ANALYSIS OF THE DIFFUSE DOMAIN APPROACH FOR A BULK-SURFACE COUPLED PDE SYSTEM *

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Abstract. We analyze a diffuse interface type approximation, known as the diffuse domain approach, of a linear coupled bulk-surface elliptic PDE system. The well-posedness of the diffuse domain approximation is shown using weighted Sobolev spaces and we prove that the solution to the diffuse domain approximation converges weakly to the solution of the coupled bulk-surface elliptic system as the approximation parameter tends to zero. Moreover, we can show strong convergence for the bulk quantity, while for the surface quantity, we can show norm convergence and strong convergence in a weighted Sobolev space.

Key words. diffuse domain method, weighted Sobolev spaces, well-posedness, diffuse interface approximation, asymptotic analysis, bulk-surface elliptic equations.

AMS subject classifications. 35J25, 35J50, 35J70, 46E35, 41A30, 41A60

1. Introduction. The diffuse domain approach [33, 28] is a method originating from the phase field methodology which approximates partial differential equations posed on domains with arbitrary geometries. The method embeds the original domain $\Omega^* \subset \mathbb{R}^n$ with complicated geometries into a larger domain Ω with a simpler geometry. Drawing on aspects of the phase field methodology, the diffuse domain method replaces the boundary Γ of Ω^* with an interfacial layer of thickness $0 < \varepsilon \ll 1$, denoted by Γ_ε. The original PDEs posed on Ω^* will have to be extended to Ω and any surface quantities or boundary terms on Γ have to be extended to fields defined on Γ_ε. The resulting PDE system, which we denote as the diffuse domain approximation, is defined on Ω and will have the same order as the original system defined in Ω^*, but with additional terms that approximate the original boundary conditions on Γ.

Our model problem will be the following elliptic coupled bulk-surface system:

$$\text{(CSI)}\quad \begin{aligned} -\nabla \cdot (\mathcal{A}\nabla u) + au &= f && \text{in } \Omega^*, \\ -\nabla_\Gamma \cdot (\mathcal{B}\nabla_\Gamma v) + bv + \mathcal{A}\nabla u \cdot \nu &= \beta g && \text{on } \Gamma, \\ \mathcal{A}\nabla u \cdot \nu &= K(v-u) && \text{on } \Gamma. \end{aligned}$$

Here, $\nabla_\Gamma v$ and $\nabla_\Gamma \cdot \boldsymbol{v}$ denote the surface gradient of v and the surface divergence of $\boldsymbol{v}$ on Γ, respectively. For a precise definition, we refer the reader to [13, Section 2] or Section 3.2 below. Meanwhile, $K, \beta \geq 0$ are nonnegative constants, and $\mathcal{A} = (a_{ij})_{1\leq i,j\leq n}$ and $\mathcal{B} = (b_{ij})_{1\leq i,j\leq n}$ denote the matrices with function coefficients a_{ij} and b_{ij}, respectively. The precise assumptions on the data and the domain will be given in Section 2.1.

We now embed $\Omega^* \cup \Gamma$ into a larger domain $\Omega \subset \mathbb{R}^n$. The location of the original boundary Γ is encoded in an order parameter φ as its zero-level set. A typical choice

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for φ is a function of the signed distance function of Γ, $d:\Omega\to\mathbb{R}$, which is defined as

$$d(x)=\begin{cases}-\inf_{z\in\Gamma}|x-z| & \text{for } x\in\Omega^*,\\ 0 & \text{for } x\in\Gamma,\\ \inf_{z\in\Gamma}|x-z| & \text{for } x\in\Omega\setminus\overline{\Omega^*}.\end{cases} \tag{1.1}$$

Let χ_{Ω^*} denote the characteristic function of Ω^* and $\delta_\Gamma := \mathcal{H}^{n-1}\llcorner\Gamma$ denote the Hausdorff measure restricted to Γ. We may define, in the sense of distributions given by the characteristic function of Ω^* and the Dirac measure of Γ,

$$\int_\Omega f\,\mathrm{d}\chi_{\Omega^*}=\int_{\Omega^*} f\,\mathrm{dx},\qquad \int_\Omega f\,\mathrm{d}\delta_\Gamma=\int_\Gamma f\,\mathrm{d}\mathcal{H}^{n-1}$$

for smooth functions $f:\Omega\to\mathbb{R}$. An equivalent distribution form for (CSI), due to [3, §2.7 and Theorem 2.8], is (see also [20, Appendix] for a derivation)

$$\begin{aligned}-\nabla\cdot(\chi_{\Omega^*}\mathcal{A}\nabla u)+\chi_{\Omega^*}au&=\chi_{\Omega^*}f+\delta_\Gamma K(v-u)\text{ in }\mathcal{D}'(\Omega),\\ -\nabla\cdot(\delta_\Gamma\mathcal{B}\nabla v)+\delta_\Gamma bv&=\delta_\Gamma\beta g-\delta_\Gamma K(v-u)\text{ in }\mathcal{D}'(\Omega).\end{aligned}$$

The diffuse domain approximation of (CSI) is derived by approximating χ_{Ω^*} and δ_Γ with more regular functions $\xi_\varepsilon(\varphi)$, $\delta_\varepsilon(\varphi)$, indexed by ε, which is related to the thickness of the interfacial layer Γ_ε. In other words, the diffuse domain approximation of (CSI) is

$$\begin{aligned}-\nabla\cdot(\xi_\varepsilon\mathcal{A}^E\nabla u^\varepsilon)+\xi_\varepsilon a^E u^\varepsilon&=\xi_\varepsilon f^E+\delta_\varepsilon K(v^\varepsilon-u^\varepsilon)\quad\text{in }\Omega,\\ -\nabla\cdot(\delta_\varepsilon\mathcal{B}^E\nabla v^\varepsilon)+\delta_\varepsilon b^E v^\varepsilon&=\delta_\varepsilon\beta g^E-\delta_\varepsilon K(v^\varepsilon-u^\varepsilon)\text{ in }\Omega,\end{aligned}\tag{CDD}$$

where $\mathcal{A}^E,\mathcal{B}^E,a^E,b^E,f^E$, and g^E denote suitable extensions of $\mathcal{A},\mathcal{B},a,b,f$, and g to the larger domain Ω. The precise assumptions on the extensions will be outlined in Section 2.3. We assume a homogeneous Neumann boundary condition for (CDD):

$$\mathcal{A}^E\nabla u^\varepsilon\cdot\nu_{\partial\Omega}=\mathcal{B}^E\nabla v^\varepsilon\cdot\nu_{\partial\Omega}=0\text{ on }\partial\Omega.$$

Formally, $\xi_\varepsilon(\varphi)\to\chi_{\Omega^*}$, $\delta_\varepsilon(\varphi)\to\delta_\Gamma$ as $\varepsilon\to0$, and so in the limit of vanishing interfacial thickness, we recover the distributional form for (CSI).

Here onward, we refer to the original problem posed on Ω^* as the SI (sharp interface) problem, and the corresponding diffuse domain approximation posed on Ω will be denoted as the DD (diffuse domain) problem.

We remark that replacing the original boundary Γ with an interfacial layer Γ_ε transforms the diffuse domain approximation into a two-scale problem. This is similar to phase field approximations of free boundary problems. The idea of adopting the phase field methodology to approximate partial differential equations has been applied to study diffusion inside a stationary cell [23], Turing patterns on a membrane [27], wave propagation in the heart [10, 18], two-phase flow [2], and soluble surfactants [20, 35]. In [7, 8], a diffuse domain type method, denoted as the smoothed boundary method, has been applied to solve partial differential equations on irregular domains with homogeneous Neumann boundary conditions using spectral methods. A generalized formulation of the smoothed boundary method can be found in [36].

The phase field methodology provides us with two candidates for $\varphi(x)$. The first is based on the smooth double-well potential $\psi_{DW}(\varphi)=\frac{1}{4}(1-\varphi^2)^2$ and leads to

$$\varphi_{DW}(x):=\tanh\left(\frac{d(x)}{\sqrt{2}\varepsilon}\right)\text{ for }x\in\Omega.$$

The other is based on the double-obstacle potential [4, 11]

$$\psi_{DO}(\varphi) = \frac{1}{2}(1-\varphi^2) + I_{[-1,1]}(\varphi), \quad I_{[-1,1]}(\varphi) = \begin{cases} 0 & \text{if } \varphi \in [-1,1], \\ +\infty & \text{otherwise} \end{cases}$$

and leads to

$$\varphi_{DO}(x) := \begin{cases} +1, & \text{if } d(x) > \varepsilon\frac{\pi}{2}, \\ \sin(d(x)/\varepsilon), & \text{if } |d(x)| \le \varepsilon\frac{\pi}{2}, \\ -1, & \text{if } d(x) < -\varepsilon\frac{\pi}{2}. \end{cases}$$

Note that in the case of the double-obstacle potential, the interfacial layer Γ_ε has finite thickness of $\varepsilon\pi$. We remark that for general phase field models φ_{DW} or φ_{DO} is the leading order approximation for the order parameter φ. In our setting, the location of the boundary Γ is known and hence we can use φ_{DW} or φ_{DO} as the order parameter itself. A common regularization of χ_{Ω^*} based on the smooth double-well potential used in [28, 34, 35] is

$$\xi_\varepsilon^{(1)}(x) = \frac{1}{2}(1-\varphi_{DW}(x)) = \frac{1}{2}\left(1-\tanh\left(\frac{d(x)}{\sqrt{2}\varepsilon}\right)\right),$$

while an alternative based on the double-obstacle potential is

$$\xi_\varepsilon^{(2)}(x) = \frac{1}{2}(1-\varphi_{DO}(x)).$$

There are many regularizations of δ_Γ available from the literature [35, 15, 33, 25]. One well-known approximation of δ_Γ is a multiple of the Ginzburg–Landau energy density (see [31])

$$\frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{\varepsilon}\psi(\varphi).$$

From the above discussions regarding the double-well and the double-obstacle potentials, we have two candidates for the regularization to δ_Γ:

$$\delta_\varepsilon^{(1)}(x) = \frac{3}{2\sqrt{2}\varepsilon}\operatorname{sech}^4\left(\frac{d(x)}{\sqrt{2}\varepsilon}\right), \quad \delta_\varepsilon^{(2)}(x) = \frac{2}{\pi\varepsilon}\cos^2\left(\frac{d(x)}{\varepsilon}\right)\chi_{\{x\in\Omega\,:\,|d(x)|\le\varepsilon\frac{\pi}{2}\}}.$$

The convergence analysis of the diffuse domain approach (with the smooth double-well potential), in the limit $\varepsilon \to 0$, has only been done in the context of recovering the original equations via formally matched asymptotics (see [28, 34, 35, 26]). A first analytical treatment of convergence in one dimension and on a half-plane in two dimensions can be found in [19], where the error between the solution to a second order system and the diffuse domain approximation in the L^∞ norm is of order $\mathcal{O}(\varepsilon^{1-\mu})$, where $\mu > 0$ arbitrarily small.

