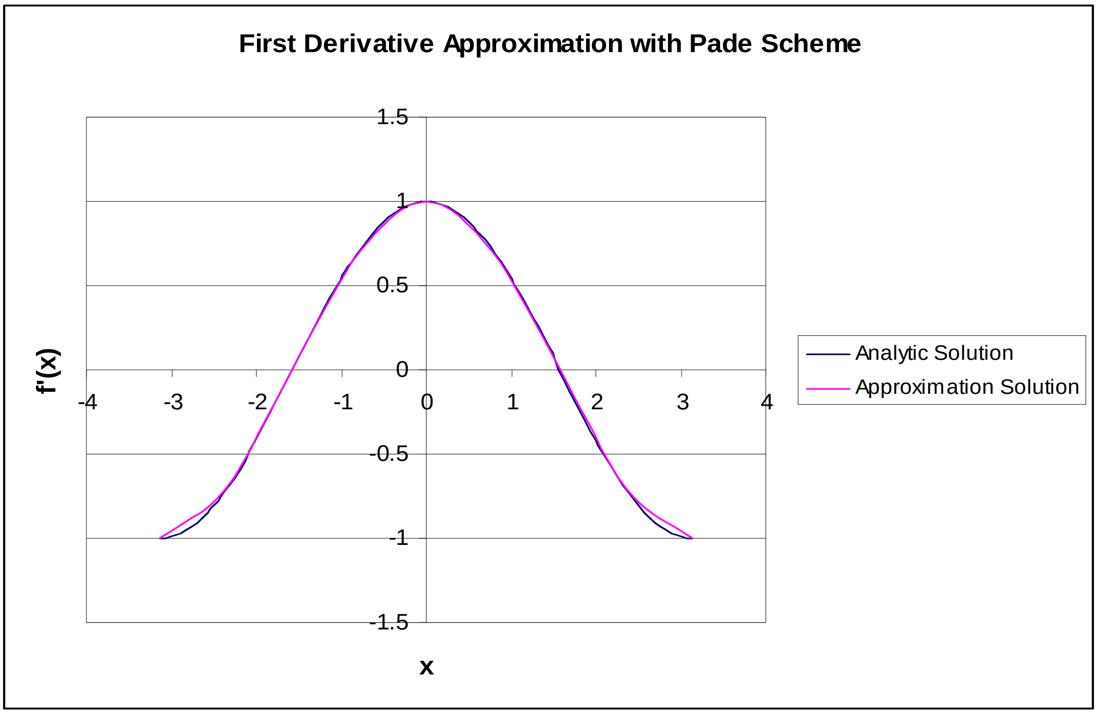


Figure 5.3 : First Derivative Approximation by Pade Compact Scheme

It has been shown in Lele's paper (Lele S.K., 1992) that the order of accuracy of Pade compact scheme is forth order which is equals to the order of accuracy of the forth-order central difference. The graph 5.3 shows that the numerical solution is very similar to that obtained from the forth-order central difference. This is the proof that whether the approximation scheme is explicit or implicit, the numerical solution is in the same order of accuracy as long as the order of accuracy of the approximation scheme is the same.

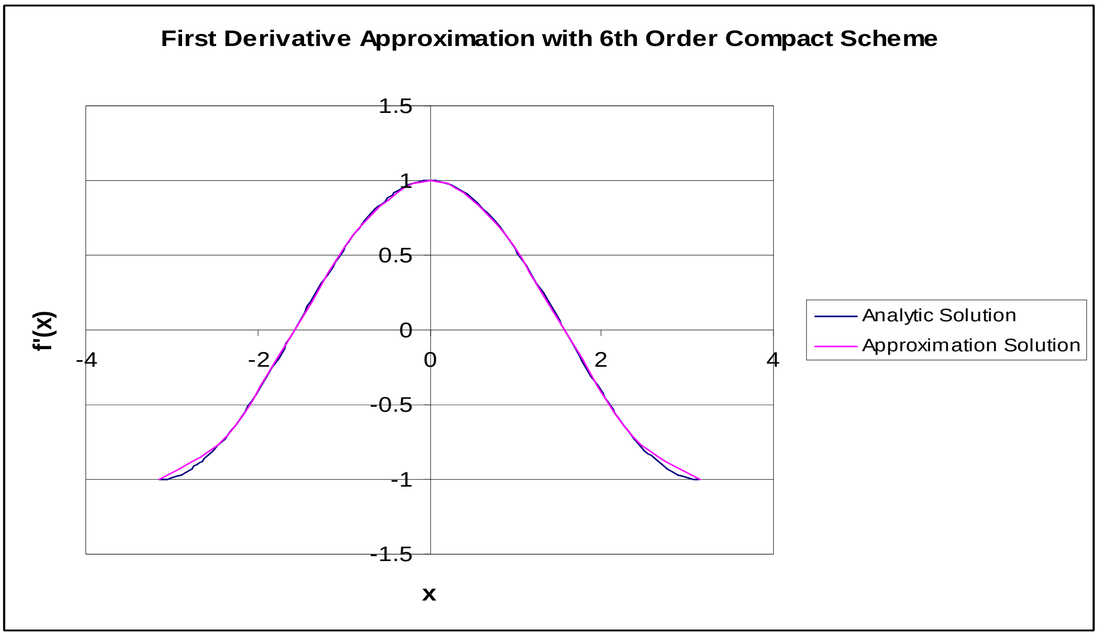


Figure 5.4 : First Derivative Approximation by Sixth-Order Compact Scheme

If the higher order approximation scheme, which is sixth-order compact scheme, is applied. Figure 5.4 is obtained. It can be seen that very small error is introduced and very accurate approximation solution is achieved. This approximation is performed under just 8 grid points.

This can be concluded that the higher order approximation scheme provides more accurate numerical solution than the lower order scheme even it is explicit or implicit when the number of grid point is fixed. In fact, the implicit method provided less accurate solution than those obtained from explicit method (Moin P., 2001) because the interation error from matrix algorithm is introduced when the implicit method is applied. However, the implicit method will be used in the aspect of unconditionally stable solution.

Effect of Matrix Algorithm on Compact Scheme

As it has been described that there are two types of matrix algorithm to be applied to solve the coefficient matrix in the approximation with compact scheme, this section will provide the two graphs of numerical solution. One is obtained by applying the normal TDMA and the other one is by applying the PTDMA.

The following matrix equation is achieved by applying the compact scheme with uniform grid spacing. The number of grid point is N+1 and computational domain is from $-\pi$ to π.

$$\begin{bmatrix} 4 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 0 & ; & . & . & 0 \\ 0 & 0 & 0 & 0 & . & . & . \\ 1 & 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \times \begin{bmatrix} f'(1) \\ f'(2) \\ f'(3) \\ . \\ . \\ . \\ f'(N) \end{bmatrix} = \frac{3}{\Delta x} \begin{bmatrix} f_2 - f_N \\ f_3 - f_1 \\ f_4 - f_2 \\ . \\ . \\ . \\ f_1 - f_{N-1} \end{bmatrix} \quad \text{-------- (5.2)}$$

The coefficient matrix obtained, when the periodic boundary condition is applied, is a bit different from the normal tri-diagonal matrix that there are two value added at the lower-left and upper-right corner. Even it is just a bit difference but it causes the error in the numerical solution when the normal TDMA is applied instead of PTDMA.

When the normal TDMA is applied, the added terms will be ignored. Then, the equation (5.2) becomes

$$\begin{bmatrix} 4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 0 & ; & . & . & 0 \\ 0 & 0 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \times \begin{bmatrix} f'(1) \\ f'(2) \\ f'(3) \\ . \\ . \\ . \\ f'(N) \end{bmatrix} = \frac{3}{\Delta x} \begin{bmatrix} f_2 - f_N \\ f_3 - f_1 \\ f_4 - f_2 \\ . \\ . \\ . \\ f_1 - f_{N-1} \end{bmatrix} \text{------- (5.3)}$$

The following graph is obtained when the equation (5.3) is solved with N = 100 by applying normal TDMA

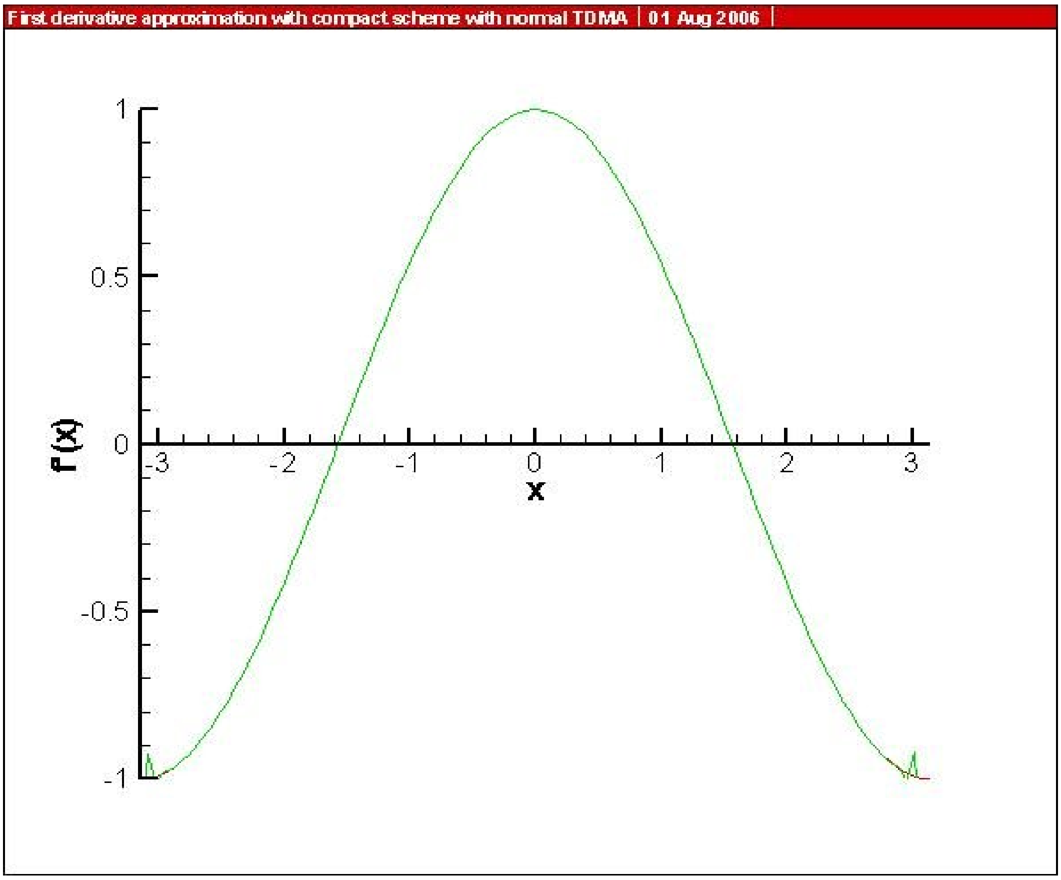


Figure 5.5 : Numerical Solution obtained by the Compact Scheme with normal TDMA applied

From figure 5.5, it can be seen that there is the big error occurred at the left and right boundary. This is because the boundary terms are ignored. However, the approximation solution is accurate at points apart from the boundary. Since the main tri-diagonal of the matrix is the same as the original coefficient matrix obtained from the compact scheme, the numerical solution is accurate. In the other word, it can be concluded that, when normal TDMA is applied, the numerical solution is accurate at the point that is not the function of the boundary point. On the other hand, the numerical solution will not be accurate at some point that is calculated from the boundary point.

