CY900 - Partial differential equations

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1. Orthogonal functions

Recall from the section on linear vector spaces that the inner product of two vectors is just the dot product in Euclidean space. This inner product definition can be extended to include vectors that are functions. An integral inner product of two functions $f_1(x)$ and $f_2(x)$, over the interval [a,b] is defined by

$$(f_1, f_2) = \int_a^b f_1(x) f_2(x) dx \tag{1}$$

Further if this inner product is zero, then $f_1(x)$ and $f_2(x)$ are said to be orthogonal over the interval [a,b].

The set of functions $\phi_1(x), \phi_2(x), \ldots, \phi_n(x)$ are orthogonal on [a,b] if

$$(\phi_n(x), \phi_m(x)) = 0 \quad \text{for all m } \neq \text{n}$$

The norm of a function is defined as

$$\|\phi_n(x)\| = \Big[\int_a^b \phi_n^2(x) dx \Big]^{\frac{1}{2}} \tag{2}$$

Finally, the set of functions $\phi_n(x)$,$n = 0, 1, 2 \ldots$ is orthogonal w.r.t a weight function $w(x)$ on the interval [a,b] if

$$\int_a^b w(x) \phi_n(x) \phi_m(x) dx = 0 \quad m \neq n$$

Now suppose $\phi_n(x)$ is an infinite orthogonal set of functions. Given a function $y = f(x)$ defined on (a,b) can we find coefficients $c_n, n = 0, 1, 2, \ldots$ for which

$$f(x) = c_0 \phi_0(x) + c_1 \phi_1(x) + c_2 \phi_2(x) + \ldots \quad ? \tag{3}$$

To find these coefficients, start by multiplying equation (3) by $\phi_m(x)$ on both sides, and integrating over the interval:

$$\int_a^b \phi_m(x) f(x) dx = c_0 \Big[\int_a^b \phi_m(x) \phi_0(x) dx \Big] + c_1 \Big[\int_a^b \phi_m(x) \phi_1(x) dx \Big] + \ldots \tag{4}$$

$$= c_m \Big[\int_a^b \phi_m(x) \phi_m(x) dx \Big] \tag{5}$$

i.e only one term on the r.h.s survives, the rest vanish because of orthogonality. Therefore, we can obtain a general expression for the coefficient c_m:

$$c_m = \frac{\int_a^b \phi_m(x) f(x) dx}{\int_a^b \phi_m^2(x)} = \frac{\int_a^b \phi_m(x) f(x) dx}{\|\phi_m(x)\|^2} \tag{6}$$

2. Fourier Series

The set of functions 1, $cos\left(\frac{\pi x}{p}\right)$, $cos\left(\frac{2\pi x}{p}\right)$, ... , $sin\left(\frac{\pi x}{p}\right)$, $sin\left(\frac{2\pi x}{p}\right)$, ... is orthogonal on the interval [-p,p]. Suppose $f(x)$ is a function defined on (-p,p) that can be expanded in this series, i.e.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n cos(\frac{n\pi x}{p}) + b_n sin(\frac{n\pi x}{p})\right) \tag{7}$$

To start with, integrate both sides of equation (7) over the interval, giving

$$\int_{-p}^{p} f(x)dx = \int_{-p}^{p} \frac{a_0}{2}dx + \sum_{n=1}^{\infty} \Big(\int_{-p}^{p} a_n cos(\frac{n\pi x}{p})dx + \int_{-p}^{p} b_n sin(\frac{n\pi x}{p})dx \Big) \tag{8}$$

Because of orthogonality with 1, only the first term on the rhs survives, giving

$$\int_{-p}^{p} f(x)dx = \int_{-p}^{p} \frac{a_0}{2}dx = pa_0 \tag{9}$$

therefore giving an expression for the coefficient a_0 as below:

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)dx \tag{10}$$

To gain a general expression for the a_n coefficients, multiply both sides of equation (7) by $cos(\frac{m\pi x}{p})$, and integrate over the interval, i.e.

$$\begin{aligned} \int_{-p}^{p} cos(\frac{m\pi x}{p})f(x)dx \;&=\; \int_{-p}^{p} cos(\frac{m\pi x}{p})\frac{a_0}{2}dx \\ &+\; \sum_{n=1}^{\infty} \Big(\int_{-p}^{p} a_n cos(\frac{m\pi x}{p})cos(\frac{n\pi x}{p})dx + \int_{-p}^{p} b_n cos(\frac{m\pi x}{p})sin(\frac{n\pi x}{p})dx \Big) \end{aligned}$$

Because of orthogonality, only one term is non-vanishing on the r.h.s, when $m = n$ in the sum for the cosine term, yielding

$$\int_{-p}^{p} cos(\frac{n\pi x}{p})f(x)dx = \int_{-p}^{p} a_n cos^2(\frac{n\pi x}{p})dx = a_n p \tag{11}$$

therefore giving an expression for the coefficient a_0 as below:

$$a_n = \frac{1}{p}\int_{-p}^{p} cos(\frac{n\pi x}{p})f(x)dx \tag{12}$$

A similar procedure can be followed to obtain the expression for b_n, by multiplying both sides of equation (7) by $sin(\frac{m\pi x}{p})$ and integrating over the interval, to obtain

$$b_n = \frac{1}{p}\int_{-p}^{p} sin(\frac{n\pi x}{p})f(x)dx \tag{13}$$

This Fourier series can be simplified by considering the parity of the function you are trying to represent. If $f(x)$ is even over (-p,p), then one can use the cosine series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos(\frac{n\pi x}{p}) \tag{14}$$

where

$$a_n = \frac{2}{p}\int_0^p cos(\frac{n\pi x}{p}) f(x) dx \tag{15}$$

and

$$a_0 = \frac{2}{p}\int_0^p f(x) dx \tag{16}$$

Similarly, if $f(x)$ is odd over (-p,p), then one can use the sine series:

$$f(x) = \sum_{n=1}^{\infty} b_n sin(\frac{n\pi x}{p}) \tag{17}$$

where

$$b_n = \frac{2}{p}\int_0^p sin(\frac{n\pi x}{p}) f(x) dx \tag{18}$$

Exercises:

1. Show that x^2 and x^3 are orthogonal over [-1,1]
2. Given a function

$$\begin{aligned} f(x) &= 0 \qquad -\pi < x < 0 \\ f(x) &= 1 \qquad 0 < x < \pi \end{aligned}$$

that is periodically repeated outside $(-\pi, \pi)$ with period 2π, expand this function in a Fourier series.