In [14], a diffuse domain type approximation for an advection-diffusion equation posed on evolving surfaces is considered. Motivated by modeling and numerical simulations, the diffuse domain approximation utilizes a double-obstacle type regularization. Note that the regularizations from the double-obstacle potential are degenerate in certain parts of the larger domain Ω (in particular they are zero outside Γ_ε for the set-up in [14]). Consequently the corresponding diffuse domain approximation becomes a degenerate equation and weighted Sobolev spaces are employed. The chief results in [14] are the well-posedness of the diffuse domain approximation and weak convergence to the solution of the original system.

For a bulk second order elliptic boundary value problem, the convergence analysis has been studied in [9]. There, a double-obstacle type diffuse domain approximation to a Robin boundary value problem is studied, with the specific choice $\delta_\varepsilon = |\nabla \xi_\varepsilon|$. The authors are able to deduce trace theorems, embedding theorems and Poincaré inequalities for Sobolev spaces weighted with ξ_ε. Under suitable assumptions on the decay of ξ_ε near $\partial\Gamma_\varepsilon$, the authors showed that there exists a $p > 2$ such that the error between the solution to the diffuse domain approximation and the solution to the Robin problem is of order $\mathcal{O}(\varepsilon^{\frac{1}{2}-\frac{1}{p}})$ in a weighted Sobolev norm. Similar results for the Neumann and Dirichlet boundary conditions are also given.

In this work, we show that weak solutions to (CDD) converge to the unique weak solution to (CSI) under appropriate assumptions. Thanks to the property of the problem, we are also able to show strong convergence in $H^1(\Omega^*)$ for the bulk quantity, while for the surface quantity we have norm convergence and strong convergence in a weighted Sobolev space.

The structure of this article is as follows. In Section 2 we introduce the assumptions and the main results. In Section 3 we prove several technical results that will simplify the proof of the main results, which are contained in Section 4. In Section 5, we compare our results with the results in [14, 9].

2. General assumptions and main results.

2.1. Assumptions on the data. We make the following assumptions on the domain and the data.

ASSUMPTION 2.1 (assumptions on domain). *We assume that Ω^* is an open bounded domain in $\mathbb{R}^n$ with compact C^3 boundary Γ and outward unit normal ν. Let Ω be an open bounded domain in $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$ such that $\overline{\Omega^*} \subset \Omega$ and $\Gamma \cap \partial\Omega = \varnothing$.*

ASSUMPTION 2.2 (assumptions on the data). *We assume that for $1 \le i, j \le n$,*

$$a_{ij}, a \in L^\infty(\Omega^*), \quad f \in L^2(\Omega^*), \quad b_{ij}, b \in L^\infty(\Gamma), \quad g \in L^2(\Gamma),$$

and there exist positive constants $\theta_0, \theta_1, \theta_2, \theta_3$ such that

$$(\mathcal{A}(x)\zeta_1) \cdot \zeta_1 \ge \theta_0 |\zeta_1|^2, \quad (\mathcal{B}(p)\zeta_2) \cdot \zeta_2 \ge \theta_1 |\zeta_2|^2, \quad a(x) \ge \theta_2, \quad b(p) \ge \theta_3,$$

for all $x \in \Omega^*$, $p \in \Gamma$, $\zeta_1 \in \mathbb{R}^n$, and $\zeta_2 \in T_p\Gamma \subset \mathbb{R}^n$.

2.2. Well-posedness of the SI problem. Let $\gamma_0 : W^{1,1}(\Omega^*) \to L^1(\Gamma)$ denote the trace operator. For convenience, let us define the following bilinear forms $a_B : H^1(\Omega^*) \times H^1(\Omega^*) \to \mathbb{R}$, $a_S : H^1(\Gamma) \times H^1(\Gamma) \to \mathbb{R}$, $l_B : L^2(\Omega^*) \times L^2(\Omega^*) \to \mathbb{R}$, and $l_S : L^2(\Gamma) \times L^2(\Gamma) \to \mathbb{R}$:

$$a_B(\varphi, \psi) := \int_{\Omega^*} \mathcal{A}\nabla\varphi \cdot \nabla\psi + a\varphi\psi \,\mathrm{dx}, \qquad l_B(\varphi, \psi) := \int_{\Omega^*} \varphi\psi \,\mathrm{dx}, \tag{2.1}$$

$$a_S(\varphi, \psi) := \int_\Gamma \mathcal{B}\nabla_\Gamma\varphi \cdot \nabla_\Gamma\psi + b\varphi\psi \,\mathrm{d}\mathcal{H}^{n-1}, \quad l_S(\varphi, \psi) := \int_\Gamma \varphi\psi \,\mathrm{d}\mathcal{H}^{n-1}. \tag{2.2}$$

THEOREM 2.1. *Let Assumptions 2.1 and 2.2 be satisfied. Then there exists a unique weak solution of* (CSI),

$$(u, v) \in H^1(\Omega^*) \times H^1(\Gamma),$$

such that for all $\varphi \in H^1(\Omega^*)$, $\psi \in H^1(\Gamma)$,

$$a_B(u,\varphi) + a_S(v,\psi) + Kl_S(v-\gamma_0(u),\psi-\gamma_0(\varphi)) = l_B(f,\varphi) + \beta l_S(g,\psi).$$

Moreover, there exists a constant C, *independent of* (u,v), *such that*

$$\|u\|^2_{H^1(\Omega^*)} + \|v\|^2_{H^1(\Gamma)} \le C(\|f\|^2_{L^2(\Omega^*)} + \|g\|^2_{L^2(\Gamma)}).$$

2.3. Assumptions on data extensions. In general, the extension operator is not unique, and so we make the following assumption.

Assumption 2.3 (extension of bulk data). *Let* $1 \le i,j \le n$. *For* $a_{ij}, a \in L^\infty(\Omega^*)$, *and* $f \in L^2(\Omega^*)$, *we assume that there exist extensions* $a_{ij}^{Ea}, a^{Ea} \in L^\infty(\Omega)$, *and* $f^{Ea} \in L^2(\Omega)$ *such that* $\mathcal{A}^{Ea} = (a_{ij}^{Ea})_{1\le i,j\le n}$ *is uniformly elliptic with constant* θ_0 *and* $a^{Ea}(x) \ge \theta_2$ *for a.e.* $x \in \Omega$.

The surface data will be extended in a specific way. Recall that by Assumption 2.1, Γ is C^3. We define the tubular neighborhood $\mathrm{Tub}^r(\Gamma)$ of Γ with width $r > 0$ as

$$\mathrm{Tub}^r(\Gamma) := \{x \in \Omega : |d(x)| < r\}.$$

Then, by [22, Lemma 14.16], there exists $\eta > 0$ such that the signed distance function d to Γ is of class $C^3(\mathrm{Tub}^\eta(\Gamma))$. Moreover, it can be shown that d is globally Lipschitz with constant 1 (see [22, Section 14.6]).

For each $y \in \Gamma$, let $T_y\Gamma$ and $\nu(y)$ denote its tangent space and outward pointing unit normal, respectively. A standard result in differential geometry shows that for η sufficiently small, there is a C^3 diffeomorphism between $\mathrm{Tub}^\eta(\Gamma)$ and $\Gamma \times (-\eta,\eta)$ given by

$$\Theta^\eta : \mathrm{Tub}^\eta(\Gamma) \to \Gamma \times (-\eta,\eta), \quad \Theta^\eta(x) = (p(x), d(x)), \tag{2.3}$$

where, for any $x \in \mathrm{Tub}^\eta(\Gamma)$, we define the closest point operator (see [30] or [13, Lemma 2.8]) $p : \mathrm{Tub}^\eta(\Gamma) \to \Gamma$ by

$$p(x) := x - d(x)\nu(p(x)). \tag{2.4}$$

Then, we also have

$$\nabla d(x) = \nu(p(x)) \text{ for } x \in \mathrm{Tub}^\eta(\Gamma). \tag{2.5}$$

Definition 2.1 (constant extension in the normal direction). *For any* $\psi \in L^q(\Gamma)$, $1 \le q \le \infty$, *we define its constant extension* ψ^e *off* Γ *to* $\mathrm{Tub}^\eta(\Gamma)$ *in the normal direction as*

$$\psi^e(x) = \psi(p(x)) \text{ for all } x \in \mathrm{Tub}^\eta(\Gamma). \tag{2.6}$$

By Corollary 3.1 below, we have that $\psi^e \in L^q(\mathrm{Tub}^\eta(\Gamma))$ if $\psi \in L^q(\Gamma)$ for $1 \le q \le \infty$, and ψ^e also can be extended to a function $\psi^{Ec} \in L^q(\Omega)$. This motivates the following assumption for the surface data.

Assumption 2.4 (extension of surface data). *Let* $1 \le i,j \le n$. *For* $b_{ij}, b \in L^\infty(\Gamma)$ *and* $g \in L^2(\Gamma)$, *let* $b_{ij}^e, b^e \in L^\infty(\mathrm{Tub}^\eta(\Gamma))$ *and* $g^e \in L^2(\mathrm{Tub}^\eta(\Gamma))$ *denote the constant extensions of* b_{ij}, b *and* g *off* Γ *to* $\mathrm{Tub}^\eta(\Gamma)$ *in the normal direction, respectively. We assume that there exist extensions* $b_{ij}^{Ec}, b^{Ec} \in L^\infty(\Omega)$, *and* $g^{Ec} \in L^2(\Omega)$ *of* b_{ij}^e, b^e *and* g^e, *respectively, such that* $\mathcal{B}^{Ec} = (b_{ij}^{Ec})_{1\le i,j\le n}$ *is uniformly elliptic with constant* θ_1 *and* $b^{Ec}(x) \ge \theta_3$ *for a.e.* $x \in \Omega$.

2.4. Assumptions on regularizations. We first introduce the functions ξ and δ, from which the regularizations ξ_ε and δ_ε are constructed by a rescaling.