Anyway, this error can be get rid by applying the periodic tri-diagonal matrix algorithm instead of the normal type. The numerical result is shown in figure 5.6. It is the evident that PTDMA do not ignore the added term in the coefficient matrix. So, numerical solution is accurate at every point.

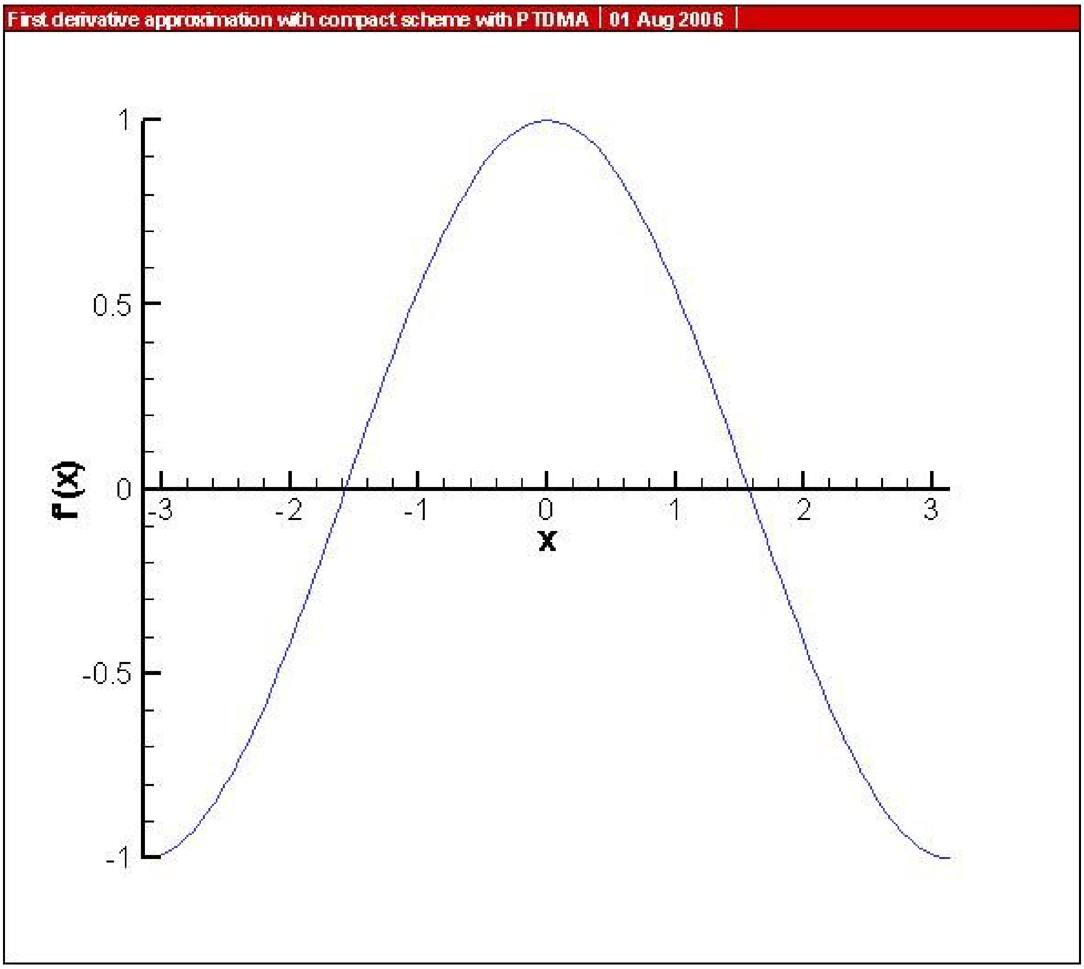


Figure 5.6 : Numerical Solution obtained by the Compact Scheme with PTDMA applied

Effect of Grid Spacing on Approximation Scheme with Uniform Distribution

In this section, the various grid spacing will be included in the approximation. Four approximation schemes will be performed to approximate the derivative of typical function with a hundred grid spacing (N = 100). The effect of grid spacing on the accuracy of the approximation scheme will be investigated on the graph between the truncation error and the grid spacing.

Since, this accuracy investigation is performed for the approximation with uniform grid spacing, so it is general that the truncation error at every point is the same. In this research, the error is calculated at point

$$x = -2.88$$

And the exact derivative of the typical function is approximately,

$$f'(x) = -0.966$$

In the computational process, more digits after the decimal point will be included by the function called *double precision* in FORTRAN. The error is calculated at this point because of some reason concerning with the error investigation of the approximation with non-uniform distribution. This reason will be described later in this report.

When the approximation schemes are applied with seven value of grid spacing, the following graph is obtained.

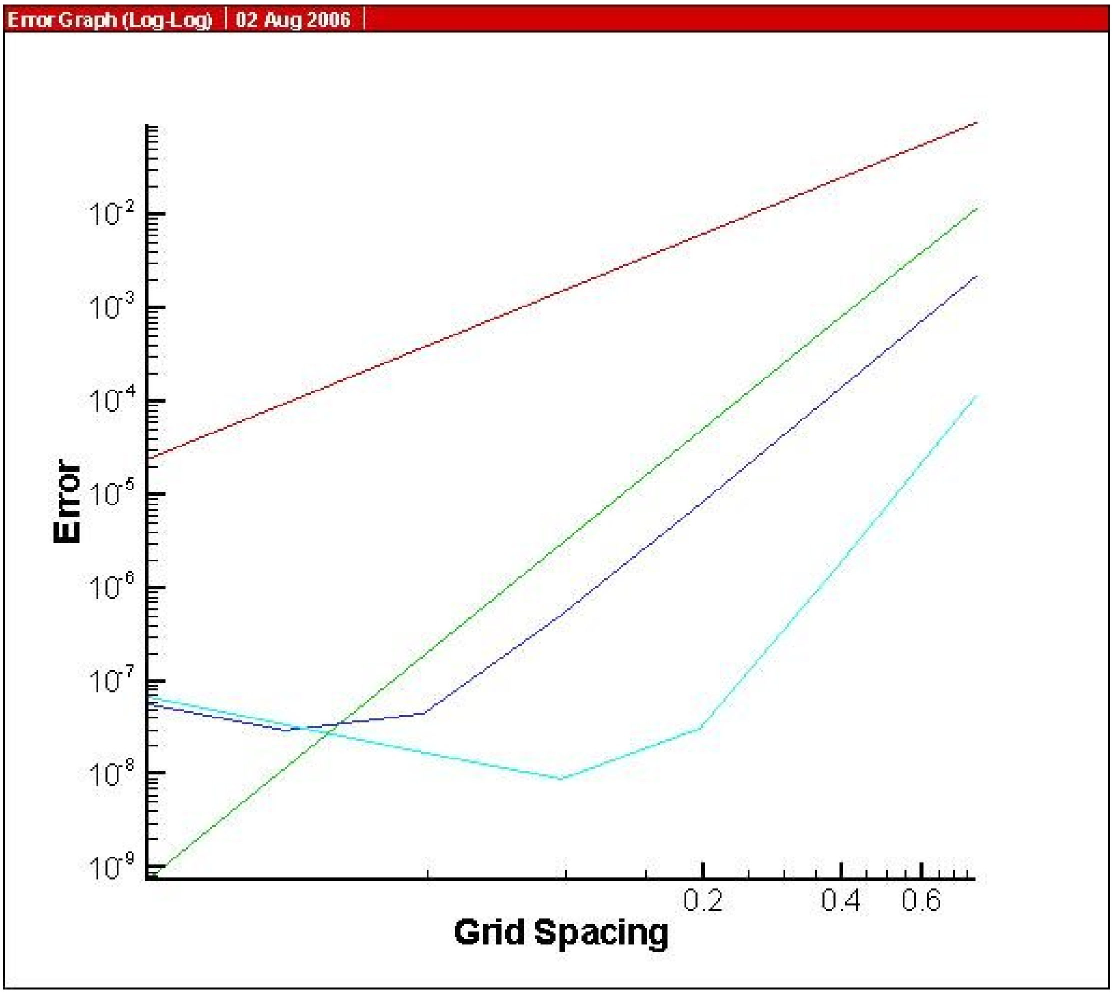


Figure 5.7 : Truncation Error V.S. Grid Spacing (Log-Log) for each approximation scheme where,
Red -- Second Order Central Difference
Green – Forth Order Central Difference
Dark Blue – Pade Compact Scheme
Blue – Sixth Order Compact Scheme

From figure 5.7, the round-off error is introduced when the compact schemes are performed. So, the truncation error graph for the compact schemes is not exactly the straight line. By the way, the analysis can still be done at the grid spacing where every approximation scheme has got exactly straight line.

First of all, the slope of the graph will be discussed. It can be seen that the slope of the truncation error graph of the second order central difference has the smallest slope and the sixth order compact scheme has the highest slope. This is the evident that the slope of the graph is directly proportional to the order of accuracy of the approximation scheme. So, the slope of the truncation error graph of forth order central difference and of Pade compact scheme are equal because the order of accuracy of those two schemes are the same. According from the slope of the truncation error graph, it can be concluded that the advantage of the higher order approximation scheme is that when the grid spacing is decreased, the error is reduced faster than the lower order scheme. This means that the grid spacing has stronger effect on the higher order scheme than the lower order scheme. In the other word, the higher order scheme is more sensitive to the effect of grid spacing than the lower order scheme.

Secondly, figure 5.7 shows that when the grid spacing is decreased, the error is decreased. On the other hand, when grid spacing is increased, the truncation error is increased as well. This is the evident that the grid spacing affect to the accuracy of the numerical approximation scheme.

Finally, the truncation error graph of the forth order central difference and the Pade compact scheme will be considered. Even the slope is the same but the accuracy is a bit difference. This can be seen from figure 5.7 that, if the grid spacing is fixed, the truncation error from the forth order central difference scheme is larger than the error from the Pade compact scheme. It can be concluded that the Pade compact scheme which is implicit method, is more accurate than forth order compact scheme although the order of accuracy are the same. The order of accuracy just shows the sensitivity to the grid spacing in this case.

