CY900 - Linear vector spaces

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1. Span, basis and dimension
You should already be familiar with representing vectors in component form, e.g. $\mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k}$ represents a vector from the origin to the point (x, y, z). The collection of all such vectors comprises 3D space (R_3), where the unit vectors $\hat{i}, \hat{i} and \hat{i}$ are a set of basis vectors for this 3D space. Similarly, you can construct a "D space from vectors $x\hat{i} + y\hat{j}$. Two linearly independent vectors can form a plane.

A set of vectors **span** a space if any vector in the space can be written as linear combinations of the spanning set. The **dimension** of the vector space is equal to the number of basis vectors that span the space.

You have already looked at how to calculate the inner product of two vectors in 3D Euclidean space, this is just the dot product of two vectors. Extending this to higher dimensional Euclidean space (say, n dimensions), the definition of a magnitude of vector $\mathbf{u}$ is

$$\mid u \mid = \sqrt{u_1^2 + u_2^2 + u_3^2 + \ldots + u_n^2}$$

and the inner product of two vectors in n-dimensional Euclidean space is

$$\mathbf{A}.\mathbf{B} = \sum_{n=1}^{\infty} A_i B_i$$

If this inner product is zero, then these two vectors are orthogonal.

2. The Schwartz inequality It is tempting to follow this extension into n-dimensional space by concluding that the dot product is the product of the magnitude of the two vectors times $cos\theta$, where θ is the angle between the two vectors. In order to be sure of this, you should check the Schwartz inequality, to ensure that $\mid cos\theta < 1.$, ie, you should check that

$$\mathbf{A}.\mathbf{B} \leq AB \leq \left[\sum_{n=1}^{\infty} A_i\right]^{\frac{1}{2}} \left[\sum_{n=1}^{\infty} B_i\right]^{\frac{1}{2}}$$

3. Building an orthonormal basis A set of vectors is orthonormal if they are mutually perpendicular and each vector is normalised. Supppose you have a set of vectors for a space – it is often the case that one seeks to construct an orthonormal basis from this set. This can be achieved using the **Gram-Schmidt** procedure. Suppose you have three vectors, **A**, **B** and **C**:

1. First normalise **A**. Call this new vector $\mathbf{v_1}$.

2. Then subtract off the projection of $\mathbf{B}$ onto $\mathbf{v_1}$ from $\mathbf{B}$, ie $\mathbf{B} - (\mathbf{B}.\mathbf{v_1})\mathbf{v_1}$ Normalise this resulting vector, and call this vector $\mathbf{v_2}$.

3. Then subtract off the projection of $\mathbf{C}$ onto $\mathbf{v_1}$, and the projection of $\mathbf{C}$ onto $\mathbf{v_2}$ from $\mathbf{C}$, ie $\mathbf{C} - (\mathbf{C}.\mathbf{v_1})\mathbf{v_1} - (\mathbf{C}.\mathbf{v_2})\mathbf{v_2}$. Normalise this vector and call it $\mathbf{v_3}$

You should be able to see how this extends to higher dimensions....

Exercises:

1. Find the Euclidean inner product of $(-1, 3, 5, 7)$ and $(5, -4, 7, 0)$
2. If $\mathbf{u} = (1, 3, -2, 7)$, what is $\mid u \mid$?
3. Apply the Gram-Schmidt procedure to transform the basis $\mathbf{u_1} = (1, 1, 1)$, $\mathbf{u_2} = (0, 1, 1)$ and $\mathbf{u_2} = (0, 0, 1)$ into an orthonormal basis.