ASSUMPTION 2.5. *We assume that $\xi : \mathbb{R} \to [0,1]$ is a continuous, monotone function such that*

$$0 \le \xi(t) \le \xi(0) = \frac{1}{2} \le \xi(s) \le 1 \text{ for all } s \le 0 \le t, \text{ and } \lim_{\varepsilon\to 0} \xi\left(\frac{x}{\varepsilon}\right) = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } x > 0, \\ \frac{1}{2} & \text{if } x = 0. \end{cases} \tag{2.7}$$

ASSUMPTION 2.6. *We assume that $\delta : \mathbb{R} \to \mathbb{R}_{\ge 0}$ is a C^1, nonnegative, even function such that*

$$\int_{\mathbb{R}} \delta(s)\,\mathrm{ds} = 1, \quad \delta(s_1) \ge \delta(s_2) \text{ if } |s_1| \le |s_2|, \tag{2.8}$$

$$\int_{\{s\in\mathbb{R}:\delta(s)>0\}} \frac{|\delta'(s)|^2}{\delta(s)}\,\mathrm{ds} + \int_{\mathbb{R}} \sqrt{\delta(s)} + \delta(s)(|s| + |s|^2)\,\mathrm{ds} =: C_{\delta,\mathrm{int}} < \infty, \tag{2.9}$$

and for any $q \ge 1$,

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^q}\delta\left(\frac{x}{\varepsilon}\right) = \begin{cases} 0 & \text{if } x \ne 0, \\ +\infty & \text{if } x = 0. \end{cases} \tag{2.10}$$

Moreover, we assume there exists a constant $C_\xi > 0$ such that

$$C_\xi \delta(t) \le \xi(t) \text{ for all } t \in \mathbb{R}. \tag{2.11}$$

DEFINITION 2.2. *Let $d(x)$ denote the signed distance function to Γ. For $x \in \Omega$ and for each $\varepsilon \in (0,1]$, we define*

$$\xi_\varepsilon(x) := \xi\left(\frac{d(x)}{\varepsilon}\right), \quad \delta_\varepsilon(x) := \frac{1}{\varepsilon}\delta\left(\frac{d(x)}{\varepsilon}\right) \tag{2.12}$$

with

$$\Omega_\varepsilon := \{x \in \Omega : \xi_\varepsilon(x) > 0\}, \quad \Gamma_\varepsilon := \{x \in \Omega : \delta_\varepsilon(x) > 0\}. \tag{2.13}$$

By (2.7), (2.10), and (2.11), we observe that

$$\Omega^* \cup \Gamma \subset \Omega_\varepsilon, \quad \Gamma \subset \Gamma_\varepsilon \subset \Omega_\varepsilon \text{ for all } \varepsilon > 0.$$

One can check that Assumptions 2.5 and 2.6 are satisfied by our candidate regularizations originating from the double-well potential,

$$\xi^{DW}(x) := \frac{1}{2}\left(1 - \tanh\left(\frac{x}{\sqrt{2}}\right)\right), \quad \delta^{DW}(x) := \frac{3}{2\sqrt{2}}\mathrm{sech}^4\left(\frac{x}{\sqrt{2}}\right),$$

and from the double-obstacle potential:

$$\begin{aligned} \xi^{DO}(x) &:= \chi_{(-\infty,-\frac{\pi}{2})}(x) + \frac{1}{2}(1 - \sin(x))\chi_{[-\frac{\pi}{2},\frac{\pi}{2}]}(x), \\ \delta^{DO}(x) &:= \frac{2}{\pi}\cos^2(x)\chi_{[-\frac{\pi}{2},\frac{\pi}{2}]}(x). \end{aligned} \tag{2.14}$$

REMARK 2.1. *For any* $\varepsilon > 0$, ξ_ε^{DW} *and* δ_ε^{DW} *derived from the double-well potential are nondegenerate in* Ω, *i.e.,* $\Omega_\varepsilon = \Gamma_\varepsilon = \Omega$ *for all* $\varepsilon > 0$. *However,* ξ_ε^{DO} *and* δ_ε^{DO} *originating from the double-obstacle potential are degenerate in* Ω. *In particular, for* (2.14),

$$\Omega_\varepsilon = \Omega^* \cup \mathrm{Tub}^{\varepsilon\frac{\pi}{2}}(\Gamma), \quad \Gamma_\varepsilon = \mathrm{Tub}^{\varepsilon\frac{\pi}{2}}(\Gamma).$$

Moreover, if $\varepsilon_1 < \varepsilon_2$, *we have* $\Omega_{\varepsilon_1} \subset \Omega_{\varepsilon_2}$ *and* $\Gamma_{\varepsilon_1} \subset \Gamma_{\varepsilon_2}$. *However, the framework of weighted Sobolev spaces is flexible enough to allow us to deduce well-posedness of the diffuse domain approximations with both the double-well and double-obstacle regularizations.*

2.5. Weighted Sobolev spaces. Due to the presence of ξ_ε and δ_ε in the diffuse domain approximations, the natural function spaces to look for well-posedness are Sobolev spaces weighted by ξ_ε and δ_ε. In the following, measurability and almost everywhere are with respect to the Lebesgue measure.

DEFINITION 2.3. *For fixed* $\varepsilon > 0$, *we define*

$$L^2(\Omega_\varepsilon, \xi_\varepsilon) := \left\{ f : \Omega_\varepsilon \to \mathbb{R} \text{ measurable s.t. } \int_{\Omega_\varepsilon} \xi_\varepsilon |f|^2 \,\mathrm{dx} < \infty \right\},$$
$$L^2(\Gamma_\varepsilon, \delta_\varepsilon) := \left\{ f : \Gamma_\varepsilon \to \mathbb{R} \text{ measurable s.t. } \int_{\Gamma_\varepsilon} \delta_\varepsilon |f|^2 \,\mathrm{dx} < \infty \right\}.$$

By Lipschitz continuity of the signed distance function d, and the continuity of $\xi(\cdot)$, we see that $\xi_\varepsilon(\cdot)$ is continuous, and consequently $\frac{1}{\xi_\varepsilon}$ is bounded in all compact sets $B \subset \Omega_\varepsilon$. Thus, $\frac{1}{\xi_\varepsilon} \in L^1_{loc}(\Omega_\varepsilon)$ and by Hölder's inequality we have the continuous embedding $L^2(\Omega_\varepsilon, \xi_\varepsilon) \subset L^1_{loc}(\Omega_\varepsilon)$ (see also [24, Theorem 1.5]). Thus, we can define derivatives for $f \in L^2(\Omega_\varepsilon, \xi_\varepsilon)$ in a distributional sense. That is, for any multi-index α, we call a function g the αth distributional derivative of f, and write $g = D^\alpha f$, if for every $\phi \in C_c^\infty(\Omega_\varepsilon)$,

$$\int_{\Omega_\varepsilon} f D^\alpha \phi \,\mathrm{dx} = (-1)^{|\alpha|} \int_{\Omega_\varepsilon} g\phi \,\mathrm{dx}.$$

We define the vector space

$$W^{1,2}(\Omega_\varepsilon, \xi_\varepsilon) := \{ f \in L^2(\Omega_\varepsilon, \xi_\varepsilon) : D^\alpha f \in L^2(\Omega_\varepsilon, \xi_\varepsilon) \text{ for } |\alpha| = 1 \}.$$

A similar definition for the vector space $W^{1,2}(\Gamma_\varepsilon, \delta_\varepsilon)$ can be made since $\frac{1}{\delta_\varepsilon} \in L^1_{loc}(\Gamma_\varepsilon)$ by the continuity of $\delta_\varepsilon(\cdot)$.

For the subsequent analysis, we will present the proofs with the double-well regularization in mind and detail any necessary modifications for the double-obstacle regularization afterward.

To streamline the presentation, it is more convenient to have a fixed domain when working with weighted Sobolev spaces, hence we introduce the following notation.

DEFINITION 2.4. *For fixed* $\varepsilon > 0$, *we define*

$$L^2(\Omega, \xi_\varepsilon) := \{ f : \Omega \to \mathbb{R} \text{ measurable s.t. } f|_{\Omega_\varepsilon} \in L^2(\Omega_\varepsilon, \xi_\varepsilon) \},$$
$$H^1(\Omega, \xi_\varepsilon) := \{ f : \Omega \to \mathbb{R} \text{ measurable s.t. } f|_{\Omega_\varepsilon} \in W^{1,2}(\Omega_\varepsilon, \xi_\varepsilon) \},$$

with inner product and induced norms:

$$\langle f,g\rangle_{L^2(\Omega,\xi_\varepsilon)} := \int_\Omega \xi_\varepsilon f g \,\mathrm{dx} = \int_{\Omega_\varepsilon} \xi_\varepsilon f g \,\mathrm{dx}, \quad \|f\|^2_{0,\xi_\varepsilon} := \langle f,f\rangle_{L^2(\Omega,\xi_\varepsilon)},$$
$$\langle f,g\rangle_{H^1(\Omega,\xi_\varepsilon)} := \int_\Omega \xi_\varepsilon (fg + \nabla f\cdot\nabla g)\,\mathrm{dx}, \qquad \|f\|^2_{1,\xi_\varepsilon} := \langle f,f\rangle_{H^1(\Omega,\xi_\varepsilon)}.$$

Here, we use the identification

$$f = g \Leftrightarrow f(x) = g(x) \text{ for a.e. } x\in\Omega_\varepsilon.$$

Similar notation and identification are used for $L^2(\Omega,\delta_\varepsilon)$ *and* $H^1(\Omega,\delta_\varepsilon)$. *Furthermore, we define*

$$\mathcal{V}_\varepsilon := \{f:\Omega\to\mathbb{R} \text{ measurable s.t. } f|_{\Omega_\varepsilon}\in W^{1,2}(\Omega_\varepsilon,\xi_\varepsilon)\} \text{ with}$$
$$\langle f,g\rangle_{\mathcal{V}_\varepsilon} := \int_\Omega \xi_\varepsilon(fg+\nabla f\cdot\nabla g) + \delta_\varepsilon f g\,\mathrm{dx}.$$

2.6. Well-posedness of the DD problem. Similar to the above, we introduce the following bilinear forms $a^\varepsilon_B : H^1(\Omega,\xi_\varepsilon)\times H^1(\Omega,\xi_\varepsilon)\to\mathbb{R}$, $a^\varepsilon_S : H^1(\Omega,\delta_\varepsilon)\times H^1(\Omega,\delta_\varepsilon)\to\mathbb{R}$, $l^\varepsilon_B : L^2(\Omega,\xi_\varepsilon)\times L^2(\Omega,\xi_\varepsilon)\to\mathbb{R}$, and $l^\varepsilon_S : L^2(\Omega,\delta_\varepsilon)\times L^2(\Omega,\delta_\varepsilon)\to\mathbb{R}$:

$$a^\varepsilon_B(\varphi,\psi) := \int_\Omega \xi_\varepsilon(\mathcal{A}^{Ea}\nabla\varphi\cdot\nabla\psi + a^{Ea}\varphi\psi)\,\mathrm{dx}, \quad l^\varepsilon_B(\varphi,\psi) := \int_\Omega \xi_\varepsilon\varphi\psi\,\mathrm{dx}, \tag{2.15}$$
$$a^\varepsilon_S(\varphi,\psi) := \int_\Omega \delta_\varepsilon(\mathcal{B}^{Ec}\nabla\varphi\cdot\nabla\psi + b^{Ec}\varphi\psi)\,\mathrm{dx}, \quad l^\varepsilon_S(\varphi,\psi) := \int_\Omega \delta_\varepsilon\varphi\psi\,\mathrm{dx}. \tag{2.16}$$

Our first main result is the well-posedness of the diffuse domain approximation.