Modified Wave Number Analysis

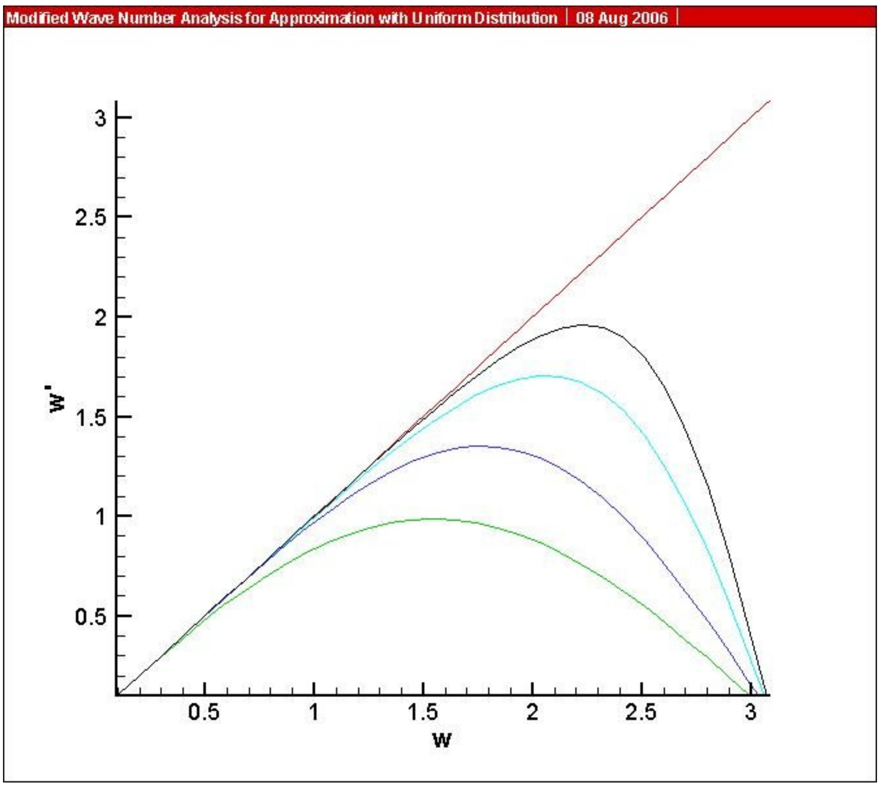


Figure 5.8 : Modified Wave Number Analysis for Numerical Approximation with Uniform Grid Spacing

In the modified wave number analysis part, the modified wave number for each numerical approximation schemes will be compared with the modified wave number of the analytic solution. The approximation is going on with the computational domain of $-\pi$ to π. The number of grid point is 64 grid points, so the grid spacing is about $\pi/32$. The analysis has been done at $x = -2.88$.

The relationship between the normalized modified wave number (w') and the modified wave number (w) of the analytic solution is exactly the straight line with the slope of 1 as shown by the red line in figure 5.8. This means that the normalized modified wave number and the modified wave number of the analytic solution always equals. In some case, it is difficult, sometime impossible, to solve the governing equation analytically, the spectral approximation can be used as an analytic solution because the spectral method provides the numerical approximation that is very close to the analytic solution (Lele S.K., 1992)

In figure 5.8, the green line expresses the modified wave number analysis for the second order central difference, the dark blue line is for the forth order central difference, the blue line is for the Pade compact scheme and the black line is for the sixth order compact scheme. According from figure 5.8, the advantage of applying the higher order approximation scheme over the lower order scheme is completely clear. It can be seen that the highest order approximation scheme, in this case, the sixth order compact scheme, can follow the exact solution over the widest range of wave number until it reaches the maximum point where then, more error is introduced. So, the graph is dropped when the wave number is still increased. For the lowest order approximation scheme, the second order central difference, the graph follows the modified wave number graph of exact solution with the minimum range. This can be concluded that the higher order scheme can stay close to the line of exact solution further than the lower order scheme.

For the scheme with the same order of accuracy, the forth order central difference and the standard Pade compact scheme, this modified wave number analysis provides the clear evident that, although the order of accuracy is the same, the Pade scheme is more accurate as shown in figure 5.8 that the Pade scheme line can follow the exact line further than of the forth order central difference.

Approximation with Non-Uniform Distribution

Numerical Solution

Four different approximation schemes will be also performed in order to calculate the first derivative of typical function. The uniform grid spacing will be transformed to non-uniform distribution by two different transformation functions which are hyperbolic tangent and hyperbolic sine. The control parameter will also be varied from 1 to 3. The numerical solution is plotted in TECHPLOT and compared with the analytic solution.

Firstly, the hyperbolic tangent with control parameter 2 will be applied as transformation function. The domain of the non-uniform grid and uniform grid are the same which is $-\pi$ to π. The number of grid point is 64 grid points. The numerical obtained from second order central difference, forth order central difference scheme, Pade compact scheme, and sixth order compact scheme are shown in figure 5.9, 5.10, 5.11, and 5.12 respectively where the green line indicates the analytic solution and the blue line indicates the approximation solution.

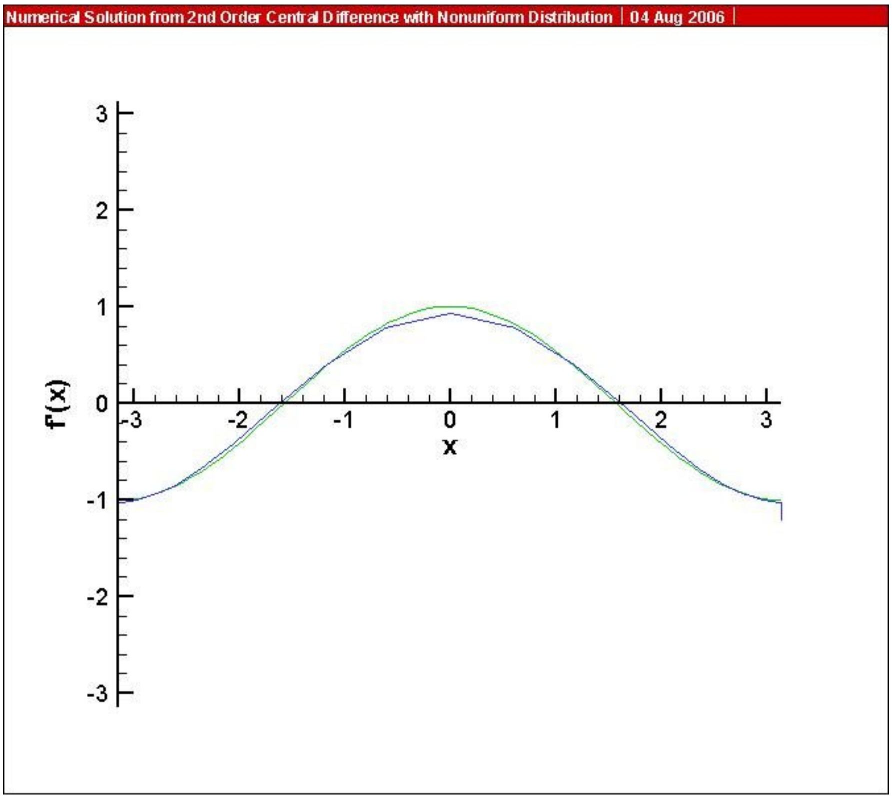


Figure 5.9 : Numerical Solution from Second-Order Central Difference with Nonuniform Distribution (Tanh2)

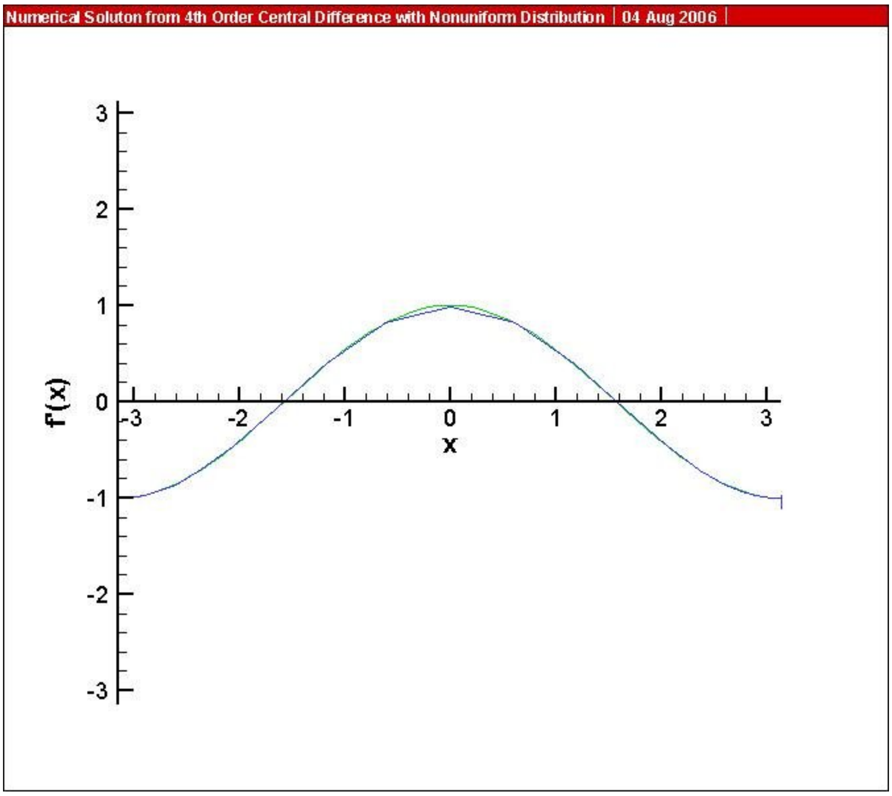


Figure 5.10 : Numerical Solution from Forth- Order Central Difference with Nonuniform Distribution(Tanh2)

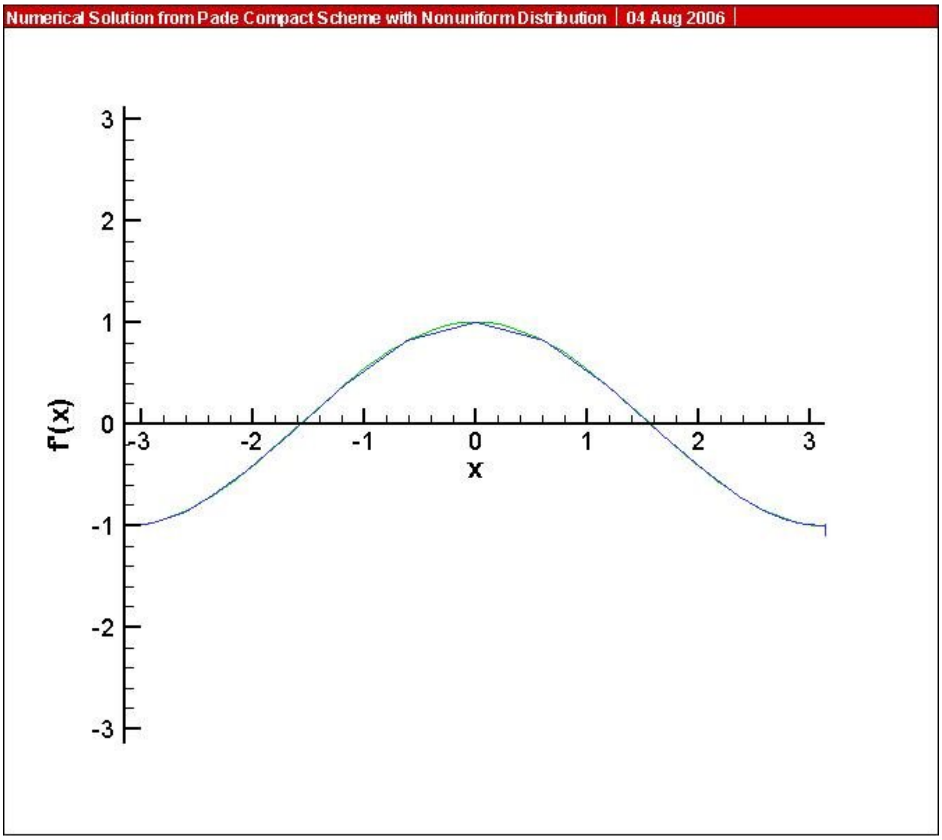


Figure 5.11 : Numerical Solution from Pade Compact Scheme with Nonuniform Distribution(Tanh2)

