CY900 - Second-order homogeneous differential equations with variable coefficients

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1. Power Series - a few reminders
Not wanting to get too far into the gory details of this subject, just a few things to remind you about a general power series

1. A power series in $(x-a)$ can be expressed as $\sum_{n=0}^{\infty} c_n(x-a)^n$
2. Every power series has an interval of convergence - this is the set of numbers for which the series converges
3. every interval of convergence has a radius of convergence, R.
4. A power series converges absolutely when $\mid x-a \mid < R$ and diverges when $\mid x-a \mid > R$
5. if $\sum_{n=0}^{\infty} c_n(x-a)^n = 0$ for all x in the interval of convergence, then $c_n = 0$ for all values of n
6. A power series represents a continuous function within its interval of convergence
7. A power series can be differentiated term by term within its interval of convergence
8. A power series can be integrated term by term within its interval of convergence
9. Two power series with a common interval of convergence can be added term by term

2. Power series solution of odes
Consider the ode $y' - 2xy = 0$ - we seek solutions by assuming the general solution exists as a power series (about $x = 0$), $y = \sum_{n=0}^{\infty} c_n x^n$. Is it possible to obtain coefficients c_n such that the ode is satisfied?

Take the first derivative of this power series, e.g.

$$y' = \sum_{n=0}^{\infty} n c_n x^{n-1} \tag{1}$$

also determine the expression for $2xy$, e.g.

$$2xy = \sum_{n=0}^{\infty} 2 c_n x^{n+1} \tag{2}$$

substitute these expressions into the ode:

$$\sum_{n=0}^{\infty} nc_n x^{n-1} - \sum_{n=0}^{\infty} 2c_n x^{n+1} = 0 \tag{3}$$

To make things clearer, you should relabel the summation indices so that your powers of x are in sync - let $k = n-1$ in the first sum and $k = n+1$ in the second sum, while also taking out the term without any x-dependence, i.e.

$$\begin{aligned} 0 &= c_1 + \sum_{k=1}^{\infty}(k+1)c_{k+1}x^k - \sum_{k=1}^{\infty} 2c_{k-1}x^k \qquad (4) \\ &= c_1 + \sum_{k=1}^{\infty}\left(\Big[(k+1)c_{k+1} - 2c_{k-1}\Big]x^k\right) \qquad (5) \end{aligned}$$

This equation implies $c_1 = 0$ and $(k+1)c_{k+1} - 2c_{k-1} = 0,$ for $k \in \mathrm{N}$.

The second of these resulting equations is a recurrence relation, rearranged as shown below:

$$c_{k+1} = \frac{2c_{k-1}}{(k+1)} \tag{6}$$

You can use this relation to find successive values of the coefficients. In this case, start by plugging in $k = 1$,

$$c_2 = \frac{2c_0}{2} = c_0 \tag{7}$$

Now try $k = 2$,

$$c_3 = \frac{2c_1}{3} = 0 \tag{8}$$

and so on....If you keep going, you end up with the solution

$$y = c_0 + 0x + c_0x^2 + 0x^3 + \frac{c_0x^4}{2!} + \ldots = c_0\sum_{n=0}^{\infty}\frac{x^{2n}}{n!} \tag{9}$$

You should recognise that this solution in fact corresponds to $y = e^{x^2}$, as you could determine without using a power series (even by inspection!). It is multiplied by a constant c_0 - so this is general solution to this ode.

3. Solutions of 2nd order odes around ordinary points

Consider the situation for a 2nd order homogeneous ode with variable coefficients,

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \tag{10}$$

that is put into standard form:

$$y'' + P(x)y' + Q(x)y = 0 \tag{11}$$

A point $x = x_0$ is an ordinary point if $P(x)$ and $Q(x)$ are both analytic at x_0, ie they both have a power series within $x - x_0$ with a positive radius of convergence. If your point fails this test, then it's a singular point (more about later...)

To solve a 2nd-order ode at an ordinary point, just follow the same kind of procedure as outlined above - seek two solutions of the form $y = \sum_{n=0}^{\infty} c_n (x - x_0)^n$, where the series solution will converge for at least $\mod x - x_0 \mod < R_1$, where R_1 is the closest singular point. In the example below, we assume an ordinary point at $x_0 = 0$.

Example: Find the power series solution to the ode $y'' - 2xy = 0$ around $x = 0$.

First determine the second derivative, by first writing down the first derivative:

$$y' = \sum_{n=0}^{\infty} n c_n x^{n-1} = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

and take the derivative again:

$$y'' = \sum_{n=1}^{\infty} n(n-1) c_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} \tag{12}$$

where the first term in each case is zero, therefore leaving c_0 and c_1 undetermined. Now put together the 2nd term of the ode, i.e.

$$2xy = \sum_{n=0}^{\infty} 2 c_n x^{n+1} \tag{13}$$

and put equations (12) and (13) together to get a power series expression for $y'' - 2xy = 0$:

$$\begin{aligned} y'' - 2xy &= \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=0}^{\infty} 2 c_n x^{n+1} \\ &= 2.1.c_2.x^0 + \sum_{n=3}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=0}^{\infty} 2 c_n x^{n+1} \\ &= 2c_2 + \sum_{n=3}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=0}^{\infty} 2 c_n x^{n+1} \\ &= 0 \end{aligned}$$

Now re-label your summation indices - let $k = n - 2$ in the first sum and $k = n + 1$ in the second sum, yielding

$$0 = 2c_2 + \sum_{k=1}^{\infty} (k+2)(k+1) c_{k+2} x^k - \sum_{k=1}^{\infty} 2 c_{k-1} x^k \tag{14}$$

$$= \left(2c_2 + \sum_{k=1}^{\infty} \Big[(k+2)(k+1) c_{k+2} - 2c_{k-1}\Big] x^k\right) \tag{15}$$