Theorem 2.2. *Let* $\Omega\subset\mathbb{R}^n$ *be an open bounded domain with Lipschitz boundary. Suppose Assumptions 2.3, 2.4, 2.5, and 2.6 are satisfied. In addition, we assume that* $\theta_3\geq K$. *Then, for each* $\varepsilon>0$, *there exists a unique weak solution*

$$(u^\varepsilon,v^\varepsilon)\in\mathcal{V}_\varepsilon\times H^1(\Omega,\delta_\varepsilon)$$

such that for all $\phi\in\mathcal{V}_\varepsilon$, $\psi\in H^1(\Omega,\delta_\varepsilon)$,

$$a^\varepsilon_B(u^\varepsilon,\phi) + a^\varepsilon_S(v^\varepsilon,\psi) + K l^\varepsilon_S(v^\varepsilon-u^\varepsilon,\psi-\phi) = l^\varepsilon_B(f^{Ea},\phi) + \beta l^\varepsilon_S(g^{Ec},\psi). \tag{2.17}$$

Moreover, the weak solutions satisfy

$$\|u^\varepsilon\|^2_{1,\xi_\varepsilon} + \|u^\varepsilon\|^2_{0,\delta_\varepsilon} + \|v^\varepsilon\|^2_{1,\delta_\varepsilon} \leq C(\|f^{Ea}\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(\Gamma)}), \tag{2.18}$$

where the constant C *is independent of* ε.

We point out that, thanks to the fact that g^{Ec} is the constant extension of g in the normal direction, the estimate in (2.18) is independent of ε.

2.7. Compactness results. Our second main result is compactness in the weighted Sobolev spaces $\mathcal{V}_\varepsilon$ and $H^1(\Omega,\delta_\varepsilon)$:

Theorem 2.3 (compactness in $\mathcal{V}_\varepsilon$ and $H^1(\Omega,\delta_\varepsilon)$). *Suppose that Assumptions 2.1, 2.4, 2.5, and 2.6 are satisfied. Let* $\{u^\varepsilon\}_{\varepsilon\in(0,1]}\subset\mathcal{V}_\varepsilon$ *and* $\{v^\varepsilon\}_{\varepsilon\in(0,1]}\subset H^1(\Omega,\delta_\varepsilon)$ *denote two bounded sequences, i.e., there exist a constant* $C>0$, *independent of* ε, *such that*

$$\|u^\varepsilon\|^2_{1,\xi_\varepsilon} + \|u^\varepsilon\|^2_{0,\delta_\varepsilon} \leq C, \quad \|v^\varepsilon\|^2_{1,\delta_\varepsilon}\leq C.$$

For $1 \le i,j \le n$, let $g^{Ec} \in L^2(\Omega)$, $b^{Ec} \in L^\infty(\Omega)$, and $b_{ij}^{Ec} \in L^\infty(\Omega)$ denote the extensions of the data $g \in L^2(\Gamma)$, $b \in L^\infty(\Gamma)$, and $b_{ij} \in L^\infty(\Gamma)$ as mentioned in Assumption 2.4.

Then, there exist $\tilde{u} \in H^1(\Omega^*)$, $\overline{v} \in H^1(\Gamma)$ such that as $\varepsilon \to 0$, $u^\varepsilon|_{\Omega^*}$ converges weakly to $\tilde{u}$ in $H^1(\Omega^*)$, along subsequences, and for any $\varphi \in H^1(\Omega)$ and $\psi \in H^1(\Gamma)$ with $\psi^{Ec} \in H^1(\Omega)$ as constructed in Corollary 3.1 below,

$$\int_\Omega \delta_\varepsilon u^\varepsilon \varphi \,\mathrm{dx} \to \int_\Gamma \gamma_0(\tilde{u})\gamma_0(\varphi)\,\mathrm{d}\mathcal{H}^{n-1}, \tag{2.19}$$

$$\int_\Omega \delta_\varepsilon v^\varepsilon g^{Ec} \,\mathrm{dx} \to \int_\Gamma \overline{v} g \,\mathrm{d}\mathcal{H}^{n-1}, \tag{2.20}$$

$$\int_\Omega \delta_\varepsilon b^{Ec} v^\varepsilon \varphi \,\mathrm{dx} \to \int_\Gamma b\overline{v}\gamma_0(\varphi)\,\mathrm{d}\mathcal{H}^{n-1}, \tag{2.21}$$

$$\int_\Omega \delta_\varepsilon \mathcal{B}^{Ec} \nabla v^\varepsilon \cdot \nabla \psi^{Ec} \,\mathrm{dx} \to \int_\Gamma \mathcal{B}\nabla_\Gamma \overline{v} \cdot \nabla_\Gamma \psi \,\mathrm{d}\mathcal{H}^{n-1}. \tag{2.22}$$

2.8. Convergence results. Recall that the unique weak solution to (CSI) is denoted by (u,v). Keeping the notation that the unique weak solutions to (CDD) are denoted by $(u^\varepsilon, v^\varepsilon)$, our third main result is the weak convergence of diffuse domain approximations:

THEOREM 2.4 (weak convergence of diffuse domain approximation). *Suppose Assumptions 2.1, 2.2, 2.3, 2.4, 2.5, and 2.6 are satisfied. In addition, we assume $\theta_3 \ge K$. Then, as $\varepsilon \to 0$,*

$$u^\varepsilon|_{\Omega^*} \text{ converge weakly to } u \text{ in } H^1(\Omega^*),$$

while, for any $\psi \in H^1(\Gamma)$ with extension $\psi^{Ec} \in H^1(\Omega)$ (as constructed in Corollary 3.1 below), as $\varepsilon \to 0$,

$$\int_\Omega \delta_\varepsilon v^\varepsilon \psi^{Ec} \,\mathrm{dx} \to \int_\Gamma v\psi \,\mathrm{d}\mathcal{H}^{n-1}, \quad \int_\Omega \delta_\varepsilon \nabla v^\varepsilon \cdot \nabla \psi^{Ec} \,\mathrm{dx} \to \int_\Gamma \nabla_\Gamma v \cdot \nabla_\Gamma \psi \,\mathrm{d}\mathcal{H}^{n-1}.$$

Here, we point out that the assertion of Theorem 2.4 for the surface quantities differs from the corresponding assertion of Theorem 2.3 in the way that the limit function $\overline{v}$ is identified as the surface part of the weak solution to (CSI).

Thanks to the coercivity of the bilinear forms a_B^ε and a_S^ε, we also obtain strong convergence with respect to the norms weighted with ξ_ε and δ_ε. For this purpose we have to extend the solution (u,v) to (CSI) to the larger domain Ω. For the bulk field u, we employ the reflection method of [16, Theorem 1, p. 254], and for the surface field v, the extension constructed in Corollary 3.1 yields a natural field to compare with.

THEOREM 2.5 (strong convergence of diffuse domain approximation). *Suppose Assumptions 2.1, 2.2, 2.3, 2.4, 2.5, and 2.6 are satisfied. In addition, we assume $\theta_3 \ge K$.*

Let $v^{Ec} \in H^1(\Omega)$ denote the constant extension of $v \in H^1(\Gamma)$ in the normal direction, as constructed in Corollary 3.1. Let $u^{Er} \in H^1(\Omega)$ denote the extension of $u \in H^1(\Omega^*)$ by the method of reflection. Then, as $\varepsilon \to 0$,

$$\|u^\varepsilon - u^{Er}\|_{1,\xi_\varepsilon}^2 + \|u^\varepsilon - u^{Er}\|_{0,\delta_\varepsilon}^2 + \|v^\varepsilon - v^{Ec}\|_{1,\delta_\varepsilon}^2 \to 0.$$

Consequently, we have the strong convergence,

$$\|u^\varepsilon|_{\Omega^*} - u\|_{H^1(\Omega^*)} \to 0 \text{ as } \varepsilon \to 0,$$

and the norm convergence:

$$\|u^\varepsilon\|_{0,\delta_\varepsilon} \to \|\gamma_0(u)\|_{L^2(\Gamma)}, \quad \|v^\varepsilon\|_{1,\delta_\varepsilon} \to \|v\|_{H^1(\Gamma)} \text{ as } \varepsilon \to 0.$$

In the next section, we present some technical results that will be essential to prove the main results.

3. Technical results. The results contained in this section can be roughly divided into three parts. The first part concerns the change of coordinates in a tubular neighborhood. In particular, Lemma 3.1 below will allow us to analyze integrals over a tubular neighborhood, while Lemma 3.2 below will allow us to decompose the Euclidean gradient into a normal part and a tangential part. This in turn allows us to extract a surface gradient from ∇v^ε in the limit $\varepsilon \to 0$ for the proof of Theorem 2.3.

The second part details some properties of surface functions that are extended constantly in the normal direction. The main aim is to show that there exists an extension of a function in $H^1(\Gamma)$ that is a valid test function for the weak formulation of (CDD) (see Lemma 3.3 below). The third part shows how integrals involving the regularizations ξ_ε and δ_ε behave as $\varepsilon \to 0$, which plays a crucial role in the proof of Theorems 2.3, 2.4, and 2.5.

3.1. Change of variables in the tubular neigborhood. Choose $\eta > 0$ and the diffeomorphism $\Theta^\eta : \mathrm{Tub}^\eta(\Gamma) \to \Gamma \times (-\eta, \eta)$ defined in (2.3). For $t \in (-\eta, \eta)$, let Γ_t denote the level set $\{x \in \Omega : d(x) = t\}$. Then, by the co-area formula [16, Theorem 5, p. 629] or [17, Theorem 2, p. 117], and that $|\nabla d| = |\nu| = 1$, we can write

$$\int_{\mathrm{Tub}^\eta(\Gamma)} f(x)\,|\nabla d(x)|\,\mathrm{dx} = \int_{\mathrm{Tub}^\eta(\Gamma)} f(x)\,\mathrm{dx} = \int_{-\eta}^{\eta} \int_{\Gamma_t} f\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{dt}\,. \tag{3.1}$$

We define the mapping $\rho_t : \Gamma \to \Gamma_t$ by

$$\rho_t(p) = p + t\nu(p) \text{ for } p \in \Gamma. \tag{3.2}$$

This map is well-defined and is injective due to the diffeomorphism Θ^η. Then, by a change of variables, we obtain

$$\int_{\Gamma_t} f\,\mathrm{d}\mathcal{H}^{n-1} = \int_\Gamma f(p + t\nu(p)) A(p,t)\,\mathrm{d}\mathcal{H}^{n-1}\,,$$

where $A(p,t)$ depends on the differential of ρ_t. To identify $A(p,t)$ as a function of t we use local coordinates.