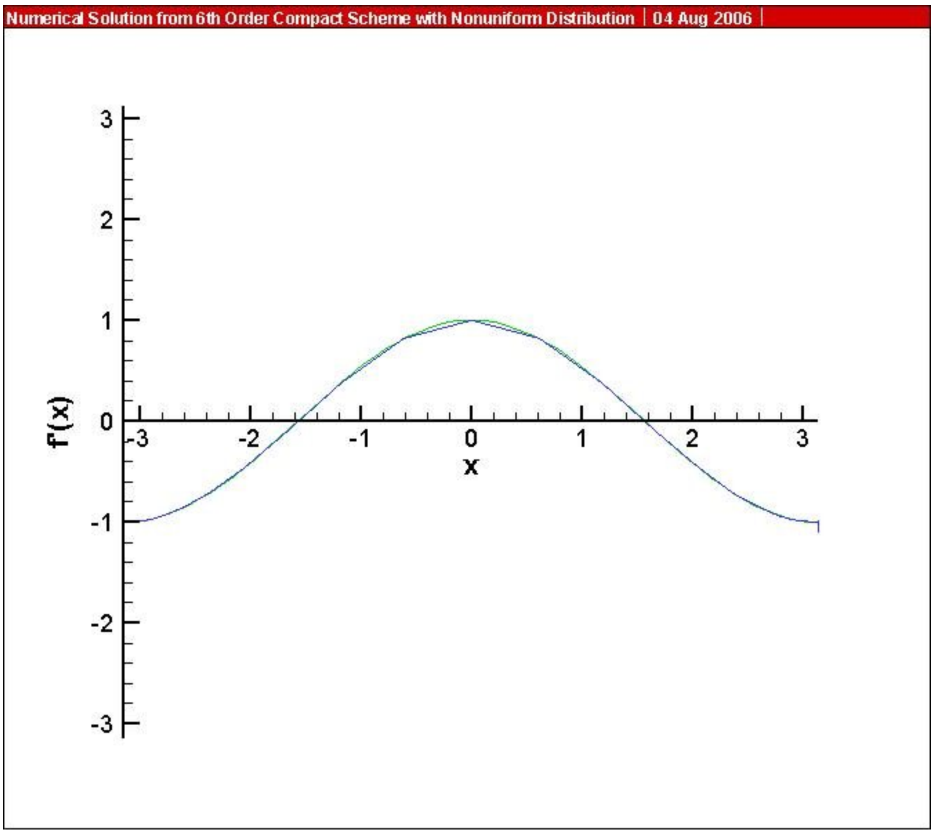


Figure 5.12: Numerical Solution from Sixth- Order Compact Scheme with Nonuniform Distribution(Tanh2)

According from the graph, the truncation error from the second order central difference is still the maximum comparing with the other scheme when the number of grid point are the same. This implies that even the grid spacing is non-uniform; the higher order scheme is more accurate than the lower order scheme. From the numerical result obtained from those four approximation scheme, the approximation is more accurate at the point near the boundary. On the other hand, it is not that accurate at the middle point. It seems the error is increased from the boundary point to the middle point and then decreased until it reaches the other boundary. In order to understand this phenomenon, the characteristic of the hyperbolic tangent grid spacing is needed to be considered.

The natural of hyperbolic tangent grid spacing is that, the grid spacing will be the smallest at the boundary and then the grid spacing is slightly increased until it reaches its maximum at the middle point of the computational domain. Then, the grid spacing is decreased with the same as increasing rate until it meets the other boundary. Due to the characteristic of hyperbolic tangent grid point, the numerical solution must be the most accurate at the boundary. The maximum error will be introduced at the middle point of the computational domain. The numerical solution shown in figure 5.9-5.12 are the evident that the approximation is the most accurate at the boundary and least accurate at the middle point.

When the transformation function has been changed to the function of hyperbolic sin, the grid non-uniformity is changed as well. The following graphs are obtained with the control parameter of 2 where the red line indicates the exact solution and the blue line indicates the approximation solution.

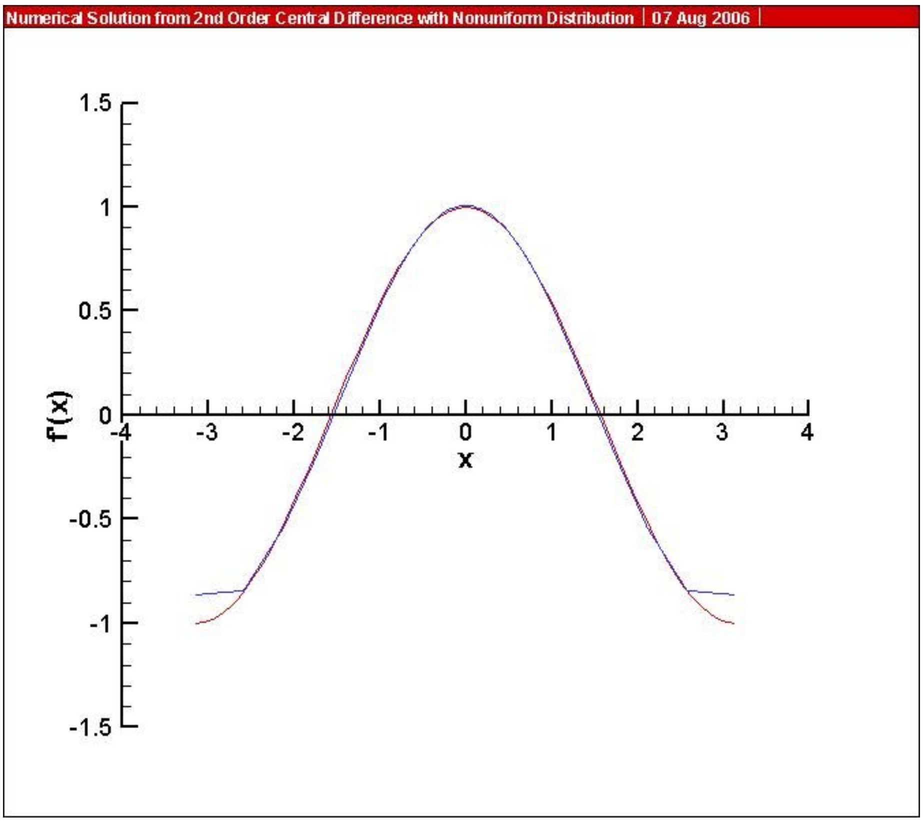
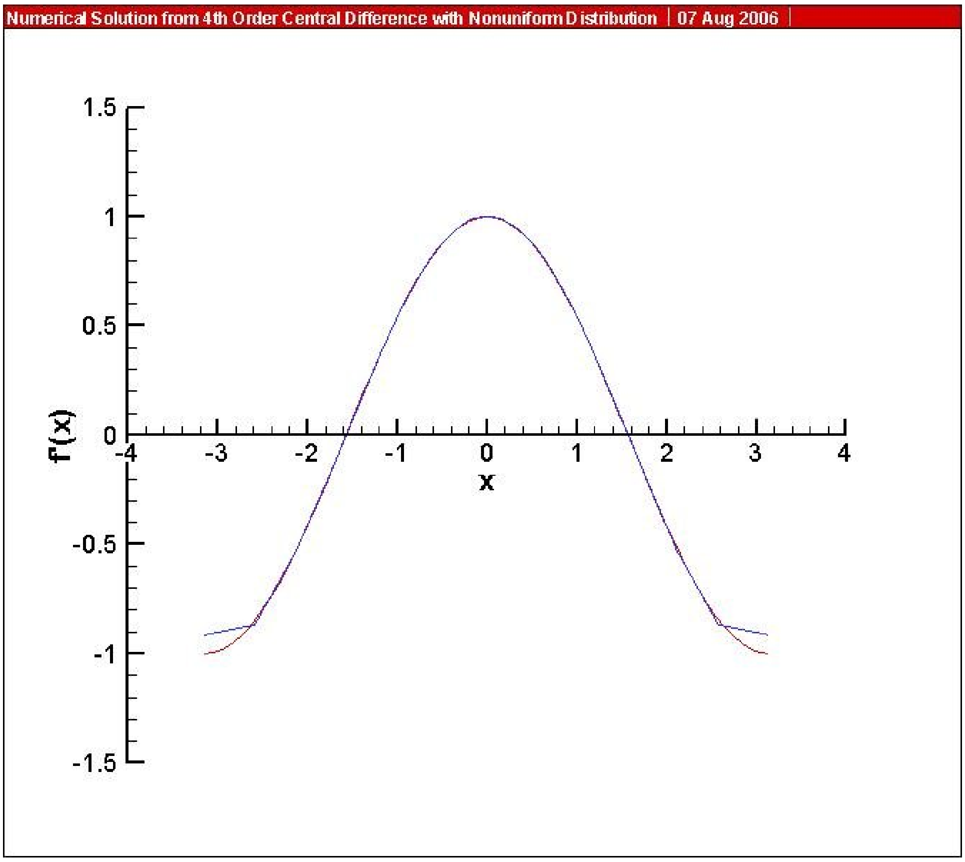


Figure 5.13 : Numerical Solution from Second-Order Central Difference with Nonuniform Distribution (Sinh2)



Figrue 5.14 : Numerical Solution from Forth-Order Central Difference with Nonuniform Distribution(Sinh2)

