LANDAU DAMPING

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1. Introduction

Landau damping is a phenomena observed in plasma wherein there is an exponential decay in the oscillations of the number density of electrons in a plasma (also referred to as Langmuir waves) and so stability is achieved in some area of the phase-space. This kind of plasma environment can be seen in fluorescent lights, plasma balls, and in the plasma rings seen in Tokamak fusion reactors, and in this report, one can visual the setting as a simplified idea of the motion of electrons in fluorescent light tube.

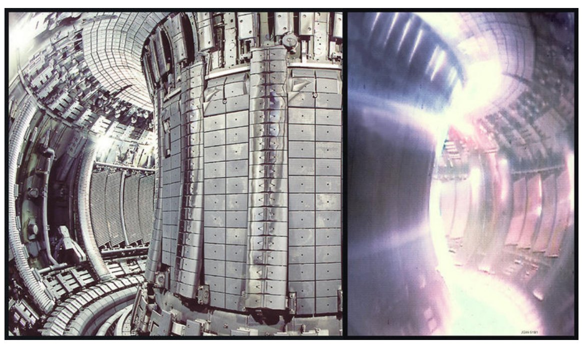

Figure 1. Plasma at work: Tokamak fusion reactor

2. Derivation

Following a similar method to [1], we consider a one-dimensional plasma that consists of heavy, positively charged ions and smaller negatively charged electrons. The density of the electrons is described by the function: $n_e \colon \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ where $n_e(\cdot, t)$ gives represents the density of the electrons at a time $t \in \mathbb{R}_{\geq 0}$. Each electron possesses a random velocity $v \in \mathbb{R}$, which are distributed according to the distribution function $f \colon \mathbb{R} \times \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$, which satisfies:

$$\int_{-\infty}^{+\infty} f(x, t, v) \, \mathrm{d}v = n_e(x, t)$$

This can be seen in Figure 2. We assume the ions are immobile, so the density of the ions is defined by the constant n_i. The assumption that the ions are fixed in space is convenient and is made as the relative velocity of the ions are much smaller in comparison to the velocity of the electrons. As a result the motion of the ions can be ignored in a simple plasma model. We also assume the electrons don't interact with each other: their size, and their individual contribution to the

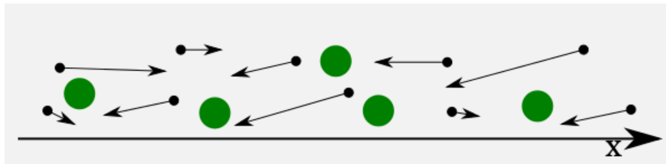

FIGURE 2. Ions and electrons in the plasma model

electric field, relative to the space they exist in, is too small to be significant. If this is the case, though, since a collisionless plasma would allow the electrons to all move completely freely, and so the oscillations in the number density would remain for all time. The calculations which follow try to prove the unintuitive notion that damping does indeed take place in reasonable cases.

We suppose the collection of electrons have mass m. Considering a small local perturbation of the distribution function f around a point (x, t, v):

$$f(x + \Delta x, t + \Delta t, v + \Delta v) - f(x, t, v)$$

This can be approximated using a first order expansion of f about (x, t, v) :

$$\begin{aligned}
&\frac{\partial f}{\partial t}(x,t,v) + \frac{\partial x}{\partial t}\frac{\partial f}{\partial x}(x,t,v) + \frac{\partial v}{\partial t}\frac{\partial f}{\partial v}(x,t,v) \\
&= \frac{\partial f}{\partial t}(x,t,v) + v\frac{\partial f}{\partial x}(x,t,v) + a\frac{\partial f}{\partial v}(x,t,v) \\
&= \frac{\partial f}{\partial t}(x,t,v) + v\frac{\partial f}{\partial x}(x,t,v) + \frac{F}{m}\frac{\partial f}{\partial v}(x,t,v)
\end{aligned}$$

using Newton's second law of motion, where F represents force acting on the electrons. Hence we can estimate the difference of a small perturbation of the electron density n_e about a point (x, t) by:

$$\begin{aligned}
&n_e(x + \Delta x, t + \Delta t) - n_e(x, t) \\
&= \int_{-\infty}^{+\infty} f(x + \Delta x, t + \Delta t, v + \Delta v)\,\mathrm{d}v \;-\; \int_{-\infty}^{+\infty} f(x, t, v)\,\mathrm{d}v \\
&= \int_{-\infty}^{+\infty} \left(f(x + \Delta x, t + \Delta t, v + \Delta v) - f(x, t, v)\right)\mathrm{d}v \\
&\approx \int_{-\infty}^{+\infty} \left(\frac{\partial f}{\partial t}(x,t,v) + v\frac{\partial f}{\partial x}(x,t,v) + \frac{F}{m}\frac{\partial f}{\partial v}(x,t,v)\right)\,\mathrm{d}v.
\end{aligned}$$