Since Γ is a compact hypersurface, we can always find a finite open cover of Γ consisting of open sets $W_i \subset \mathbb{R}^n$, $1 \le i \le N$, such that $\Gamma \subset \bigcup_{i=1}^N W_i$. For each $1 \le i \le N$, let $\alpha_i(s)$ denote a regular parameterization of $W_i \cap \Gamma$ with parameter domain $\mathcal{S}_i \subset \mathbb{R}^{n-1}$, i.e., $\alpha_i : \mathcal{S}_i \to W_i \cap \Gamma$ is a local regular parameterization of Γ (the existence of such local regular parameterizations follows from the regularity of Γ). By the injectivity of ρ_t, Γ_t is also a compact hypersurface with a finite open cover $\{\rho_t(W_i \cap \Gamma)\}_{i=1}^N$. In addition, $\rho_t \circ \alpha_i$ is a local parameterization of Γ_t. Let

$$\begin{aligned} J_{i,0}(s) &:= (\partial_{s_1}\alpha_i(s), \dots, \partial_{s_{n-1}}\alpha_i(s), \nu(\alpha_i(s))) \in \mathbb{R}^{n\times n}, \\ B_i(s) &:= (\partial_{s_1}\nu(\alpha_i(s)), \dots, \partial_{s_{n-1}}\nu(\alpha_i(s)), \mathbf{0}) \in \mathbb{R}^{n\times n}, \\ J_{i,\eta}(s,t) &:= J_{i,0}(s) + tB_i(s). \end{aligned}$$

A short calculation shows that

$$\begin{aligned}\det J_{i,\eta} &= \det(J_{i,0} + tB_i) = (\det J_{i,0})(\det(I + tJ_{i,0}^{-1}B_i)) = (\det J_{i,0})t^n \det\left(t^{-1}I + J_{i,0}^{-1}B_i\right)\\ &= (\det J_{i,0})t^n\left(\frac{1}{t^n} - \frac{1}{t^{n-1}}\mathrm{tr}\left(-J_{i,0}^{-1}B_i\right) + \cdots + (-1)^n \det(-J_{i,0}^{-1}B_i)\right)\\ &= (\det J_{i,0})\left(1 + \mathrm{tr}\left(tJ_{i,0}^{-1}B_i\right) + \cdots + (-1)^{2n}\det(tJ_{i,0}^{-1}B_i)\right),\end{aligned}$$

where $\mathrm{tr}\,(A)$ denotes the trace of a matrix A, and we used the well-known fact that the coefficients of the monic characteristic polynomial

$$\det(xI - A) = p_A(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

are given by

$$a_{n-k} = (-1)^k \sum_{1\le j_1<\cdots<j_k\le n} \lambda_{j_1}\lambda_{j_2}\ldots\lambda_{j_k}, \quad k = 1,\ldots,n-1, \quad a_0 = (-1)^n\det(A),$$

where $\{\lambda_j\}_{j=1}^n$ are the eigenvalues of A (see [6]). We define

$$Z_i(t,s) := t^n p_{-J_{i,0}^{-1}B_i}(1/t) - 1$$

so that

$$\det J_{i,\eta}(s,t) = (\det J_{i,0}(s))(1 + Z_i(t,s)).$$

Since α_i is a regular parameterization, the tangent vectors $\{\partial_{s_j}\alpha_i\}_{1\le j\le n-1}$ are linearly independent and hence $\det J_{i,0} \ne 0$. Then, for any $f \in L^1_{loc}(\Gamma)$,

$$\int_{W_i\cap\Gamma} f\,\mathrm{d}\mathcal{H}^{n-1} = \int_{\mathcal{S}_i} f(\alpha_i(s))\,|\det J_{i,0}|\,\mathrm{d}s\,, \tag{3.3}$$

and for any $f \in L^1_{loc}(\Gamma_t)$,

$$\begin{aligned}\int_{\rho_t(W_i\cap\Gamma)} f\,\mathrm{d}\mathcal{H}^{n-1} &= \int_{\mathcal{S}_i} f(\alpha_i(s) + t\nu(\alpha_i(s)))\,|\det J_{i,\eta}(s,t)|\,\mathrm{d}s\\ &= \int_{\mathcal{S}_i} f(\alpha_i(s) + t\nu(\alpha_i(s)))\,|\det J_{i,0}(s)|\,|1 + Z_i(t,s)|\,\mathrm{d}s\,.\end{aligned} \tag{3.4}$$

By Assumption 2.1, Γ is a compact C^3 hypersurface, and so the eigenvalues of $J_{i,0}(s)$ and $J_{i,0}^{-1}B_i(s)$ are bounded uniformly in $s \in \mathcal{S}_i$. Hence, for each $1 \le i \le N$, there exists a constant c_i, depending on n, η, and the eigenvalues, such that

$$|Z_i(t,s)| = \left|t^n p_{-J_{i,0}^{-1}B_i}(1/t) - 1\right| \le C(\lambda_{j_k})\,|t|\,(1 + \eta + \cdots + \eta^{n-1}) \le c_i\,|t|\,.$$

Moreover, by the compactness of Γ, there are only a finite number of c_i, and so we can deduce that there exists a constant $\tilde{c}$ such that, for all $t \in (-\eta,\eta), s \in \mathcal{S}_i, 1 \le i \le N$,

$$\max_{1\le i\le N} |Z_i(t,s)| \le \tilde{c}\,|t|\,.$$

We note that $\tilde{c}$ can be chosen independently of η. For instance, let η_0 denote the maximal value such that Θ^{η_0} is a diffeomorphism. Then, for any $0 < \eta < \eta_0$, we have $c_i(\eta) \le c_i(\eta_0)$. We choose $\tilde{c} > \max_{1\le i\le N} c_i(\eta_0)$; then the above holds true for all $\eta < \eta_0$.

Let $\{\mu_i\}_{i=1}^N$ be a partition of unity subordinate to the covering $\{W_i \cap \Gamma\}_{i=1}^N$ of Γ. Consequently, by the diffeomorphism Θ^η, we observe that $\{\mu_i \circ \rho_t^{-1}\}_{i=1}^N$ is a partition of unity subordinate to the covering $\{\rho_i(W_i \cap \Gamma)\}_{i=1}^N$ of Γ_t for $t \in (-\eta, \eta)$. We define

$$Z(t,p) := \sum_{i=1}^N \mu_i(p) Z_i(t, \alpha_i^{-1}(p)) \text{ for } p \in \Gamma.$$

From the above discussion, we see that $Z(t,p)$ is uniformly bounded in $p \in \Gamma$ and

$$|Z(t,p)| \le \tilde{c}|t| \text{ for all } |t| < \eta.$$

Then, from (3.3), for any $f \in L^1(\Gamma)$ we have

$$\int_\Gamma f \, \mathrm{d}\mathcal{H}^{n-1} = \sum_{i=1}^N \int_{W_i \cap \Gamma} \mu_i f \, \mathrm{d}\mathcal{H}^{n-1} = \sum_{i=1}^N \int_{\mathcal{S}_i} (\mu_i f)(\alpha_i(s)) \, |\det J_{i,0}(s)| \, \mathrm{ds},$$

and similarly from (3.4), for any $f \in L^1(\mathrm{Tub}^\eta(\Gamma))$,

$$\begin{aligned}
\int_{\mathrm{Tub}^\eta(\Gamma)} f(x) \, \mathrm{dx} &= \int_{-\eta}^\eta \int_{\Gamma_t} f \, \mathrm{d}\mathcal{H}^{n-1} \, \mathrm{dt} = \int_{-\eta}^\eta \sum_{i=1}^N \int_{\rho_t(W_i \cap \Gamma)} (\mu_i \circ \rho_t^{-1}) f \, \mathrm{d}\mathcal{H}^{n-1} \, \mathrm{dt} \\
&= \int_{-\eta}^\eta \sum_{i=1}^N \int_{\mathcal{S}_i} \mu_i(\alpha_i(s)) f(\alpha_i(s) + t\nu(\alpha_i(s))) \, |\det J_{i,0}(s)| \, |1 + Z_i(t,s)| \, \mathrm{ds} \, \mathrm{dt} \\
&= \int_{-\eta}^\eta \sum_{i=1}^N \int_{W_i \cap \Gamma} \mu_i(p) f(p + t\nu(p)) \, |1 + Z(t,p)| \, \mathrm{d}\mathcal{H}^{n-1}(p) \, \mathrm{dt} \\
&= \int_{-\eta}^\eta \int_\Gamma f(p + t\nu(p)) \, |1 + Z(t,p)| \, \mathrm{d}\mathcal{H}^{n-1}(p) \, \mathrm{dt}.
\end{aligned}$$

Hence, we can identify

$$A(p,t) = |1 + Z(t,p)|.$$

We summarize our findings of this section in the following.

Lemma 3.1. *Suppose Assumption 2.1 is satisfied. There exists $\tilde{c} > 0$ such that for any $\eta < (\tilde{c})^{-1}$, and any $f \in L^1(\mathrm{Tub}^\eta(\Gamma))$, we have*

$$\int_{\mathrm{Tub}^\eta(\Gamma)} f(x) \, \mathrm{dx} = \int_{-\eta}^\eta \int_\Gamma f(p + t\nu(p)) \, |1 + Z(t,p)| \, \mathrm{d}\mathcal{H}^{n-1} \, \mathrm{dt}, \tag{3.5}$$

where

$$|Z(t,p)| \le \tilde{c}|t| \text{ for all } |t| < \eta. \tag{3.6}$$

Consequently,

$$\begin{aligned}
\frac{1}{1 + \tilde{c}\eta} \int_{\mathrm{Tub}^\eta(\Gamma)} |f(x)| \, \mathrm{dx} &\le \int_{-\eta}^\eta \int_\Gamma |f(p + t\nu(p))| \, \mathrm{d}\mathcal{H}^{n-1} \, \mathrm{dt} \\
&\le \frac{1}{1 - \tilde{c}\eta} \int_{\mathrm{Tub}^\eta(\Gamma)} |f(x)| \, \mathrm{dx}.
\end{aligned} \tag{3.7}$$

3.2. Coordinates in a scaled tubular neighborhood. In the subsequent convergence analysis, we will use a tubular neighborhood whose width scales with ε^k for some $0 < k \leq 1$, i.e., we consider $X^\varepsilon := \mathrm{Tub}^{\varepsilon^k \eta}(\Gamma)$. For this section, we take $X^\varepsilon = \mathrm{Tub}^{\varepsilon\eta}(\Gamma)$, i.e., $k = 1$, to derive some technical results.

