CY900 - Linear vector spaces

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1. Matrix operators

Consider the linear equations

$$\begin{aligned} X &= ax + by \\ Y &= cx + by \end{aligned}$$

or

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

or

$$\mathbf{R} = \mathsf{M}\mathbf{r}$$

For each point (x, y), this gives us a way to move this point to (X, Y). However, not all points under this **transformation** will move - e.g. the origin will not move. This process is also known as a **mapping**.

Any matrix (such as M) can be considered as on operator on a conformable (same shape) column matrix/vector $\mathbf{r}$. In the following section, we will consider matrices as linear operators, i.e.

$$\mathsf{M}(\mathbf{r_1} + \mathbf{r_2}) = \mathsf{M}\mathbf{r_1} + \mathsf{M}\mathbf{r_2}$$

The set of linear equations (the first equation in this Section) and the resulting transformation to 2D space can be thought of in 2 ways:

1. that we have one set of axes, and we move the vector (have 2 different vectors, (x, y) and (X, Y)), or,
2. that we have one vector, and move the axes (have two sets of axes (x, y) and (x', y')

In this latter case, vectors $\mathbf{r} = \mathbf{r}'$, with components relative to each set of axes. The latter transformation can be achieved by:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

If several transformations are to be applied, say, M_1 is applied to a vector first, followed by M_2, this is written as

$$\mathsf{M}_2\mathsf{M}_1 \begin{pmatrix} X \\ Y \end{pmatrix}$$

You should already know that in general matrix multiplications do not commute, so in general this will yield a different result to applying M_2 first, then M_1.

Operations such as rotation, reflection, and shear can all be described by matrices.

2. Orthogonal transformations Orthogonal transformations are operations that preserve the norm of a vector. Such operations can be represented by an orthogonal matrix, ie a matrix where $\mathsf{M}^{-1} = \mathsf{M}^{\mathsf{T}}$. The determinant of an orthogonal matrix is ± 1.

Rotations are an important class of orthogonal transformation. In 2D Euclidean space, the rotation of a point in the x, y plane is given by

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Application of this transformation to a vector emanating from the origin will result in a rotation of this vector by θ degrees in a anit-clockwise sense.

Alternatively, you can think of this transformation as being a rotation of the coordinate axes, such that the coordinates in your new axis system (x', y') are related to the coordinates in your old coordinate system (x, y) via the relation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

where the axes have been rotated by θ degrees in an anti-clockwise sense. Clearly the effect of rotating a vector clockwise by a fixed amount will have the same effect as rotating the axis frame anti-clockwise by the same amount.

The extension to 3D space is trivial – the following example gives the rotation in 3D space of a vector about the z-axis:

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Transformations that involve the axis frame moving are also referred to as 'change of basis' transformations. The matrix associated with this transformation is called a transition matrix. To find the transition matrix for a given 'change of basis', the columns of the transition matrix will be the coordinates of the new basis vectors relative to the old basis.

Exercises:

1. Find a matrix that maps in 2D Euclidean space corresponding to first a shear by a factor of 2 in the x-direction, followed by a reflection about $y = x$. Then find the matrix that describes the reflection followed by the shear.

2. Find a single matrix that performs the successive operations in 2D Euclidean space of first a rotation by 30° anticlockwise, followed by reflection about the x-axis.

3. Consider the bases B $= \{\mathbf{u_1}, \mathbf{u_2}\}$ and B$' = \{\mathbf{u'_1}, \mathbf{u'_2}\}$, where $\mathbf{u_1} = (1, 0)$, $\mathbf{u_2} = (0, 1)$, $\mathbf{u_1}' = (1, 1)$ and $\mathbf{u_1}' = (2, 1)$. Find the transition matrix in going from B$'$ to B.