Hence, by solving the partial differential equation

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{F}{m}\frac{\partial f}{\partial v} = 0 \tag{1}$$

we can guarantee that

$$\int_{-\infty}^{+\infty} n_e(x,t)\,\mathrm{d}x = \text{ constant} \qquad \text{for all } \; t \in \mathbb{R}_{\geq 0}$$

so the electron mass is conserved. Equation (1) is called the Vlasov equation.

In our model, we assume that the electrons do not collide, so the force that the electrons experience is given by the Lorentz force:

$$F = -e(E + v \cdot B)$$

where $-e$ is the charge of each electron, E and B represent the mean electrical and the mean magnetic fields that the electrons that each electron experiences, respectively. We assume further that there is no magnetic field acting on the particles, so $B = 0$. This simplifies the expression of the force acting on the electrons. However, in plasma physics experiments, the plasma is confined using powerful electromagnets, so one way to increase the relevance of the model is to relax the condition that $B = 0$.

Using the first of Maxwell's equations, we have:

$$\nabla \cdot E = \frac{\partial E}{\partial x} = -\frac{\rho}{\varepsilon_0}$$

where ρ denotes the total charge density of the system and ε_0 is a constant called the permittivity of free space, which describes how well electricity conducts through empty space. This is relevant as a fully ionised plasma consists of only ions and electrons, the rest of the space is empty. If we let Qe denote the mean charge of the ions in the plasma, we have:

$$\rho = -e(n_e - Qn_i)$$

The constant term Q is necessary as the model makes no assumptions about the matter that has been ionised to form the plasma.

We can consider the electric field as the gradient of a electric potential, which we will denote by $\phi \colon \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$:

$$E = -\frac{\partial \phi}{\partial x}$$

which we can use in Maxwell's equation and the Vlasov equation to give:

$$\frac{\partial^2 \phi}{\partial x^2} = -\frac{\partial E}{\partial x} = \frac{e}{\varepsilon_0}(Qn_i - n_e) \tag{2a}$$

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{e}{m}\frac{\partial \phi}{\partial x}\frac{\partial f}{\partial v} = 0. \tag{2b}$$

This system of non-linear equations can be non-dimensionalised by rescaling the variables and the functions. After doing this we obtain the following system of equations:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{\partial \phi}{\partial x}\frac{\partial f}{\partial v} = 0 \tag{3a}$$

$$\frac{\partial^2 \phi}{\partial x^2} = 1 - n, \tag{3b}$$

where

$$\int_{-\infty}^{+\infty} f(x,t,v)\,\mathrm{d}v = n(x,t). \tag{3c}$$

3. Cold electron limit

In the cold electron limit, we make the assumption that the electrons are less energetic, resulting in the electrons have purely deterministic velocities. This can be expressed as follows:

$$f(x,t,v) = n_e(x,t)\delta(v - \bar{v}(x,t))$$

where $v\colon \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ gives the velocity of an electron with position x at time t. Here δ defines the Dirac delta distribution, which can be integrated against functions belonging to the space of test functions, $C_c^\infty(\mathbb{R} \times \mathbb{R}_{\geq 0} \times \mathbb{R})$. A simple calculation shows the following equation holds:

$$\bar{v}(x,t) = \frac{1}{n_e(x,t)} \int_{-\infty}^{+\infty} v f(x,t,v)\,\mathrm{d}v$$

so $\bar{v}$ can be thought of as the mean velocity of the electrons as a function of space and time. The space of test functions that we will be using is $C_c^\infty(\mathbb{R} \times \mathbb{R}_{\geq 0} \times \mathbb{R})$. We use ψ to denote an arbitrary function belonging to this function space.

By integrating the Vlasov equation against an arbitrary test function ψ over space, time and velocity variables we obtain:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(n\bar{v}) = 0,$$

using the Fundamental Lemma of the Calculus of Variations. Similarly, we integrating the Vlasov equation against test functions of the form $v\psi$ we obtain the following equation:

$$\frac{\partial}{\partial t}(n\bar{v}) + \frac{\partial}{\partial x}(n\bar{v}^2) = \frac{\partial \phi}{\partial x}.$$

These two equations relate the system to fluid dynamics, as these are two of the Euler equations of fluid dynamics. Each of the equations indicates that the system conserves some physical quantity. The first equation shows the conservation of mass and the second shows the conservation of momentum of the system.