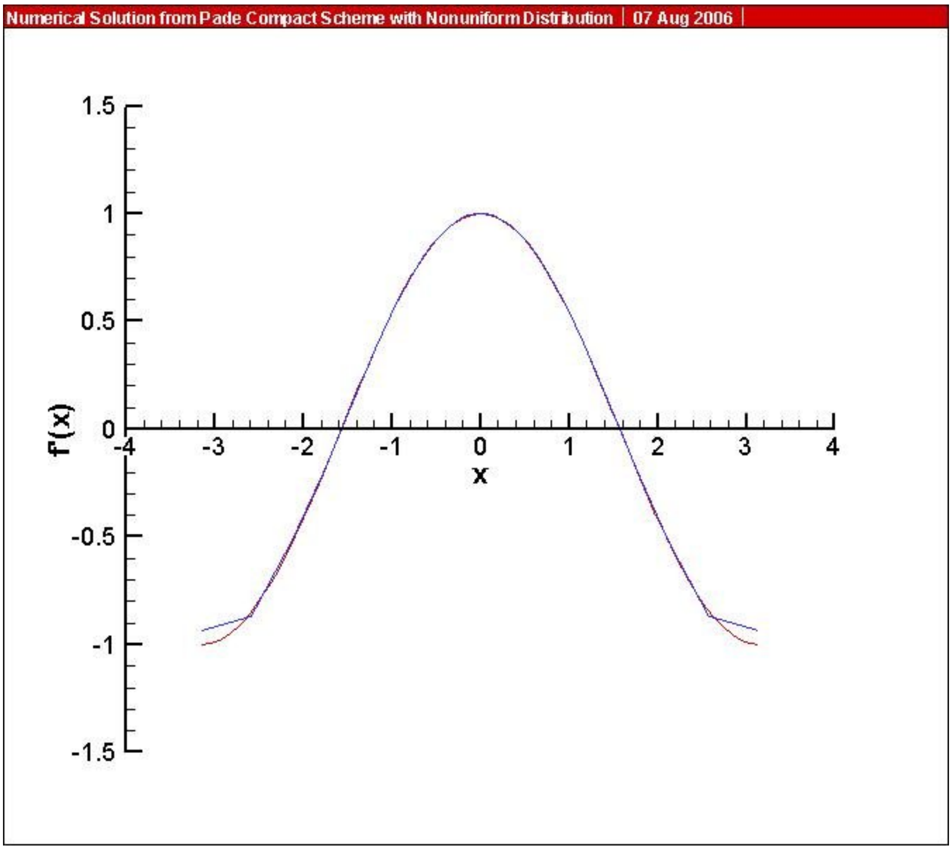


Figure 5.15 : Numerical Solution from Pade Compact Scheme with Nonuniform Distribution(Sinh2)

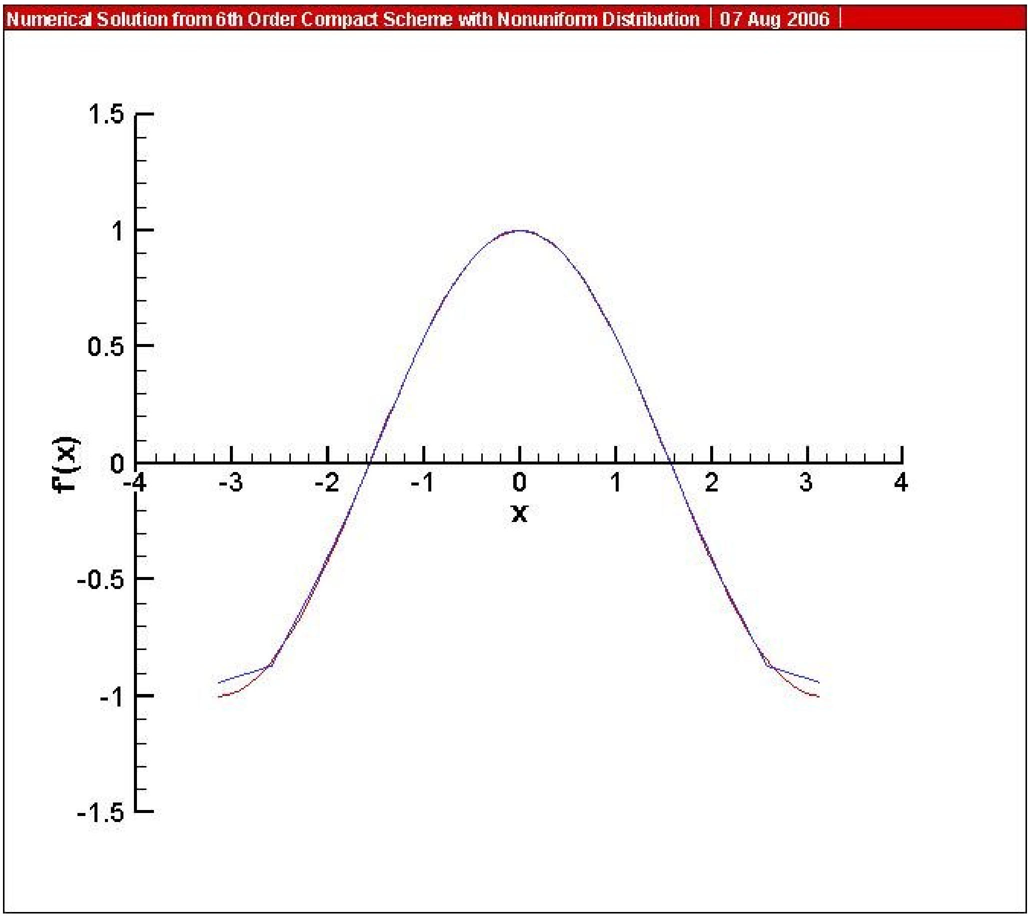


Figure 5.16 : Numerical Solution from Sixth Order Compact Scheme with Nonuniform Distribution(Sinh2)

From figure 5.13 to 5.16, the truncation error is the maximum at the boundary and it becomes the minimum at the middle point of the computational domain. This is because of the natural of the hyperbolic sine grid. The natural of hyperbolic sine grid is completely opposite to the hyperbolic tangent grid that the grid spacing will be the largest at the boundary and then slightly decreased with respect to the value of control parameter. The smallest grid spacing is occurred at the middle point of the computational domain. It has been known that the truncation error will be increased as long as the grid spacing is increased. This is the proof that the hyperbolic sine grid provides the accurate numerical solution at the middle point and rough approximation at both boundaries.

Effect of Grid Spacing on Approximation Scheme with Non-uniform Distribution

In this section, the comparison between the truncation error generated from the uniform distribution and from the non-uniform distribution will be performed. The number of grid point will be fixed at 64 grid points and the computational domain is from $-\pi$ to π. For the approximation with non-uniform distribution, the control parameter will be set to 2.

Since the grid spacing for the non-uniform approximation is not equals at every grid point, the numerical solution obtained in the previous section is the evident that the truncation error at each point is definitely difference. This is the difficulty in order to compare the accuracy between them. The method used in this research is that, firstly, the grid spacing of the uniform grid spacing will be calculated.

Then, it will be compared with the grid spacing of the non-uniform distribution. The point where the grid spacing is very close together will be considered. From the comparison, the considering grid point is at $x = -2.88$

The truncation error graph comparing between the numerical result from the approximation with second order central difference with uniform grid spacing and non-uniform grid spacing is shown below where the red line is for the uniform approximation and the green line is for the non-uniform one.

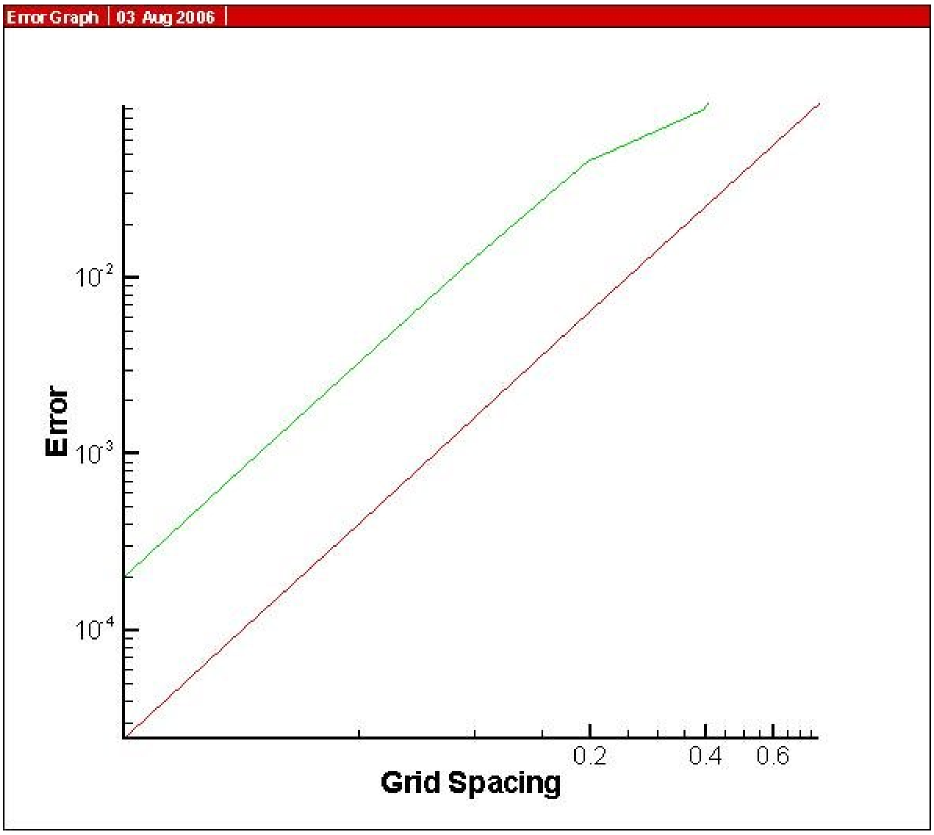


Figure5.17 : Truncation Error V.S. Grid Spacing for Second-Order Central Difference
Red – Uniform Distribution
Green – Non-uniform Distribution

From the figure, the slope of the truncation error graph for both estimations is the same. This means that the transformation function does not affect the sensitivity to the grid spacing of the scheme itself. In the accuracy aspect, it can be noticed that, at the same grid spacing, the truncation error from the non-uniform approximation is greater than the uniform distribution. The uniform approximation is more accurate than the non-uniform approximation.

The graph between truncation error and grid spacing from the forth order central difference are following.

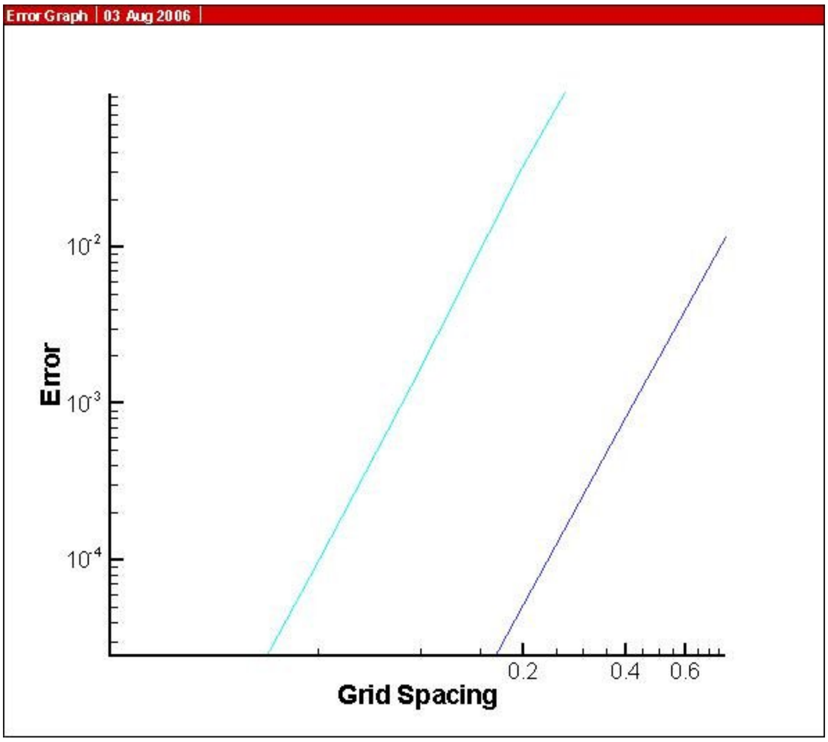


Figure 5.18 : Truncation Error V.S. Grid Spacing for Forth-Order Central Difference
Red – Uniform Distribution
Green – Non-uniform Distribution

It has been confirmed that the transformation function does not affect to the sensitivity to the grid spacing of the approximation scheme. For approximation with forth-order central difference, the space of the graph between the uniform approximation and non-uniform approximation is bigger than of the second-order central difference. This implies that the higher order scheme is more sensitive to the quality of the grid. It means that, for the higher order scheme, when the grid points lose their uniformity, more truncation error is generated.

