"JUST THE MATHS"

UNIT NUMBER

19.1

PROBABILITY 1
(Definitions and rules)

by

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UNIT 19.1 - PROBABILITY 1 - DEFINITIONS AND RULES

19.1.1 INTRODUCTION

To introduce the definition of probability, suppose 30 high-strength bolts became mixed with 25 ordinary bolts by mistake, all of the bolts being identical in appearance.

We would like to know how sure we can be that, in choosing a bolt, it will be a high-strength one. Phrases like "quite sure" or "fairly sure" are useless, mathematically, and we define a way of measuring the certainty.

We know that, in 55 simultaneous choices, 30 will be of high strength and 25 will be ordinary; so we say that, in one choice, there is a $\frac{30}{55}$ chance of success; that is, approximately, a 0.55 chance of success.

Obviously, in one single choice, we haven't any idea what the result will be; but experience has proved that, in a significant number of choices, just over half will most likely give a high-strength bolt.

Such predictions can be used, for example, to estimate the cost of mistakes on a production line.

DEFINITION 1.
The various occurrences which are possible in a statistical problem are called **"events"**.
If we are interested in one particular event, it is termed **"successful"** when it occurs and **"unsuccessful"** when it does not.

ILLUSTRATION

If, in a collection of 100 bolts, there are 30 high-strength, 25 ordinary and 45 low-strength, then we have three possible events according to which type is chosen.
We can make 100 **"trials"** and, in each trial, one of three events will occur.

DEFINITION 2.
If, in n possible trials, a successful event occurs s times, then the number $\frac{s}{n}$ is called the **"probability of success in a single trial"**. It is also known as the **"relative frequency of success"**.

ILLUSTRATIONS

1. From a bag containing 7 black balls and 4 white balls, the probability of drawing a white ball is $\frac{4}{11}$.

2. In tossing a perfectly balanced coin, the probability of obtaining a head is $\frac{1}{2}$.
3. In throwing a die, the probability of getting a six is $\frac{1}{6}$.
4. If 50 chocolates are identical in appearance, but consist of 15 soft-centres and 35 hard-centres, the probability of choosing a soft-centre is $\frac{15}{50} = 0.3$.

19.1.2 APPLICATION OF PROBABILITY TO GAMES OF CHANCE

If a competitor in a game of chance has a probability, p, of winning, and the prize money is £m, then £mp is considered to be a fair price for entry to the game.

The quantity mp is known as the **"expectation"** of the competitor.

19.1.3 EMPIRICAL PROBABILITY

So far, all the problems discussed on probability have been "descriptive"; that is, we know all the possible events, the number of successes and the number of failures. In other problems, called **"inference"** problems, it is necessary either (a) to take **"samples"** in order to infer facts about a total **"population'** (for example, a public census or an investigation of moon-rock); or (b) to rely on past experience (for example past records of heart deaths, road accidents, component failure).

If the probability of success, used in a problem, has been inferred by samples or previous experience, it is called **"empirical probability"**.

However, once the probability has been calculated, the calculations are carried out in the same way as for descriptive problems.

19.1.4 TYPES OF EVENT

DEFINITION 3.
If two or more events are such that not more than one of them can occur in a single trial, they are called "mutually exclusive".

ILLUSTRATION

Drawing an Ace or drawing a King from a pack of cards are mutually exclusive events; but drawing an Ace and drawing a Spade are not mutually exclusive events.

DEFINITION 4.
If two or more events are such that the probability of any one of them occurring is not affected by the occurrence of another, they are called **"independent"** events.

ILLUSTRATION

From a pack of 52 cards (that is, Jokers removed), the event of drawing and immediately replacing a red card will have a probability of $\frac{26}{52} = 0.5$; and the probability of this occurring a second time will be exactly the same. They are independent events.