Let $W \subset \mathbb{R}^n$ be one of the open sets in a finite open cover of Γ, with associated local regular parameterization $\alpha : \mathcal{S} \to W \cap \Gamma$ and parameter domain $\mathcal{S} \subset \mathbb{R}^{n-1}$. We define the metric tensor $\mathcal{G}_0 = (g_{0,ij})_{1\leq i,j\leq n-1}$ by

$$g_{0,ij}(p) := \partial_{s_i}\alpha(s) \cdot \partial_{s_j}\alpha(s) \text{ for } 1 \leq i,j \leq n-1 \text{ if } p = \alpha(s) \in \Gamma \cap W, \tag{3.8}$$

with inverse metric tensor $\mathcal{G}_0^{-1} = (g_0^{ij})_{1\leq i,j\leq n-1}$. Then, for a scalar function $f : \Gamma \to \mathbb{R}$, the surface gradient of f on Γ at a point $p = \alpha(s) \in \Gamma \cap W$ for some $s \in \mathcal{S}$ is defined as

$$\nabla_\Gamma f(\alpha(s)) = \sum_{j=1}^{n-1} \left(\sum_{i=1}^{n-1} g_0^{ij}(\alpha(s)) \partial_{s_i} f(\alpha(s)) \right) \partial_{s_j}\alpha(s). \tag{3.9}$$

One can show that the definition (3.9) does not depend on the parameterization. There is also an alternate and equivalent definition of the surface gradient in terms of level sets; we refer the reader to [13, §2.2] for more details.

Recall the signed distance function d defined in (1.1). We introduce the rescaled distance variable

$$z = \frac{d}{\varepsilon}.$$

For $z \in (-\eta, \eta)$, we define a parallel hypersurface at distance εz away from Γ as

$$\Gamma^{\varepsilon z} := \{p + \varepsilon z \nu(p) : p \in \Gamma\}. \tag{3.10}$$

Let $p(x)$ denote the closest point operator of $x \in X^\varepsilon$ as defined in (2.4) such that

$$x = p(x) + \varepsilon z \nu(p(x)) \text{ for some } z \in (-\eta, \eta). \tag{3.11}$$

Then, by the injectivity of the closest point operator, we have

$$\nu(y) = \nu(p(y)) \text{ for } y \in \Gamma^{\varepsilon z}. \tag{3.12}$$

For any scalar function $f : X^\varepsilon \to \mathbb{R}$, we define its representation $F_\varepsilon(p, z)$ in the (p, z) coordinate system by

$$F_\varepsilon(p, z) := f(p + \varepsilon z \nu(p)) \text{ for } p \in \Gamma, z \in (-\eta, \eta). \tag{3.13}$$

Locally, for $x \in X^\varepsilon \cap W$, we have

$$x = G_\varepsilon(s, z) := \alpha(s) + \varepsilon z \nu(\alpha(s)) \text{ for some } s \in \mathcal{S}, z \in (-\eta, \eta). \tag{3.14}$$

Then, we can define the local representation $\tilde{F}_\varepsilon(s, z)$ of f in the (s, z) coordinate system by

$$\tilde{F}_\varepsilon(s, z) := F_\varepsilon(\alpha(s), z) = f(\alpha(s) + \varepsilon z \nu(\alpha(s))) \text{ for } s \in \mathcal{S}, z \in (-\eta, \eta). \tag{3.15}$$

Let $(s_1, \ldots, s_{n-1}) \in \mathcal{S}$ and $s_n := z$. Then, by (3.14),

$$\partial_{s_i} G_\varepsilon(s, z) = \partial_{s_i}\alpha(s) + \varepsilon z \partial_{s_i}\nu(\alpha(s)) \text{ for } 1 \leq i \leq n-1, \quad \partial_{s_n} G_\varepsilon = \varepsilon \nu(\alpha(s)), \tag{3.16}$$

and $\{\partial_{s_i} G_\varepsilon\}_{i=1}^n$ is a basis of $\mathbb{R}^n$ locally around $\Gamma^{\varepsilon z}$. For $1 \le i, j \le n-1$, we define the metric tensor $\mathcal{G}_{\varepsilon z} = (g_{\varepsilon,ij})_{1\le i,j\le n}$ in these new coordinates as

$$\begin{aligned} g_{\varepsilon,ij}(p,z) &= (\partial_{s_i}\alpha(s) + \varepsilon z \partial_{s_i}\nu(\alpha(s))) \cdot (\partial_{s_j}\alpha(s) + \varepsilon z \partial_{s_j}\nu(\alpha(s))), \\ g_{\varepsilon,in}(p,z) &= g_{\varepsilon,ni}(p,z) = (\partial_{s_i}\alpha(s) + \varepsilon z \partial_{s_i}\nu(\alpha(s))) \cdot \varepsilon\nu(\alpha(s)) = 0, \\ g_{\varepsilon,nn}(p,z) &= (\varepsilon\nu(\alpha(s))) \cdot (\varepsilon\nu(\alpha(s))) = \varepsilon^2 \end{aligned} \tag{3.17}$$

for $p = \alpha(s) \in \Gamma \cap W$, and we have used that $\partial_{s_i}\nu \cdot \nu = \frac{1}{2}\partial_{s_i}|\nu|^2 = 0$.

Let $\mathcal{G}_{\varepsilon z}^{-1} = (g_\varepsilon^{ij})_{1\le i,j\le n}$ denote the inverse metric tensor. Then, for any scalar function $f : X^\varepsilon \to \mathbb{R}$, with the representation $\tilde{F}_\varepsilon(s,z)$ in the (s,z) coordinate system, we can express the surface gradient $\tilde{\nabla}_{\Gamma^{\varepsilon z}}\tilde{F}_\varepsilon(s,z)$ on the parallel hypersurface $\Gamma^{\varepsilon z}$ in local coordinates as

$$\tilde{\nabla}_{\Gamma^{\varepsilon z}}\tilde{F}_\varepsilon(s,z) := \sum_{j=1}^{n-1}\left(\sum_{i=1}^{n-1} g_\varepsilon^{ij}(\alpha(s),z)\partial_{s_i}\tilde{F}_\varepsilon(s,z)\right)\partial_{s_j}G_\varepsilon(s,z). \tag{3.18}$$

Similarly, we can define the surface gradient $\tilde{\nabla}_\Gamma \tilde{F}_\varepsilon(s,z)$ on the original hypersurface Γ as

$$\tilde{\nabla}_\Gamma \tilde{F}_\varepsilon(s,z) = \sum_{j=1}^{n-1}\left(\sum_{i=1}^{n-1} g_0^{ij}(\alpha(s))\partial_{s_i}\tilde{F}_\varepsilon(s,z)\right)\partial_{s_j}\alpha(s). \tag{3.19}$$

Using (3.13) and (3.15) to switch to the (p,z) coordinate system, we can define the surface gradient $\nabla_{\Gamma^{\varepsilon z}}F_\varepsilon(p,z)$ on the parallel hypersurface $\Gamma^{\varepsilon z}$ via the formula

$$\nabla_{\Gamma^{\varepsilon z}}F_\varepsilon(p,z) := \tilde{\nabla}_{\Gamma^{\varepsilon z}}\tilde{F}_\varepsilon(\alpha^{-1}(p),z) \text{ for } p = \alpha(s) \in \Gamma \cap W \tag{3.20}$$

and the surface gradient $\nabla_\Gamma F_\varepsilon(p,z)$ on the original hypersurface Γ as

$$\nabla_\Gamma F_\varepsilon(p,z) := \tilde{\nabla}_\Gamma \tilde{F}_\varepsilon(\alpha^{-1}(p),z) \text{ for } p = \alpha(s) \in \Gamma \cap W. \tag{3.21}$$

It is convenient to define the remainder both in local and global representations:

$$\begin{aligned} \tilde{\nabla}_{\varepsilon z}\tilde{F}_\varepsilon(s,z) &:= \tilde{\nabla}_{\Gamma^{\varepsilon z}}\tilde{F}_\varepsilon(s,z) - \tilde{\nabla}_\Gamma \tilde{F}_\varepsilon(s,z), \\ \nabla_{\varepsilon z}F_\varepsilon(p,z) &:= \nabla_{\Gamma^{\varepsilon z}}F_\varepsilon(p,z) - \nabla_\Gamma F_\varepsilon(p,z), \end{aligned} \tag{3.22}$$

with the relation

$$\nabla_{\varepsilon z}F_\varepsilon(p,z) = \tilde{\nabla}_{\varepsilon z}\tilde{F}_\varepsilon(\alpha^{-1}(p),z) \text{ for } p = \alpha(s) \in \Gamma \cap W.$$

We now state a result that decomposes the Euclidean gradient ∇f into a surface component and a component in the normal direction.

Lemma 3.2. *Suppose that Assumption 2.1 is satisfied. Let $f : X^\varepsilon \to \mathbb{R}$ be a C^1 function with representation F_ε in the (p,z) coordinate system and representation $\tilde{F}_\varepsilon$ in the (s,z) coordinate system, as defined in* (3.13) *and* (3.15)*, respectively. Then, for $p \in \Gamma$, $s \in \mathcal{S}$, $z \in (-\eta,\eta)$ such that $p = \alpha(s)$, and $x = \alpha(s) + \varepsilon z\nu(\alpha(s)) = p + \varepsilon z\nu(p) \in X^\varepsilon$,*

$$\begin{aligned} \nabla f(x) &= \frac{1}{\varepsilon}\nu(p)\partial_z F_\varepsilon(p,z) + \nabla_{\Gamma^{\varepsilon z}}F_\varepsilon(p,z) \\ &= \frac{1}{\varepsilon}\nu(\alpha(s))\partial_z \tilde{F}_\varepsilon(s,z) + \tilde{\nabla}_{\Gamma^{\varepsilon z}}\tilde{F}_\varepsilon(s,z), \end{aligned} \tag{3.23}$$

where $\nabla_{\Gamma^{\varepsilon z}}(\cdot)$ *and* $\tilde{\nabla}_{\Gamma^{\varepsilon z}}(\cdot)$ *are defined in* (3.20) *and* (3.18), *respectively.*