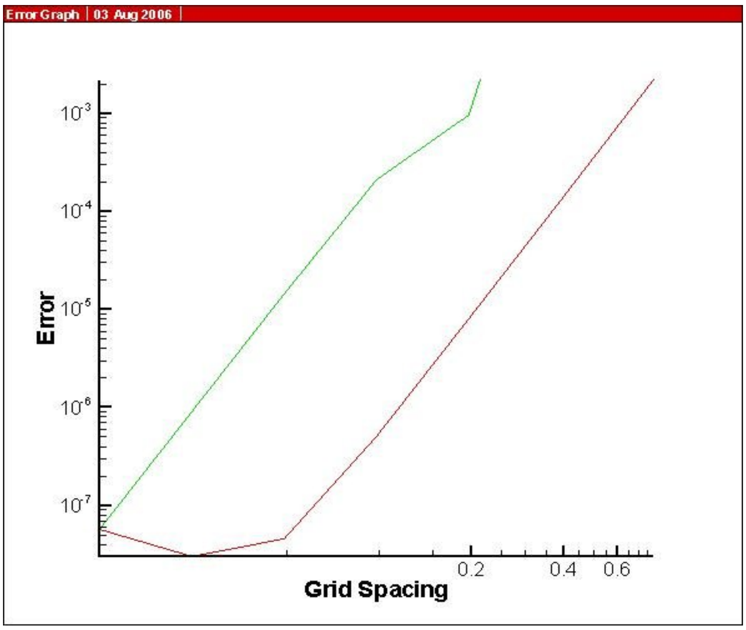


Figure 5.19 : Truncation Error V.S. Grid Spacing for Pade Compact Scheme
Red – Uniform Distribution
Green – Non-uniform Distribution

To confirm this effect of the quality of the grid point on the approximation scheme, the error graph of the Pade compact scheme and sixth-order compact scheme are needed. The truncation error graph for the Pade compact scheme is shown in figure 5.19,

Since, the order of accuracy of the Pade compact scheme and the forth-order central difference are equal, so the spacing between the truncation error graph of the uniform and non-uniform approximation are the same. This can be concluded that the sensitivity to the effect of the grid quality of those two approximation schemes is at the same level. Finally, the truncation error graph for the sixth-order compact scheme is shown below.

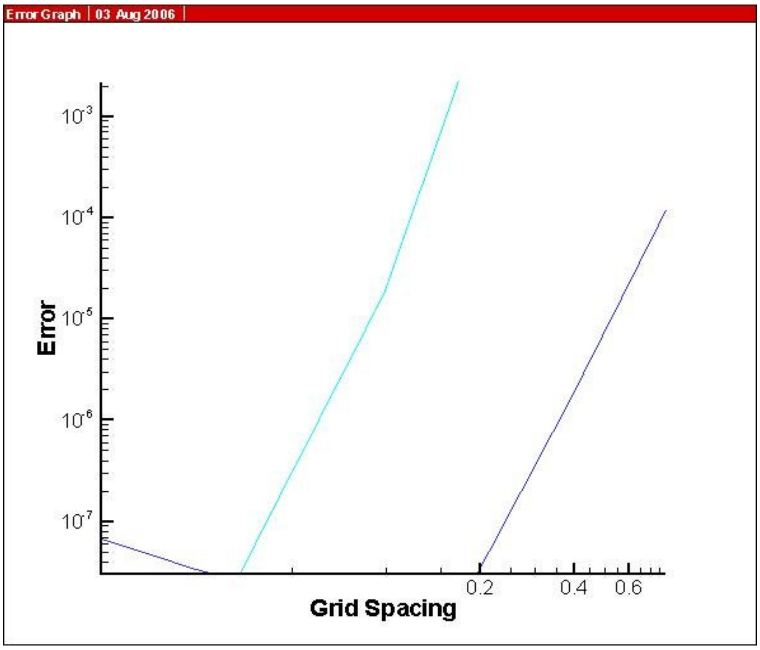


Figure 5.20 : Truncation Error V.S. Grid Spacing for Sixth-Order Compact Scheme
Red – Uniform Distribution
Green – Non-uniform Distribution

The truncation error graph for the sixth-order compact scheme is also the evident that the higher order approximation scheme is more sensitive to the grid quality. It can be seen that the spacing between the lines is the biggest since it is the highest order scheme being considered in this research.

In short, it can be concluded that the second-order central difference is the most insensitive to the grid quality and the sixth-order compact scheme has the highest sensitivity to the grid quality.

Modified Wave Number Analysis

The modified wave number analysis for the non-uniform distribution will be performed. The effect of the transformation function and the control parameter on the modified wave number will be investigated. First, the hyperbolic tangent transformation function will be considered, then following by the hyperbolic sine function. In order to investigate the effect of control parameter, three values of control parameter will be considered which are 1, 2 and 3.

Hyperbolic Tangent Transformation Function

The hyperbolic tangent transformation function will be applied to the uniform approximation. The number of grid point is 100 grid points and the modified wave number analysis will be performed at $x = -2.88$

For the second order central difference, the relationship between the normalized modified wave number and the modified wave number is shown in the following figure where the red line indicates the exact solution, the green line is for approximation with uniform grid spacing, the dark blue line is for the non-uniform approximation with control parameter of 1 and the blue and the pink line indicate the non-uniform approximation with control parameter of 2 and 3 respectively.

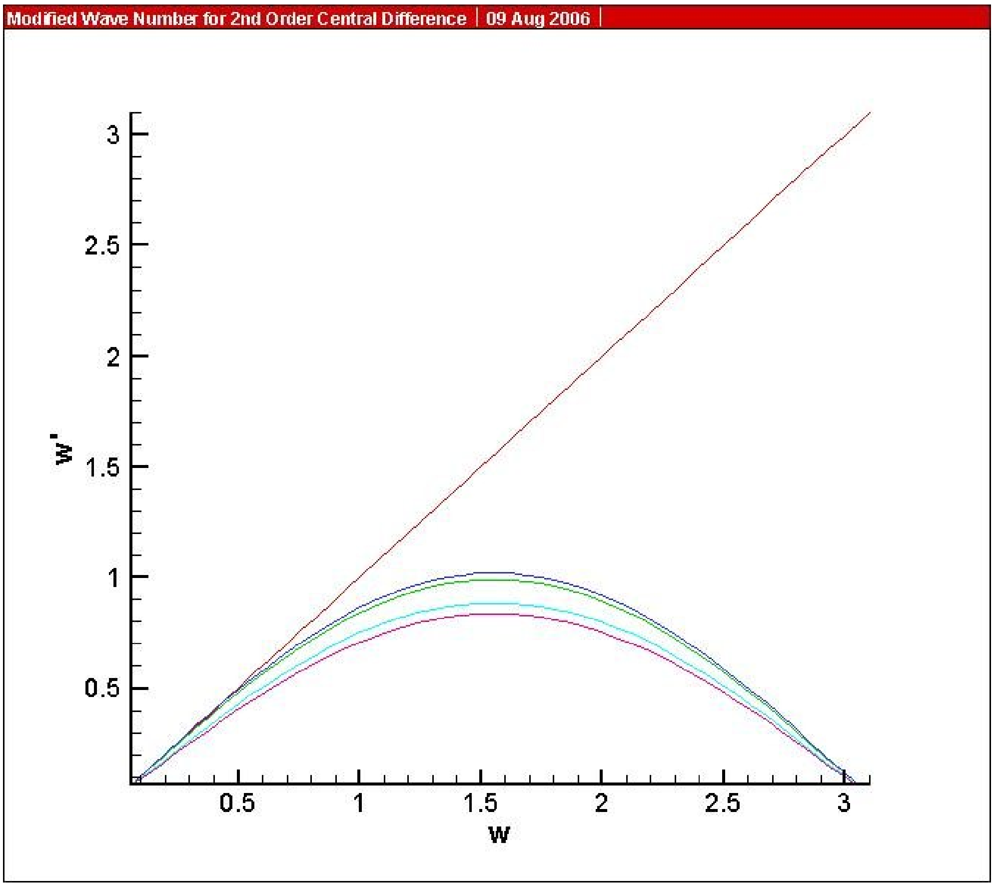


Figure 5.21 : The Modified Wave Number Analysis for Second-Order Central Difference (Tanh)

The effect of the control parameter is investigated here that when the uniform approximation is transformed by the hyperbolic tangent function with control parameter of 1, the modified wave number is slightly increased. However when the control parameter is increased, the modified wave number becomes lower than the modified wave number of the uniform approximation. In order to study more about the effect of the control parameter on the approximation scheme, the modified wave number analysis for the higher order scheme is required.

The modified wave number for the forth order central difference and the Pade compact scheme are shown in figure 5.22 and 5.23 respectively.

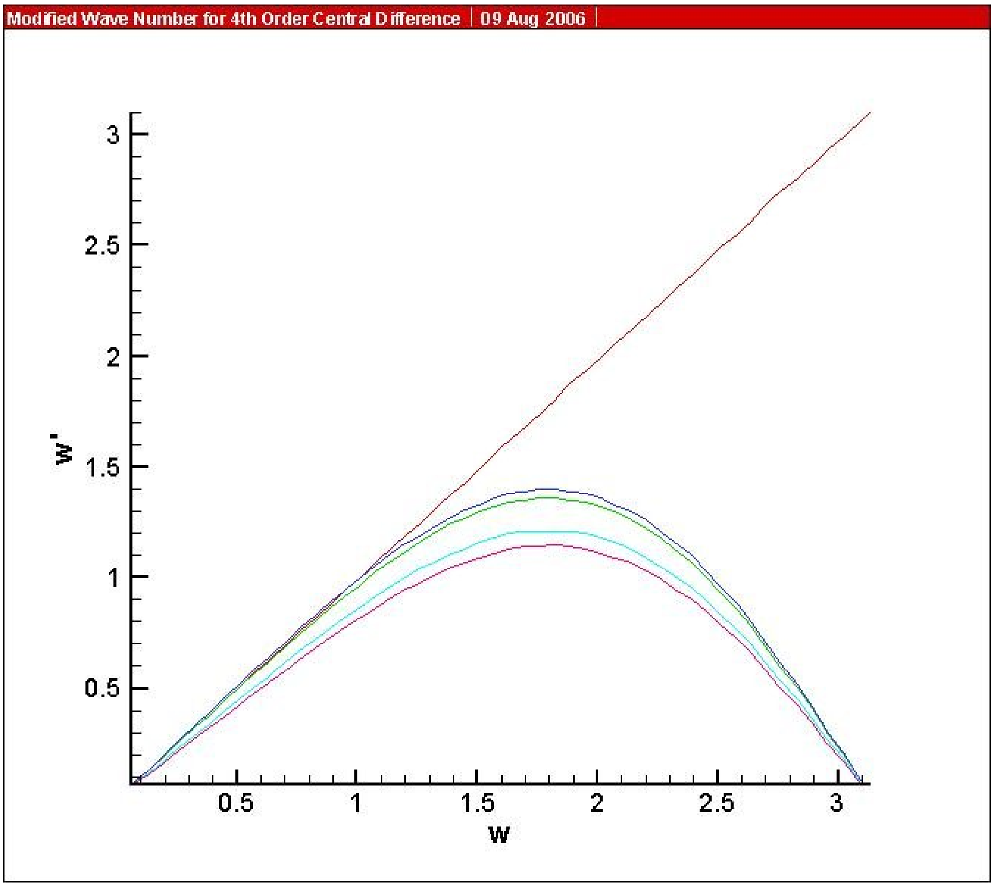


Figure 5.22 : The Modified Wave Number Analysis for Forth-Order Central Difference (Tanh)