However, two successive events of drawing a red card **without** replacing it are **not** independent. If the first card drawn is red, the probability that the second is red will be $\frac{25}{51}$; but, if the first card drawn is black, the probability that the second is red will be $\frac{26}{51}$

19.1.5 RULES OF PROBABILITY

1. If $p_1, p_2, p_3, ..., p_r$ are the separate probabilities of r mutually exclusive events, then the probability that some **one** of the r events will occur is

   $$p_1 + p_2 + p_3 + ... + p_r.$$

   ILLUSTRATION

   Suppose a bag contains 100 balls of which 1 is red, 2 are blue and 3 are black. The probability of choosing any one of these three colours will be

   $$0.06 = 0.01 + 0.02 + 0.03$$

   However, the probability of drawing a spade or an ace from a pack of 52 cards will not be $\frac{13}{52} + \frac{4}{52} = \frac{17}{52}$ but $\frac{16}{52}$ since there are just 16 cards which are either a spade or an ace.

2. If $p_1, p_2, p_3, ..., p_r$ are the separate probabilities of r independent events, then the probability that **all** will occur in a single trial is

   $$p_1.p_2.p_3......p_r.$$

   ILLUSTRATION

   Suppose there are three bags, each containing white, red and blue balls. Suppose also that the probabilities of drawing a white ball from the first bag, a red ball from the second bag and a blue ball from the third bag are respectively p_1, p_2 and p_3. The probability of making these three choices in succession is $p_1.p_2.p_3$ because they are independent events.

   However, if three cards are drawn, without replacing, from a pack of 52 cards, the probability of drawing a 3, followed by an ace, followed by a red card will not be $\frac{4}{52}.\frac{4}{52}.\frac{26}{52}$.

19.1.6 CONDITIONAL PROBABILITIES

For dependent events, the multiplication rule requires a knowledge of the **new** probabilities of successive events in the trial, after the previous ones have been dealt with. These are called **"conditional probabilitities"**.

EXAMPLE

From a box, containing 6 white balls and 4 black balls, 3 balls are drawn at random without replacing them. What is the probability that there will be 2 white and 1 black ?

Solution

The cases to consider, together with their probabilities are as follows:

(a) White, White, Black..............Probability $= \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8} = \frac{120}{720} = \frac{1}{6}$.

(b) Black, White, White..............Probability $= \frac{4}{10} \times \frac{6}{9} \times \frac{5}{8} = \frac{120}{720} = \frac{1}{6}$.

(c) White, Black, White..............Probability $= \frac{6}{10} \times \frac{4}{9} \times \frac{5}{8} = \frac{120}{720} = \frac{1}{6}$.

The probability of any one of these three outcomes is therefore

$$\frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}.$$

19.1.7 EXERCISES

1. A card is drawn at random from a deck of 52 playing cards. What is the probability that it is either an Ace or a picture card ?
2. If a die is rolled, what is the probability that the roll yields either a 3 or a 4 ?
3. In a single throw of two dice, what is the probability that a 9 or a doublet will be thrown ?
4. Ten balls, numbered 1 to 10, are placed in a bag. One ball is drawn and not replaced; and then a second ball is drawn. What are the probabilities that
   (a) the balls numbered 3 and 7 are drawn;
   (b) neither of these two balls are drawn ?
5. On a gaming machine, there are three reels with ten digits 0,1,2,3,4,5,6,7,8,9 plus a star on each reel. When a coin is inserted, and the machine started, the three reels revolve independently before coming to rest.

(a) What is the probabiity of getting a particular sequence of numbers ?

(b) What is the probability of getting three stars ?

(c) What is the probability of getting the same number on each reel ?

6. Three balls in succession are drawn, without replacement, from a bag containing 8 black, 8 white and 8 red balls. If a prize of £5 is awarded for drawing no black balls, what is the expectation ?

7. Three persons A,B and C take turns to throw three dice once. If the first one to throw a total of 11 is awarded a prize of £200, what are the expectations of A,B and C ?

19.1.8 ANSWERS TO EXERCISES

1. $\frac{4}{13}$.
2. $\frac{1}{3}$.
3. $\frac{5}{18}$.
4. (a)
$$\frac{1}{45};$$
(b)
$$\frac{28}{45}.$$
5. (a)
$$\frac{1}{11^3} \simeq 0.00075;$$
(b)
$$\frac{1}{11^3} \simeq 0.00075;$$
(c)
$$\frac{10}{11^3} \simeq 0.0075$$
6. The expectation is £1.38
7. For A,B and C, the expectations are £25, £21.88 and £19.14 respectively since the probabilities are $\frac{1}{8}$, $\frac{7}{8^2}$ and $\frac{7}{8^3}$ respectively.