CY900 - Partial differential equations

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1. Introduction to pdes

In this section, only linear equations in 2 variables will be considered, e.g.

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

where A through to G are functions of x and y, as is $u = u(x, y)$. If $G(x, y) = 0$ then the pde is homogeneous, if not, then the pde is inhomogeneous.

The key idea to recall here is that integration of a partial derivative does **not** lead to a constant, but rather to an arbitrary function, e.g. if $\frac{\partial u}{\partial x} = 0$, then

$$u = \int \frac{\partial u}{\partial x} dx = f(y) \tag{1}$$

For example, the pde $\frac{\partial^2 u}{\partial y^2} = 0$ can be solved by integrating twice. Integrating the first time yields

$$\frac{\partial u}{\partial y} = f(x) \tag{2}$$

and integrating again yields

$$u(x, y) = yf(x) + g(x) \tag{3}$$

Example: Solve the following pde using an integrating factor approach:

$$\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} = 1 \tag{4}$$

First substitute $v = \frac{\partial u}{\partial y}$, s.t. equation (4) now becomes

$$\frac{\partial v}{\partial x} + v = 1 \tag{5}$$

Using the integrating factor

$$\frac{\partial}{\partial x}(e^x v) = e^x$$

leads to

$$e^x v = e^x + g(y) \quad \text{so that} \tag{6}$$

$$v = 1 + g(y)e^{-x} \quad \text{and therefore} \tag{7}$$

$$\frac{\partial u}{\partial y} = 1 + g(y)e^{-x} \tag{8}$$

Integrating equation(8) yields the solution to the pde,

$$u(x,y) = y + f(y)e^{-x} + h(x)$$

where

$$f(y) = \int g(y)dy$$

2. Separation of variables
It is sometimes possible to find particular solutions of a pde in the form of a product

$$u(x,y) = X(x)Y(y) \tag{9}$$

leading to the following shorthand notation:

$$\frac{\partial u}{\partial x} = X'Y \quad , \quad \frac{\partial u}{\partial y} = XY'$$

$$\frac{\partial^2 u}{\partial x^2} = X''Y \quad , \quad \frac{\partial^2 u}{\partial y^2} = XY''$$

To see the method of separation of variables in action, take the example of the pde

$$\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial u}{\partial y} \tag{10}$$

now seek solutions of this pde using a product solution

$$u(x,y) = X(x)Y(y) \tag{11}$$

In shorthand notation equation (10) can be written as

$$X''Y = 4XY' \tag{12}$$

where the above equation can be rearranged to give

$$\frac{X''}{4X} = \frac{Y'}{Y} \tag{13}$$

Since the l.h.s is independent of y and the r.h.s is independent of x, it is safe to conclude that both sides are equal to a constant. It is convenient to write this constant as λ^2 or $-\lambda^2$. This constant is sometimes referred to as the **separation constant**. There are three cases of the separation constant to explore:

Case I: $\lambda^2 > 0$

In this case, we now have two odes to solve; one in x and one in y:

$$X'' - 4X\lambda^2 = 0 \text{ and } Y' - Y\lambda^2 = 0 \tag{14}$$

This will yield solutions (that you should be easily able to obtain by now)

$$X = c_1 cosh(2\lambda x) + c_2 sinh(2\lambda x) \tag{15}$$

$$Y = c_3 e^{\lambda^2 y} \tag{16}$$

Yielding a product solution

$$u(x,y) = A_1 e^{\lambda^2 y} cosh(2\lambda x) + B_1 e^{\lambda^2 y} sinh(2\lambda x) \tag{17}$$

where $A_1 = c_1 c_3$ and $B_1 = c_2 c_3$.

Case II: $-\lambda^2 < 0$

The two odes to solve are now:

$$X'' + 4X\lambda^2 = 0 \quad \text{and} \quad Y' + Y\lambda^2 = 0 \tag{18}$$

yielding solutions

$$X = c_4 cos(2\lambda x) + c_5 sin(2\lambda x) \tag{19}$$

$$Y = c_6 e^{-\lambda^2 y} \tag{20}$$

Yielding a product solution

$$u(x,y) = A_2 e^{-\lambda^2 y} cos(2\lambda x) + B_2 e^{-\lambda^2 y} sin(2\lambda x) \tag{21}$$

where $A_2 = c_4 c_6$ and $B_2 = c_5 c_6$.