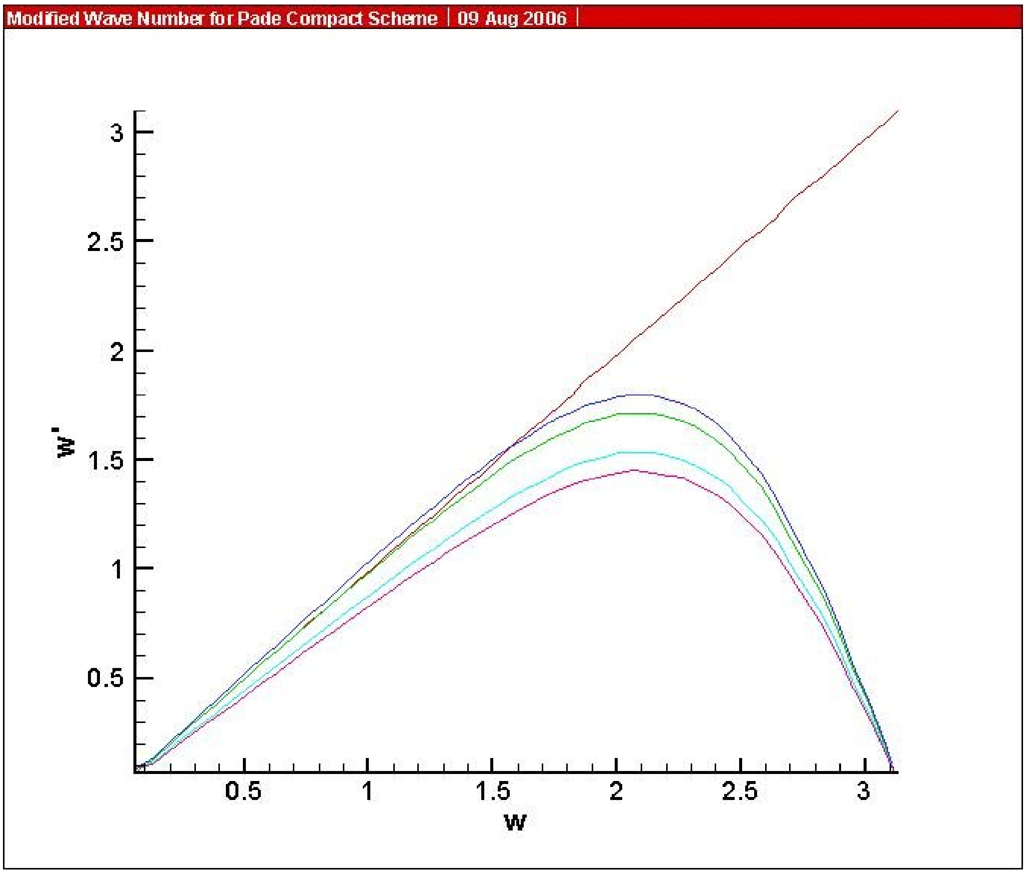


Figure 5.23 : The Modified Wave Number Analysis for Pade Compact Scheme (Tanh)

The effect of the control parameter on the forth order central difference scheme and the standard Pade compact scheme are exactly the same as the second order central difference, as shown figure 5.22 and 5.23, that when the control parameter is 1, the modified wave number is increased while the modified wave number is decreased when the control parameter is increased to 2 and 3. However, for the higher order approximation scheme, it seems the effect of the control parameter is larger than the lower approximation scheme. To proof this effect, the modified wave number for sixth order compact scheme is generated and is shown below.

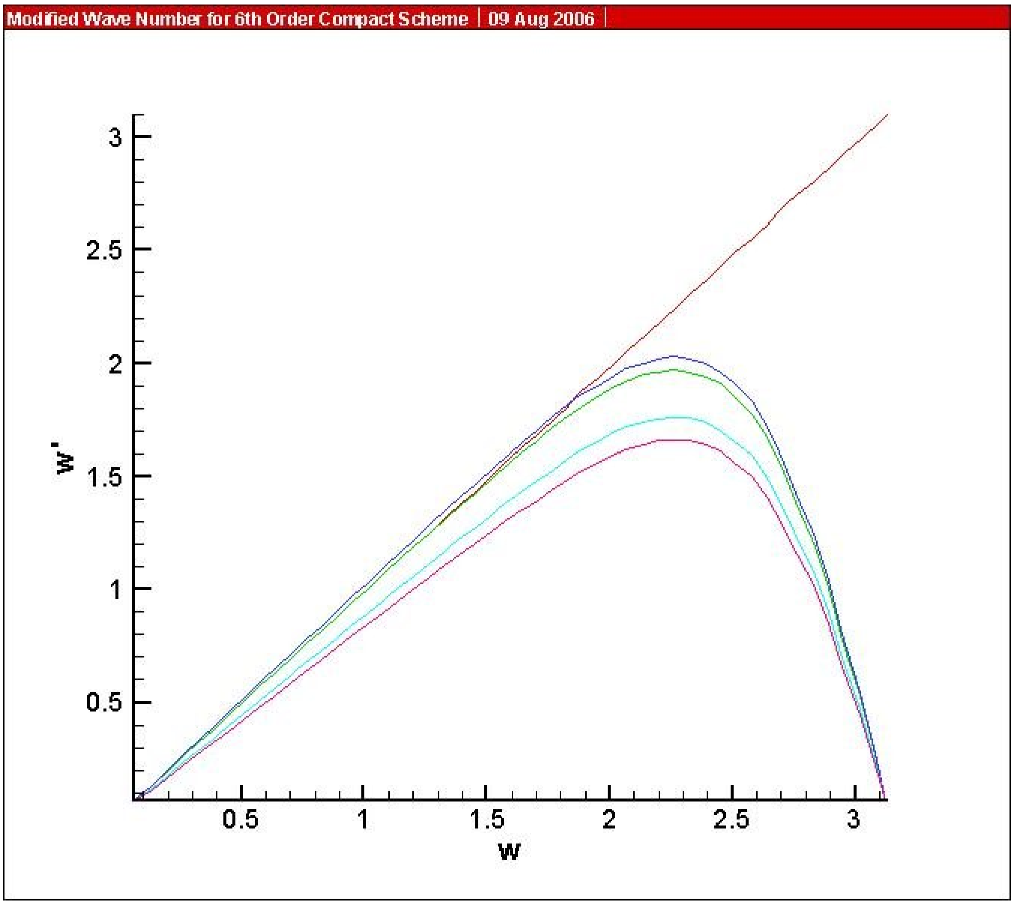


Figure 5.24 : The Modified Wave Number Analysis for Sixth-Order Compact Scheme (Tanh)

The figure 5.24 shows the evident that the effect of the control parameter becomes larger for the higher order approximation scheme.

As it has been known that the control parameter is the parameter to control the non-uniformity of the grid point. This means that when the non-uniformity of grid point is increased, the numerical approximation loses their accuracy. According from the figure 5.22-5.24, it can be concluded that the lower order approximation scheme has less sensitivity to the grid non-uniformity. This means that when the grid non-uniformity is increased, the change in modified wave number is small. For the higher order approximation scheme, the sensitivity to the grid non-uniformity is higher. It means the modified wave number is changed more when the grid point is increased their non-uniformity.

Hyperbolic Sine Transformation Function

Now, the grid transformation function will be changed to hyperbolic sine function. Three difference value of control parameter are also applied. The number of grid point will be fixed at 100 grid points and the modified wave number will be plotted at $x = 0.327\,x$. The following graphs are the modified wave number analysis for the second order and forth order central difference, Pade compact scheme and the sixth order compact scheme respectively where the red line is for the exact solution, the green line is for the uniform approximation, the dark blue, blue, and pink line are for non-uniform approximation with control parameter of 1, 2 and 3 respectively.

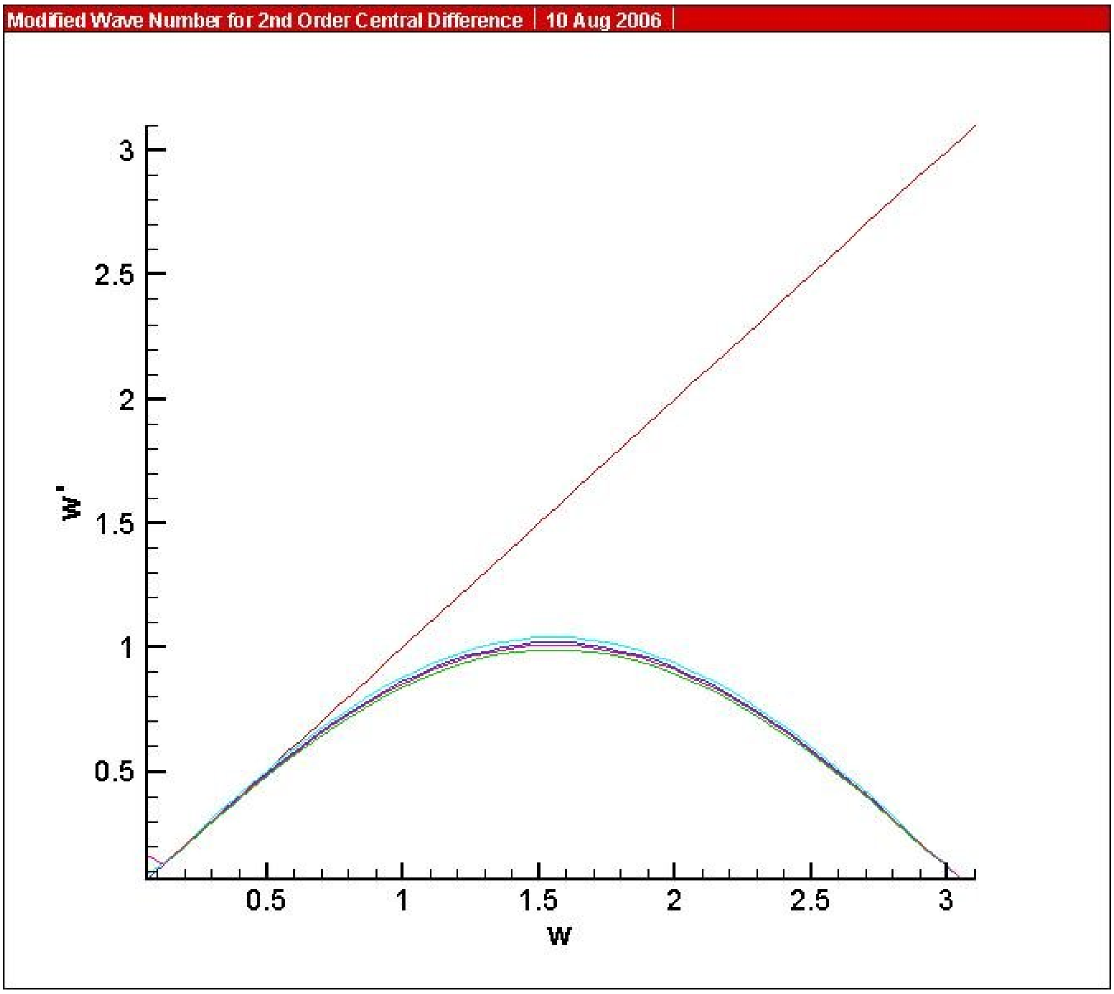


Figure 5.25 : The Modified Wave Number Analysis for Second-Order Central Difference (Sinh)