4. Linearised Vlasov equation

One possible option to simplify the system of equations is to linearize the Vlasov equation. This can be done by considering a perturbation of an undisturbed plasma by a wave. An undisturbed plasma is in equilibrium, with uniform distribution of electrons which has no dependence on the position or time. In this case, the distribution function, denoted by f_0, represents the distribution of the velocities of the electrons. We write the distribution function as:

$$f(x,t,v) = f_0(v) + \varepsilon\,\tilde{f}(x,t,v).$$

Here $\tilde{f}$ is the distribution of the perturbation. Similarly, we can corresponding equations for the electron density function n_e:

$$n_e(x,t) = 1 + \varepsilon\,\tilde{n}(x,t).$$

In the undisturbed plasma, as the distribution function is independent of space and time, we have that:

$$n_0(x,t) = \int_{-\infty}^{+\infty} f_0(x,t,v)\,\mathrm{d}v = 1 \quad \text{for all } (x,t) \in \mathbb{R} \times \mathbb{R}_{\geq 0}$$

and $\tilde{n}$ is the density of electrons corresponding to the distribution function $\tilde{f}$. Also, we consider the perturbed electrical potential:

$$\phi(x,t) = \phi_0(x,t) + \varepsilon\,\tilde{\phi}(x,t).$$

We derive a first-order perturbation equation from the Vlasov equation by collecting all the terms that have ε as a co-efficient. This gives a linear-system of integro-differential equations:

$$\frac{\partial \tilde{f}}{\partial t} + v\frac{\partial \tilde{f}}{\partial x} + f_0'(v)\frac{\partial \tilde{\phi}}{\partial x} = 0 \tag{4a}$$

$$\frac{\partial^2 \tilde{\phi}}{\partial x^2} = -\tilde{n} \tag{4b}$$

where

$$\tilde{n}(x,t) = \int_{-\infty}^{+\infty} \tilde{f}(x,t,v)\,\mathrm{d}v. \tag{4c}$$

Now this system of equations is linear, we can now attempt to solve the system of equations. These solution of this problem give a simple model for a plasma that exhibits Landau damping.

We now take this simplified linearized model, where the electron distribution is approximately $f_0(v)$ as before, and we have an initial condition for f given by $F_0(v,x)$. In order to solve this equation to get an expression for f, we need to seperate out some of the variables to get something we can solve. Our plan is to tackle it in the following way, as seen in [2]:

- Apply the Fourier Transform in x to f and the number density $n(x,t)$.
- Integrate in v to get an integral equation for $\hat{n}(k,t)$
- Apply the Laplace transform in t to $\hat{n}(k,t)$ to get $\tilde{n}(k,p)$ and rearrange to get a function we can solve for n.

Once we have this, in order to get back n itself, we have to find the set of singularities to ensure that we can perform the inverse Laplace Transform given by:

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \tilde{n}(k,p)e^{pt}dp$$

where the line $\mathrm{Re}(k) = \gamma$ is to the right of all the singularities. However, this knowledge is itself sufficient to tell us a lot about the model. If the singularities all have $\mathrm{Re}(p) < 0$ for all values of k, then we get an exponential decay in the amplitude of the density oscillations at any specific point. This comes from the fact that poles of the form $p = a + ib$ introduce a term of the form $Ce^{at}\cos(bt+d)$.

We define the Fourier Transform here to be:

$$\hat{g}(k) = \int_{-\infty}^{\infty} g(x)e^{ikx}dx$$

Looking at the Fourier Transform of f, n, and ϕ, and using the fact that under the Fourier Transform derivatives becomes multiples, or in other words, taking the transform with respect to x:

$$\widehat{\left(\frac{\partial f}{\partial x}\right)} = (2\pi ik)\hat{f}$$

we get the following variations of the Vlasov equation with initial conditions and the number density expression:

$$\frac{\partial \hat{f}}{\partial t}(k,t,v) - ikv\hat{f}(k,t,v) - ikf_0'(v)\hat{\phi}(k,t) = 0$$
$$\int_{-\infty}^{\infty} \hat{f}(k,t,v)dv = \hat{n}(k,t) = -k^2\hat{\phi}$$
$$\hat{f}(k,0,v) = \hat{F}_0(k,t)$$