In addition, the remainders $\nabla_{\varepsilon z} F_\varepsilon$ *and* $\tilde{\nabla}_{\varepsilon z}\tilde{F}_\varepsilon$ *defined in* (3.22) *satisfy*

$$\nabla_{\varepsilon z} F_\varepsilon(p,z)\cdot\nu(p) = 0, \quad \tilde{\nabla}_{\varepsilon z}\tilde{F}_\varepsilon(s,z)\cdot\nu(\alpha(s)) = 0, \tag{3.24}$$

$$\nabla_{\varepsilon z} F_\varepsilon(p,z) = \mathcal{O}(\varepsilon), \quad \tilde{\nabla}_{\varepsilon z}\tilde{F}_\varepsilon(s,z) = \mathcal{O}(\varepsilon) \text{ as } \varepsilon \to 0. \tag{3.25}$$

Proof. For the first assertion, we follow the proof given in [1, Appendix]. The equivalent result in two dimensions can be found in [21, Appendix B] and in [14]. Let $f : X^\varepsilon \to \mathbb{R}$ be a C^1 function. Recall the metric tensor $\mathcal{G}_{\varepsilon z} = (g_{\varepsilon,ij})_{1\le i,j\le n}$ as defined in (3.17). If we denote the submatrix $\tilde{\mathcal{G}}_{\varepsilon z}$ and its inverse $\tilde{\mathcal{G}}_{\varepsilon z}^{-1}$ by

$$\tilde{\mathcal{G}}_{\varepsilon z} = (g_{\varepsilon,ij})_{1\le i,j\le n-1}, \quad \tilde{\mathcal{G}}_{\varepsilon z}^{-1} = (g_\varepsilon^{ij})_{1\le i,j\le n-1},$$

then, we observe that

$$\mathcal{G}_{\varepsilon z} = \begin{pmatrix} \tilde{\mathcal{G}}_{\varepsilon z} & \mathbf{0} \\ \mathbf{0}^T & \varepsilon^2 \end{pmatrix}, \quad \mathcal{G}_{\varepsilon z}^{-1} = \begin{pmatrix} \tilde{\mathcal{G}}_{\varepsilon z}^{-1} & \mathbf{0} \\ \mathbf{0}^T & \varepsilon^{-2} \end{pmatrix},$$

where $\mathbf{0} \in \mathbb{R}^{n-1}$ is the zero column vector of length $n-1$. Moreover, for $f(x) = \tilde{F}_\varepsilon(s(x), z(x))$, we have from (3.16) and (3.18),

$$\nabla f(x) = \sum_{j=1}^{n}\left(\sum_{i=1}^{n} g_\varepsilon^{ij}\partial_{s_i}\tilde{F}_\varepsilon\right)\partial_{s_j}G_\varepsilon = \frac{1}{\varepsilon^2}\partial_z\tilde{F}_\varepsilon\partial_z G_\varepsilon + \tilde{\nabla}_{\Gamma^{\varepsilon z}}\tilde{F}_\varepsilon = \frac{1}{\varepsilon}\partial_z\tilde{F}_\varepsilon\nu(\alpha(s)) + \tilde{\nabla}_{\Gamma^{\varepsilon z}}\tilde{F}_\varepsilon.$$

For the assertion regarding the remainder, let

$$\begin{aligned} \mathcal{C}_{ij}(s) &:= \partial_{s_i}\nu(\alpha(s))\cdot\partial_{s_j}\alpha(s) + \partial_{s_i}\alpha(s)\cdot\partial_{s_j}\nu(\alpha(s)), \quad & \mathcal{C} &:= (\mathcal{C}_{ij})_{1\le i,j\le n-1}, \\ \mathcal{D}_{ij}(s) &:= \partial_{s_i}\nu(\alpha(s))\cdot\partial_{s_j}\nu(\alpha(s)), & \mathcal{D} &:= (\mathcal{D}_{ij})_{1\le i,j\le n-1}, \end{aligned}$$

so that by (3.17) and (3.8),

$$\begin{aligned} g_{\varepsilon,ij}(\alpha(s),z) &= g_{0,ij}(\alpha(s)) + \varepsilon z\mathcal{C}_{ij}(s) + (\varepsilon z)^2\mathcal{D}_{ij}(s), \\ \tilde{\mathcal{G}}_{\varepsilon z} &= \mathcal{G}_0 + \varepsilon z\mathcal{C} + (\varepsilon z)^2\mathcal{D}. \end{aligned}$$

A calculation involving the ansatz

$$\tilde{\mathcal{G}}_{\varepsilon z}^{-1} = (\mathcal{G}_0 + \varepsilon z\mathcal{C} + (\varepsilon z)^2\mathcal{D})^{-1} = \mathcal{G}_0^{-1} + \mathcal{E} \tag{3.26}$$

will yield that

$$\mathcal{E} = -(I + \mathcal{G}_0^{-1}(\varepsilon z\mathcal{C} + (\varepsilon z)^2\mathcal{D}))^{-1}(\mathcal{G}_0^{-1}(\varepsilon z\mathcal{C} + (\varepsilon z)^2\mathcal{D})\mathcal{G}_0^{-1})$$

if $\tilde{\mathcal{G}}_{\varepsilon z}$, $\mathcal{G}_0$ and $I + \mathcal{G}_0^{-1}(\varepsilon z\mathcal{C} + (\varepsilon z)^2\mathcal{D})$ are invertible. Here, I denotes the identity matrix.

Since Γ is a compact C^3 hypersurface, all entries in the matrices $\mathcal{G}_0$, $\mathcal{C}$, and $\mathcal{D}$ are bounded. For a matrix H and ε sufficiently small so that the absolute values of the eigenvalues of εH are less than 1, we have

$$(I + \varepsilon H)^{-1} = I - \varepsilon H + \varepsilon^2 H^2 - \dots.$$

Hence, we can express

$$\tilde{\mathcal{G}}_{\varepsilon z}^{-1} = \mathcal{G}_0^{-1} - \varepsilon z\mathcal{G}_0^{-1}\mathcal{C}\mathcal{G}_0^{-1} + \mathcal{O}(\varepsilon^2) \text{ as } \varepsilon \to 0.$$

Consequently, looking at (3.18), and (3.19), we see that for $s \in \mathcal{S}, z \in (-\eta, \eta)$,

$$
\begin{aligned}
&\tilde{\nabla}_{\Gamma^{\varepsilon z}} \tilde{F}_\varepsilon(s,z) = \sum_{i,j=1}^{n-1} g_\varepsilon^{ij}(\alpha(s),z)\partial_{s_i}\tilde{F}_\varepsilon(s,z)\partial_{s_j}G_\varepsilon(s,z) \\
&= \sum_{i,j=1}^{n-1} g_0^{ij}(\alpha(s))\partial_{s_i}\tilde{F}_\varepsilon(s,z)\partial_{s_j}\alpha(s) + \varepsilon z \sum_{i,j=1}^{n-1} g_0^{ij}(\alpha(s))\partial_{s_i}\tilde{F}_\varepsilon(s,z)\partial_{s_j}\nu(\alpha(s)) \\
&- \varepsilon z \sum_{i,j=1}^{n-1} (\mathcal{G}_0^{-1}\mathcal{C}\mathcal{G}_0^{-1})_{ij}(s)\partial_{s_i}\tilde{F}_\varepsilon(s,z)\partial_{s_j}\alpha(s) + \text{ h.o.t.} \\
&= \tilde{\nabla}_\Gamma \tilde{F}_\varepsilon(s,z) + \tilde{\nabla}_{\varepsilon z}\tilde{F}_\varepsilon(s,z) = \tilde{\nabla}_\Gamma \tilde{F}_\varepsilon(s,z) + \mathcal{O}(\varepsilon) \text{ as } \varepsilon \to 0,
\end{aligned} \tag{3.27}
$$

where h.o.t. denotes terms of higher order in ε. By definition and by (3.12),

$$
\tilde{\nabla}_{\varepsilon z}\tilde{F}_\varepsilon(s,z)\cdot\nu(\alpha(s)) = \tilde{\nabla}_{\Gamma^{\varepsilon z}}\tilde{F}_\varepsilon(s,z)\cdot\nu(\alpha(s)) - \tilde{\nabla}_\Gamma\tilde{F}_\varepsilon(s,z)\cdot\nu(\alpha(s)) = 0. \tag{3.28}
$$

The analogous statements for the representation in the (p,z) coordinate system can be obtained from (3.27) and (3.28) by applying the relations (3.20), (3.21), and (3.22). □

We point out that, for fixed $\varepsilon > 0$ sufficiently small, such that $\partial X^\varepsilon \cap \partial\Omega = \varnothing$, $\partial X^\varepsilon = \{x \in \Omega : |d(x)| = \varepsilon\eta\}$ has the same regularity as Γ, i.e., ∂X^ε is a C^3 boundary. Using that $C^1(\overline{X^\varepsilon})$ functions are dense in $H^1(X^\varepsilon)$, we can extend the assertions of Lemma 3.2 to $H^1(X^\varepsilon)$ functions, where (3.23) hold for weak derivatives, and $\partial_z(\cdot)$ and $\nabla_{\Gamma^{\varepsilon z}}(\cdot)$ are to be understood in the weak sense, and (3.24) and (3.25) hold almost everywhere. We will use this for the weak derivatives of φ and ψ in the proof of Theorem 2.3.

Using (3.5), (3.23), and a change of variables $t \mapsto \varepsilon z$, we have for any $0 < k \le 1$,

$$
\begin{aligned}
\|f\|^2_{H^1(\mathrm{Tub}^{\varepsilon^k\eta}(\Gamma))} &= \int_{-\varepsilon^k\eta}^{\varepsilon^k\eta}\int_\Gamma (|f|^2 + |\nabla f|^2)(p + t\nu(p))\,|1 + Z(t,p)|\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{dt}\,, \\
&= \int_{\frac{-\eta}{\varepsilon^{1-k}}}^{\frac{\eta}{\varepsilon^{1-k}}}\int_\Gamma \frac{1}{\varepsilon}|\partial_z F_\varepsilon|^2(p,z)\,|1 + Z(\varepsilon z,p)|\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{dz} \\
&+ \int_{\frac{-\eta}{\varepsilon^{1-k}}}^{\frac{\eta}{\varepsilon^{1-k}}}\int_\Gamma \varepsilon(|F_\varepsilon|^2 + |\nabla_{\Gamma^{\varepsilon z}}F_\varepsilon|^2)(p,z)\,|1 + Z(\varepsilon z,p)|\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{dz}\,.
\end{aligned} \tag{3.29}
$$