From figure 5.25, it can be seen that the effect of the control parameter is very small when the hyperbolic sine function is used compared with the effect of the control parameter on the hyperbolic tangent function. The modified wave number is slightly increased from the uniform approximation line when the control parameter is equal to 1 and then slightly increases when it has been changed to 2. The modified wave number is decreased when the control parameter is then increased to 3. The modified wave number for the control parameter of 1 and 3 are closed together. The modified wave number for forth order central difference and Pade compact scheme are shown next.

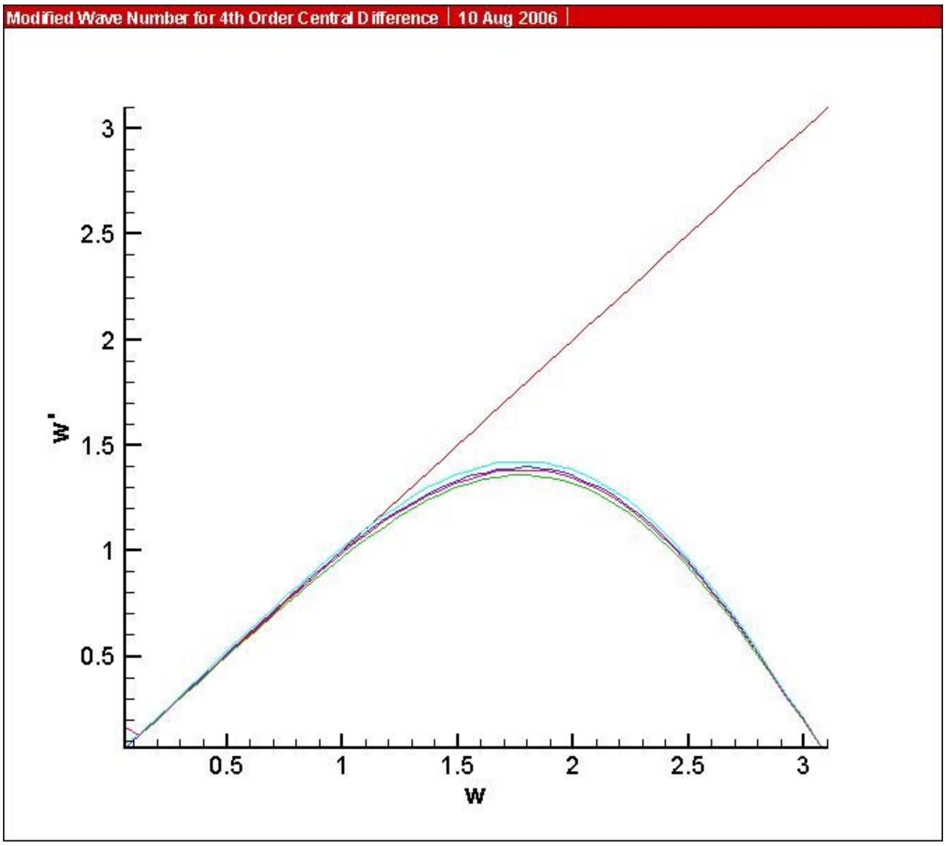


Figure 5.26 : The Modified Wave Number Analysis for Forth-Order Central Difference (Sinh)

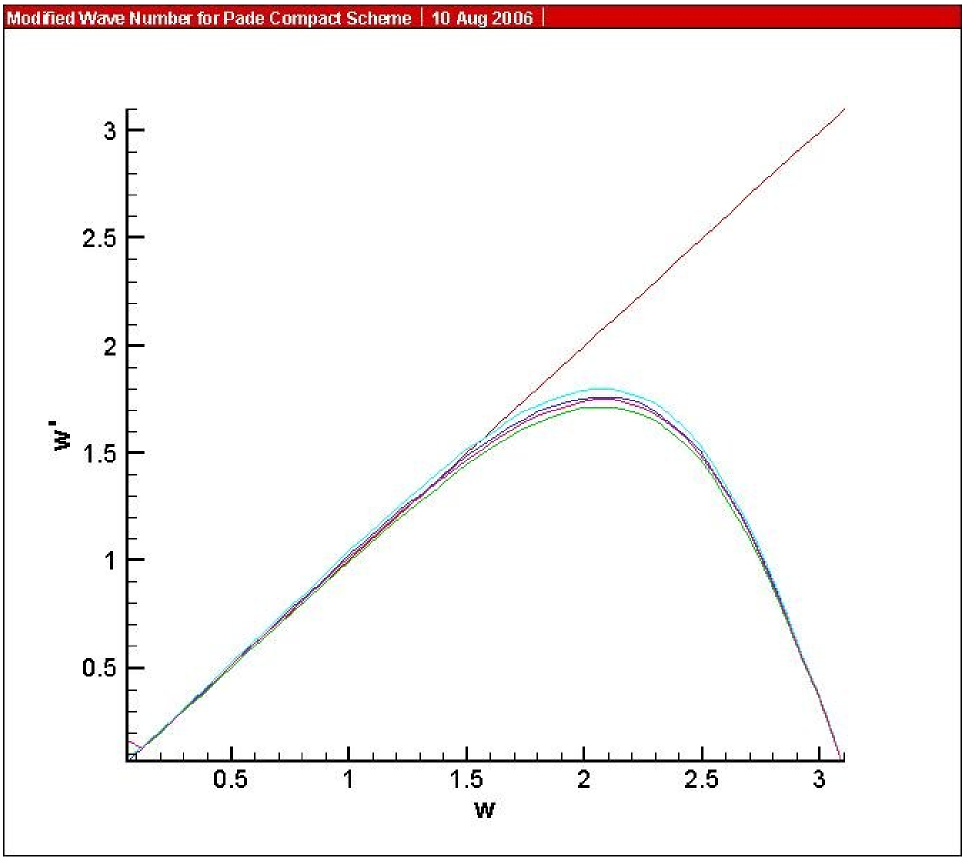


Figure 5.27 : The Modified Wave Number Analysis for Pade Compact Scheme (Sinh)

From figure 5.26 and 5.27, the effect of the control parameter on the hyperbolic sine function is larger when the higher order approximation scheme is applied. Then the modified wave number of sixth order compact scheme is plotted and shown below.

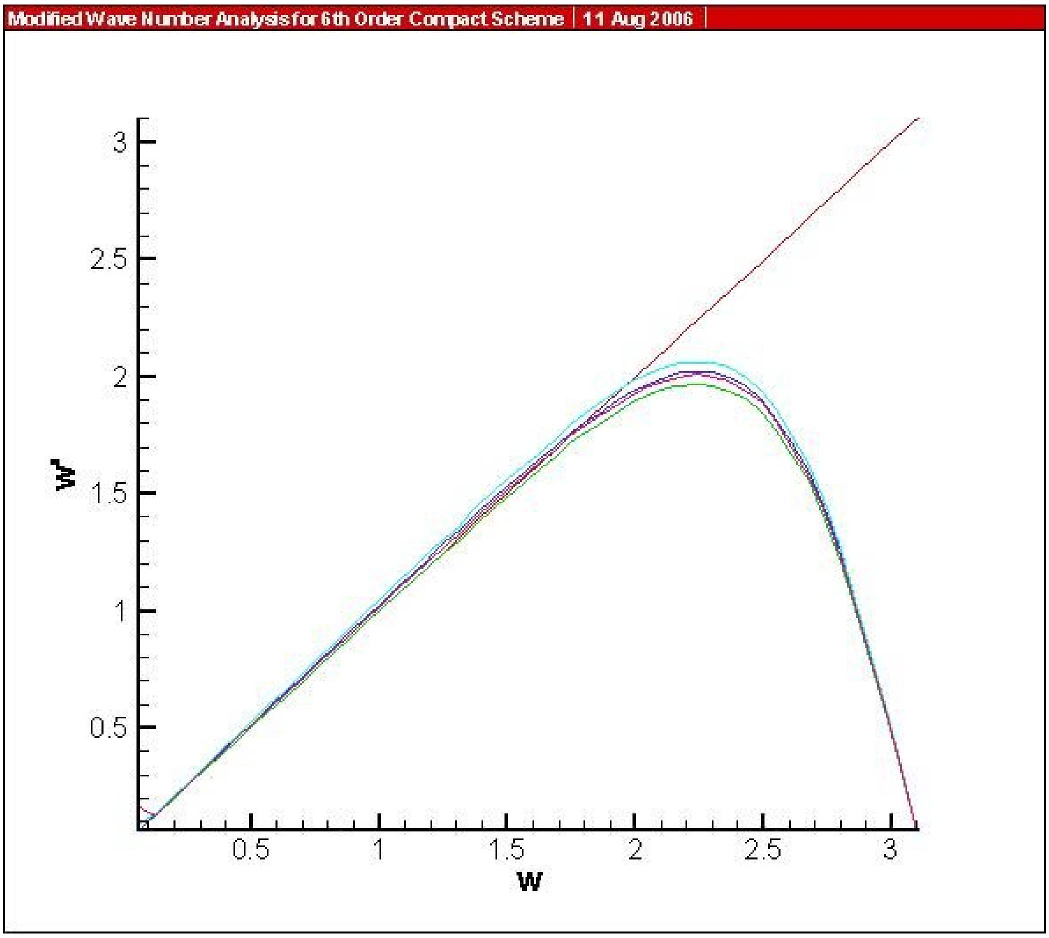


Figure 5.28 : The Modified Wave Number for Sixth-Order Compact Scheme (Sinh)

According from the modified wave number analysis, it can be concluded that the effect of the control parameter on the approximation with hyperbolic sine grid is that when the modified wave number will be increased when the control parameter is equal to 1 and 2, then it is decreased when the control parameter is increased to 3. The amount of change in modified wave number is directly proportional to the order of accuracy of the approximation scheme. In the other word, the higher order scheme is more sensitive to the value of control parameter or the non-uniformity of the grid than the lower order scheme as it has been concluded in the hyperbolic tangent transformation function part.