Equating powers of k to zero implies that $2c_2 = 0$ and

$$(k+2)(k+1) c_{k+2} - 2c_{k-1} = 0 \tag{16}$$

with this last equation yielding a recurrence relation:

$$c_{k+2} = \frac{2c_{k-1}}{(k+2)(k+1)} \tag{17}$$

By applying this recurrence relation, bearing in mind that $c_2 = 0$, the solution for the ode can be written down:

$$\begin{aligned} y &= c_0 + c_1 x + 0 + \frac{2}{3.2} c_0 x^3 + \frac{2}{4.3} c_1 x^4 + 0 + \frac{2^2}{6.5.3.2} c_0 x^6 + \frac{2^2}{7.6.4.3} c_1 x^7 + \ldots \\ &= c_0 \Big[1 + \frac{2}{3.2} x^3 + \frac{2^2}{6.5.3.2} x^6 + \ldots \Big] \\ &+ c_1 \Big[x + \frac{2}{4.3} x^4 + \frac{2^2}{7.6.4.3} x^7 + \ldots \Big] \end{aligned}$$

As expected, this yields two independent solutions:

$$y_1(x) = c_0 \Big[1 + \sum_{k=1}^{\infty} \frac{2^k [1.4.7 \ldots (3k-2)]}{(3k)!} x^{3k} \Big] \tag{18}$$

and

$$y_2(x) = c_1 \Big[x + \sum_{k=1}^{\infty} \frac{2^k [2.5.8 \ldots (3k-1)]}{(3k+1)!} x^{3k+1} \Big] \tag{19}$$

4. Solutions of 2nd order odes around singular points

The case of regular singular points only will be considered here. A regular singular point is defined as follows – take a general 2nd order homogeneous ode:

$$a_2(x) y'' + a_1(x) y' + a_0(x) y = 0 \tag{20}$$

that is put into standard form:

$$y'' + P(x) y' + Q(x) y = 0 \tag{21}$$

A point $x = x_0$ is a regular singular point if both $(x - x_0) P(x)$ and $(x - x_0)^2 Q(x)$ are analytic at x_0. If this is not the case, then the singular point is an irregular singular point. To solve an ode such as equation (20) about a regular singular point, the method of Froebenius can be used. In this case, you try to expand your solution to this ode in a series that involves a parameter, s, such that your general expression for y is:

$$y = x^s \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+s} \tag{22}$$

Example: Find the power series solution to the ode $3xy'' + y' - y = 0$ around the regular singular point $x = 0$.

To start, try a solution of the form

$$y = x^s \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+s} \tag{23}$$

and take derivatives of this expression, e.g.

$$y' = \sum_{n=0}^{\infty}(n+s)c_n x^{n+s-1} \tag{24}$$

$$y'' = \sum_{n=0}^{\infty}(n+s-1)(n+s)c_n x^{n+s-2} \tag{25}$$

now substitute these expressions your ode, to give

$$\begin{aligned}
0 &= \sum_{n=0}^{\infty} 3(n+s-1)(n+s)c_n x^{n+s-1} + \sum_{n=0}^{\infty}(n+s)c_n x^{n+s-1} - \sum_{n=0}^{\infty} c_n x^{n+s} \\
&= \sum_{n=0}^{\infty}\Big[3(n+s-1)(n+s) + (n+s)\Big]c_n x^{n+s-1} - \sum_{n=0}^{\infty} c_n x^{n+s} \\
&= x^s\Big(\sum_{n=0}^{\infty}(n+s)(3n+3s-2)c_n x^{n-1} - c_n x^n\Big) \\
&= x^s\Big(s(3s-2)c_0 x^{-1} + \sum_{n=1}^{\infty}(n+s)(3n+3s-2)c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n\Big) \\
&= x^s\Big(s(3s-2)c_0 x^{-1} + \sum_{k=0}^{\infty}(k+s+1)(3k+3s+1)c_{k+1} x^k - \sum_{k=0}^{\infty} c_k x^k\Big)
\end{aligned}$$

Equating like powers of x to zero, we find two equations:

$$s(3s-2)c_0 = 0 \quad \text{and,} \tag{26}$$

$$(k+s+1)(3k+3s+1)c_{k+1} - c_k = 0 \tag{27}$$

The first of these, equation (26) is known as an **indicial equation**, and it yields in this case two values for s. We must suppose that $c_0 \neq 0$ in this case, yielding two values of s: $s = 0$ and $s = \frac{2}{3}$.

For each value of s obtained, you will get a different recurrence relation when substituted into equation (27).

For $s = 0$ the recurrence relation is

$$c_{k+1} = \frac{c_k}{(k+1)(3k+1)} \tag{28}$$

and for $s = 2/3$ the recurrence relation is

$$c_{k+1} = \frac{c_k}{(3k+5)(k+1)} \tag{29}$$

Each of these recurrence relations yields a solution to the ode – if you persist you should get the solutions

$$y_1(x) = c_0 x^{\frac{2}{3}}\Big[1 + \sum_{n=1}^{\infty}\frac{x^n}{n!5.8.11\ldots(3n+2)}\Big] \quad \text{and,} \tag{30}$$

$$y_2(x) = c_1 x^0 \Big[1 + \sum_{n=1}^{\infty} \frac{x^n}{n!1.4.7\ldots(3n-2)}\Big] \tag{31}$$

Added together, these two solutions yield the general solution for this ode.

Exercises: Find power series solutions for the following odes.

1. $y'' + xy' + y = 0$
2. $y'' + y = 0$, given $y(0) = 0$ and $y'(0) = 1$
3. $x^2y'' + 4xy' + (x^2 + 2)y = 0$, about $x = 0$, given this is a regular singular point.