This is now an ordinary differential equation in t, and so using the integrating factor e^{ikv}, we get the solution $\hat{f}$:

$$\hat{f}(k,t,v) = \hat{F}_0(v,k)e^{-ikvt} + \frac{i}{k}f_0'e^{ikvt}\int_0^t \hat{n}(k,t)e^{-ikv\tau}\,\mathrm{d}\tau$$

Now integrating with respect to v gives us an equation for $\hat{n}$:

$$\hat{n}(k,t) = \int_{-\infty}^{\infty} \hat{F}_0(v,k)e^{ikvt}\,\mathrm{d}v + \frac{i}{k}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_0'(v)\hat{n}(k,\tau)e^{ikv(t-\tau)}\,\mathrm{d}\tau\,\mathrm{d}v \tag{5}$$

Unfortunately, this integral equation isn't easy to solve in its current form, and deducing any damping properties at this stage isn't likely either. However, we do note that the final term in the left-hand-side of (5) is a convolution of transforms, which equates to a product of convolutions.

In order to get something constructive, we apply the Laplace Transform to $\hat{n}$, where the transform is defined by:

$$\mathcal{L}[g(t)](p) :\, = \int_0^{\infty} e^{-pt}f(t)dt$$

Applying this transform to $\hat{n}$, and using Fubini to change the order of integration, we get the following equation, which we can rearrange to get an explicit form:

$$\tilde{n}(k,p,v) :\, = \mathcal{L}[\hat{n}](k,p,v) = \int_{-\infty}^{\infty} \frac{f_0'(v)dv}{p+ikv} + \frac{\tilde{n}(k,p)}{k^2}\int_{-\infty}^{\infty}\frac{f_0'(v)dv}{v-ip/k}$$
$$= \frac{\left(\int_{-\infty}^{\infty} \frac{f_0'(v)dv}{p+ikv}\right)}{\left(1 - \frac{1}{k^2}\int_{-\infty}^{\infty}\frac{f_0'(v)dv}{v-ip/k}\right)}$$

Singularities of this equation are precisely where the expression in the denominator vanishes, or in other words at values of (k,p) which solve the following equation attributed to Landau:

$$1 = \frac{1}{k^2}\int_{-\infty}^{\infty}\frac{f_0'(v)dv}{v-ip/k} \tag{6}$$

Practically, one would want to look at the Gaussian initial conditions for the density given by $f_1(v) = (1/\varepsilon)e^{-v^2/\varepsilon^2}$. This is a very natural choice, and for small values of ε we can see that this approaches a point mass at zero in v, meaning that all the electrons would be stationary initially. However, this is clearly not easy to work with analytically to find singularities and discuss damping, so we consider a similarly behaved function: $f_0(v) = (\varepsilon(v^2+\varepsilon^2))^{-1}$. If we want to find values of p which solve our above expression (6) for fixed values of k, first note that for $k=0$ we run into all sorts of problems with our definitions of the transforms of f and n.

However, we do get for $k = 0$ that $\hat{f}$ must be constant in time, and so the density remains constant over time as well. For $k \neq 0$, we get that:

- for $\mathrm{Re}(p/k) < 0$ we get the following solution for $\tilde{n}$, using f_0:
$$\tilde{n}(k,p) = \frac{-\pi}{\varepsilon^2(p - k\varepsilon)^2}$$
which has solutions in p of $k\varepsilon + i\frac{\sqrt{\pi}}{\varepsilon}$ and $k\varepsilon - i\frac{\sqrt{\pi}}{\varepsilon}$
- For $\mathrm{Re}(p/k) > 0$ we get:
$$\tilde{n}(k,p) = \frac{-\pi}{\varepsilon^2(p + k\varepsilon)^2}$$
which has solutions in p of $-k\varepsilon - i\frac{\sqrt{\pi}}{\varepsilon}$ and $-k\varepsilon + i\frac{\sqrt{\pi}}{\varepsilon}$

It can be seen that in both of these cases, the solutions p, don't match up with the stipulations of the sign of k/p, and so all in fact non of these solutions hold, leading us to the conclusion that we have no singularites at all. Hence, we can choose the curve of our inverse Laplace transform to be in the left-half-plane, and therefore the system must be damped.

This damping is in the Fourier transform, which means that the number density of electrons must converge exponentially fast to its mean, and so the force from the electric field converges exponentially fast to zero, leaving a system where the electrons are evenly distributed and can move without interacting with each other, and only interacting with the comparatively stationary ions. This entire argument works in a very similar way in higher dimensions, and the model is equally effective in those settings. One could look further into stability of the density over time, dependent on our initial conditions of the velocity distribution of the electrons.

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