Case III: $\lambda^2 = 0$

The two odes to solve are now:

$$X'' = 0 \quad \text{and} \quad Y' = 0 \tag{22}$$

yielding solutions

$$X = c_7 x + c_8 \tag{23}$$

$$Y = c_9 \tag{24}$$

Yielding a product solution

$$u(x,y) = A_3 x + B_3 \tag{25}$$

where $A_2 = c_7 c_9$ and $B_2 = c_8 c_9$.

3. Superposition principle

In the following, the assumption is made that whenever there is an infinite set of solutions to a pde, $u_1, u_2, u_3 \ldots$, then we can construct another solution u, by forming the infinite series

$$u(x,y) = \sum_{k=1}^{\infty} = u_k(x,y)$$

4. Boundary value problems

Boundary value problems are pdes that are to be solved by applying boundary conditions and initial conditions. Examples are the heat equation, the wave equation and Laplace's equation. When solving boundary value problems, you will find that your choice of the separation constant becomes important – the wrong choice of separation constant will lead to either unphysical results or to not being able to find a solution according to your boundary conditions and initial conditions.

Example: Here the heat equation is solved for the following conditions. Consider a rod of length L, with an initial temperature= $f(x)$ throughout the rod, with the ends of the rod at a fixed temperature of 0 degrees for all time. The temperature of the rod is given as a function of time, t, and distance along the rod, x, ie temperature= $u(x,t)$. Of course there are many assumptions being made about this system, such that heat is only moving in the x-direction and so on. The pde for this situation is given by

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad k > 0, \quad 0 < x < L, \quad t > 0 \tag{26}$$

where k is proportional to the thermal conductivity of the rod and the boundary conditions and initial conditions are:

$$u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0 \tag{27}$$

and,

$$u(x,0) = f(x), \quad 0 < x < L \tag{28}$$

Try a product solution of the form

$$u(x,t) = X(x)T(t) \tag{29}$$

with a separation constant of $-\lambda^2$. This choice of separation constant will lead to a sinusoidal function in x (the equilibrium solution when time $\to \infty$) and a decaying exponential function in t (the transient solution in time). Therefore this choice of separation constant makes physical sense. In shorthand notation, the heat equation can be re-expressed as

$$\frac{X''}{X} = \frac{T'}{kT} \tag{30}$$

This yields two odes to solve:

$$X'' + X\lambda^2 = 0 \text{ and } T' + kT\lambda^2 = 0 \tag{31}$$

yielding solutions

$$X = c_1 cos(\lambda x) + c_2 sin(\lambda x) \tag{32}$$

$$T = c_3 e^{-k\lambda^2 t} \tag{33}$$

Before taking the product to form the solution $u(x,t)$, consider first some of the boundary conditions. The condition $u(0,t) = 0$, implies that $c_1 = 0$. Therefore,

$$X = c_2 sin(\lambda x) \tag{34}$$

The condition $u(L,t)=0$ gives rise to a non-trivial solution provided that

$$\lambda = \frac{n\pi}{L}, \quad n = 1, 2, 3 \ldots \tag{35}$$

such that

$$X = A_n sin(\frac{n\pi x}{L}) \tag{36}$$

Using the superposition principle gives the solution

$$u(x,t) = \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} A_n e^{\left(-\frac{kn^2\pi^2 t}{L^2}\right)} sin(\frac{n\pi x}{L}) \tag{37}$$

Expressions for the A_n can be obtained by applying the initial conditions $u(x,0) = f(x)$. Therefore

$$u(x,0) = \sum_{n=1}^{\infty} A_n sin(\frac{n\pi x}{L}) \tag{38}$$

Recalling the Fourier sine series, we can get an expression for A_n:

$$A_n = \frac{2}{L}\int_0^L f(x) sin(\frac{n\pi x}{L}) dx \tag{39}$$

giving us a final expression for the solution to the heat equation as:

$$u(x,t) = \sum_{n=1}^{\infty} \left[\frac{2}{L}\int_0^L f(x) sin(\frac{n\pi x}{L}) dx\right] e^{\left(-\frac{kn^2\pi^2 t}{L^2}\right)} sin(\frac{n\pi x}{L}) \tag{40}$$

Exercises:

1. Using the method of undetermined coefficients, solve $\frac{\partial^2 u}{\partial x^2} - y^2 u = e^x$
2. Solve Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad 0 < x < a, \quad 0 < y < b \tag{41}$$

to find the steady-state temperature $u(x,y)$ of a 2D plate, subject to the following boundary conditions and initial conditions:

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=a} = 0, \quad 0 < y < b$$

and

$$u(x,0) = 0, \quad u(x,b) = f(x), \quad 0 < x < a$$

Be sure to obtain expressions for any coeffcients you might use in superposition.