CY900 - Second-order inhomogeneous differential equations

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1. Variation of parameters

In this section, ways to get solutions of 2nd order odes with variable coefficients are covered - not a million miles away from the integrating factor approach for 1st-order equations. Given a 2nd order ode with variable coefficients,

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x) \tag{1}$$

first start with putting this equation into standard form by dividing through both sides by $a_2(x)$, e.g.

$$y'' + P(x)y' + Q(x)y = f(x) \tag{2}$$

Now suppose you know two functions that are solutions of the homogeneous counterpart of equation(2), called y_1 and y_2, such that, e.g

$$y_1'' + P(x)y_1' + Q(x)y_1 = 0 \tag{3}$$

and similarly for y_2. We now ask the question: is it possible to construct a particular solution of equation (1), y_p, of the form:

$$y_p = y_1(x)u_1(x) + y_2(x)u_2(x) \quad ? \tag{4}$$

Taking first and second derivatives of y_p yields

$$y_p' = y_1'u_1 + y_1u_1' \quad \text{and} \tag{5}$$

$$y_p'' = y_1'u_1' + y_1''u_1 + y_2'u_2' + y_2''u_2 \tag{6}$$

where to obtain equation(5), we also had to demand that

$$u_1'y_1 + u_2'y_2 = 0 \tag{7}$$

By substituting the expressions in equations (5) and (6) into equation (2), we get

$$y_p'' + P(x)y_p' + Q(x)y_p = u_1[y_1'' + P(x)y_1' + Q(x)y_1] + u_2[y_2'' + P(x)y_2' + Q(x)y_2] + y_1'u_1' + y_2'u_2'$$

Now, since the expressions in square brackets are zero (e.g. see equation (3)], the equation above becomes

$$\begin{aligned} y_p'' + P(x)y_p' + Q(x)y_p &= y_1'u_1' + y_2'u_2' && (8) \\ &= f(x) && (9) \end{aligned}$$

Equations (7) and (9) consitute a set of linear equations, which can be solved (for u_1' and u_2') via Cramer's Rule, i.e. $u_1' = \frac{W_1}{W}$ and $u_2' = \frac{W_2}{W}$, where W is the Wronskian

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

and W_1 is defined as

$$\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$$

and W_2 is defined as

$$\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

To find u_1 and u_2, all you have to do is integrate u_1' and u_2' (sometimes this has to be done numerically).

Example: Using variation of parameters, find the total solution to the ode:

$$y'' - 3y' + 4y = (x+1)e^{2x} \tag{10}$$

The y_c solution (which you should have no problems in obtaining) is

$$y_c = c_1 e^{2x} + c_2 x e^{2x} \tag{11}$$

The Wronskian is therefore e^{4x}, and therefore

$$u_1' = -\frac{y_2}{W} = -x^2 - x \tag{12}$$

and

$$u_2' = \frac{y_1}{W} = x + 1 \tag{13}$$

Integrating equations (12) and (13) lead to $u_1 = -\frac{x^3}{3} - \frac{x^2}{2}$ and $u_2 = \frac{x^2}{2} + x$. Substituting these expressions into equation(4) yields

$$y_p = -(\frac{x^3}{3} - \frac{x^2}{2})e^{2x} + (\frac{x^2}{2} + x)xe^{2x} \tag{14}$$

$$= (\frac{x^3}{6} + \frac{x^2}{2})e^{2x} \tag{15}$$

and gives the total solution

$$y = c_1 e^{2x} + c_2 x e^{2x} + (\frac{x^3}{6} + \frac{x^2}{2})e^{2x} \tag{16}$$

Exercises: Find the solutions to the following odes.

1. $y'' - 2y' + y = \frac{e^x}{x^2+1}$
2. $4y'' + 36y = cosec(3x)$
3. $xy'' - (x+1)y' + y = x^2$, given that y_1 and y_2 are known: $y_1 = e^x$ and $y_2 = x + 1$