In addition, using that for $\varepsilon \in (0,1]$,

$$
1 - \tilde{c}\varepsilon^k\eta \ge 1 - \tilde{c}\eta, \quad 1 + \tilde{c}\varepsilon^k\eta \le 1 + \tilde{c}\eta,
$$

we find that an analogous statement to (3.7) holds for $\mathrm{Tub}^{\varepsilon^k\eta}(\Gamma)$ for any $0 < k \le 1$:

$$
\begin{aligned}
\frac{1}{1+\tilde{c}\eta}\int_{\mathrm{Tub}^{\varepsilon^k\eta}(\Gamma)}|f|\,\mathrm{dx} &\le \int_{-\varepsilon^k\eta}^{\varepsilon^k\eta}\int_\Gamma |f(p + t\nu(p))|\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{dt} \\
&\le \frac{1}{1-\tilde{c}\eta}\int_{\mathrm{Tub}^{\varepsilon^k\eta}(\Gamma)}|f|\,\mathrm{dx}\,.
\end{aligned} \tag{3.30}
$$

3.3. On functions extended constantly along the normal direction. Let $f \in H^1(\Gamma)$, and let f^e denote its constant extension off Γ in the normal direction to $\mathrm{Tub}^\eta(\Gamma)$, as defined in (2.6). A short calculation shows the following relation between ∇f^e and $\nabla_\Gamma f$ (see [13, proof of Theorem 2.10] for more details):

$$
\nabla f^e(x) = (\boldsymbol{I} - d(x)\boldsymbol{H}(x))\nabla_\Gamma f(p(x)) \text{ for } x \in \mathrm{Tub}^\eta(\Gamma), \tag{3.31}
$$

where $\boldsymbol{H}$ denotes the Hessian of the signed distance function d, and $\boldsymbol{I}$ denotes the identity tensor. Consequently

$$\nabla f^e(y) = \nabla_\Gamma f(y) \text{ for } y \in \Gamma. \tag{3.32}$$

Corollary 3.1. *Let Assumption 2.1 be satisfied, and choose $\eta > 0$ with the diffeomorphism $\Theta^\eta : \mathrm{Tub}^\eta(\Gamma) \to \Gamma \times (-\eta, \eta)$ as defined in (2.3). Let $1 \le q \le \infty$, $f \in W^{1,q}(\Gamma)$, and let f^e denote its constant extension in the normal direction off Γ to $\mathrm{Tub}^\eta(\Gamma)$, as defined in (2.6). Then, $f^e \in W^{1,q}(\mathrm{Tub}^\eta(\Gamma))$, and there exists a constant $C > 0$, independent of f, such that*

$$\|f^e\|_{L^q(\mathrm{Tub}^\eta(\Gamma))} \le C\|f\|_{L^q(\Gamma)}, \quad \|\nabla f^e\|_{L^q(\mathrm{Tub}^\eta(\Gamma))} \le C\|\nabla_\Gamma f\|_{L^q(\Gamma)}.$$

Moreover, let f^{Ec} denote the extension of f^e from $\mathrm{Tub}^\eta(\Gamma)$ to Ω by the method of reflection. Then $f^{Ec} \in W^{1,q}(\Omega)$ and there exists a constant $C > 0$, independent of f, such that

$$\|f^{Ec}\|_{L^q(\Omega))} \le C\|f\|_{L^q(\Gamma)}, \quad \|\nabla f^{Ec}\|_{L^q(\Omega))} \le C\|\nabla_\Gamma f\|_{L^q(\Gamma)}. \tag{3.33}$$

Proof. Since Γ is a C^3 hypersurface, we see that

$$\|\boldsymbol{H}\|_{C^0(\mathrm{Tub}^\eta(\Gamma))} \le \|d\|_{C^2(\mathrm{Tub}^\eta(\Gamma))} =: \|d\| < \infty. \tag{3.34}$$

Let $1 \le q < \infty$, $f \in W^{1,q}(\Gamma)$ and let f^e denote its constant extension in the normal direction off Γ to $\mathrm{Tub}^\eta(\Gamma)$, as defined in (2.6). Then, by (3.31), (3.5), (3.6), and (3.7), we have

$$\begin{aligned}
\|f^e\|^q_{L^q(\mathrm{Tub}^\eta(\Gamma))} &= \int_{\mathrm{Tub}^\eta(\Gamma)} |f(p(x))|^q \,\mathrm{dx} \\
&= \int_{-\eta}^{\eta} \int_\Gamma |f(p)|^q \,|1 + Z(t,p)| \,\mathrm{d}\mathcal{H}^{n-1} \,\mathrm{dt} \le C(\tilde{c}\eta)\|f\|^q_{L^q(\Gamma)}, \\
\|\nabla f^e\|^q_{L^q(\mathrm{Tub}^\eta(\Gamma))} \,\mathrm{dx} &= \int_{\mathrm{Tub}^\eta(\Gamma)} |\nabla_\Gamma f(p(x)) - d(x)\boldsymbol{H}(x)\nabla_\Gamma f(p(x))|^q \,\mathrm{dx} \\
&\le C(1 + \|d\|) \int_{\mathrm{Tub}^\eta(\Gamma)} |\nabla_\Gamma f(p(x))|^q \,\mathrm{dx} \le C(\|d\|, \tilde{c}\eta)\|\nabla_\Gamma f\|^q_{L^q(\Gamma)}.
\end{aligned}$$

Thus, $f^e \in W^{1,q}(\mathrm{Tub}^\eta(\Gamma))$.

For the case $q = \infty$, thanks to (2.6) and (3.31), we have

$$\|f^e\|_{L^\infty(\mathrm{Tub}^\eta(\Gamma))} = \|f\|_{L^\infty(\Gamma)}, \quad \|\nabla f^e\|_{L^\infty(\mathrm{Tub}^\eta(\Gamma))} \le C(\tilde{c}\eta, \|d\|)\|\nabla_\Gamma f\|_{L^\infty(\Gamma)}.$$

By the extension theorem [16, Theorem 1, p. 254], we can extend f^e to $f^{Ec} \in W^{1,q}(\Omega)$ with

$$\begin{aligned}
\|f^{Ec}\|_{L^q(\Omega)} &\le C\|f^e\|_{L^q(\mathrm{Tub}^\eta(\Gamma))} \le C\|f\|_{L^q(\Gamma)}, \\
\|\nabla f^{Ec}\|_{L^q(\Omega)} &\le C\|\nabla f^e\|_{L^q(\mathrm{Tub}^\eta(\Gamma))} \le C\|\nabla_\Gamma f\|_{L^q(\Gamma)},
\end{aligned}$$

where C is independent of f. □

The next lemma allows us to test with extensions of $H^1(\Gamma)$ functions as constructed in Corollary 3.1 in the weak formulation of (CDD).

Lemma 3.3. *Suppose Assumption 2.1 and 2.6 are satisfied. Let $f \in H^1(\Gamma)$ and let f^{Ec} denote its extension to Ω as constructed in Corollary 3.1. Then, for all $\varepsilon > 0$,*

$$f^{Ec} \in H^1(\Omega, \delta_\varepsilon),$$

and there exists a constant $C > 0$, independent of f and ε, such that

$$\|f^{Ec}\|_{0,\delta_\varepsilon} \le C\|f\|_{L^2(\Gamma)}, \quad \|\nabla f^{Ec}\|_{0,\delta_\varepsilon} \le C\|\nabla_\Gamma f\|_{L^2(\Gamma)}.$$

Furthermore, let b^{Ec} and $\mathcal{B}^{Ec}$ denote the extensions of the data b and $\mathcal{B}$ as mentioned in Assumption 2.4. Then, as $\varepsilon \to 0$,

$$\begin{aligned} \int_\Omega \delta_\varepsilon b^{Ec} \left|f^{Ec}\right|^2 \,\mathrm{dx} &\to \int_\Gamma b|f|^2 \,\mathrm{d}\mathcal{H}^{n-1}, \\ \int_\Omega \delta_\varepsilon \mathcal{B}^{Ec}\nabla f^{Ec}\cdot\nabla f^{Ec}\,\mathrm{dx} &\to \int_\Gamma \mathcal{B}\nabla_\Gamma f\cdot\nabla_\Gamma f\,\mathrm{d}\mathcal{H}^{n-1}. \end{aligned} \tag{3.35}$$

Consequently, as $\varepsilon \to 0$,

$$\int_\Omega \delta_\varepsilon \left|f^{Ec}\right|^2 \mathrm{dx} \to \int_\Gamma |f|^2 \,\mathrm{d}\mathcal{H}^{n-1}, \quad \int_\Omega \delta_\varepsilon \left|\nabla f^{Ec}\right|^2 \mathrm{dx} \to \int_\Gamma |\nabla_\Gamma f|^2 \,\mathrm{d}\mathcal{H}^{n-1}. \tag{3.36}$$

Proof. We note that (3.36) follows from (3.35) if we consider $b(p) \equiv 1$ with $b^{Ec}(x) = 1$, and $b_{ij}(p) = \delta_{ij}$ with $b_{ij}^{Ec}(x) = \delta_{ij}$, for $p \in \Gamma$, $x \in \Omega$, $1 \le i,j \le n$, and δ_{ij} is the Kronecker delta.

Choose $\eta > 0$ and the diffeomorphism Θ^η as defined in (2.3). Thanks to (2.10) with $q = 1$,

$$\|\delta_\varepsilon\|_{L^\infty(\Omega\setminus\mathrm{Tub}^\eta(\Gamma))} \to 0 \text{ as } \varepsilon \to 0, \tag{3.37}$$

and so, for $\varepsilon \in (0,1]$,

$$\|\delta_\varepsilon\|_{L^\infty(\Omega\setminus\mathrm{Tub}^\eta(\Gamma))} \le \sup_{\varepsilon\in(0,1]} \|\delta_\varepsilon\|_{L^\infty(\Omega\setminus\mathrm{Tub}^\eta(\Gamma))} =: C_{\mathrm{sup}}. \tag{3.38}$$

Then, by (3.5) with a change of variables $t \mapsto \varepsilon s$, (2.6), (3.6), (2.8), and (3.33) with $q = 2$,

$$\begin{aligned} &\int_\Omega \delta_\varepsilon \left|f^{Ec}\right|^2 \mathrm{dx} \le \int_{\mathrm{Tub}^\eta(\Gamma)} \delta_\varepsilon |f^e|^2 \,\mathrm{dx} + \|\delta_\varepsilon\|_{L^\infty(\Omega\setminus\mathrm{Tub}^\eta(\Gamma))}\|f^{Ec}\|^2_{L^2(\Omega\setminus\mathrm{Tub}^\eta(\Gamma))} \\ &\le \int_{-\eta}^{\eta}\int_\Gamma \frac{1}{\varepsilon}\delta\left(\frac{t}{\varepsilon}\right)|f^e(p+t\nu(p))|^2\,|1+Z(t,p)|\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{dt} + C_{\mathrm{sup}}\|f^{Ec}\|^2_{L^2(\Omega)} \\ &\le \int_{\frac{-\eta}{\varepsilon}}^{\frac{\eta}{\varepsilon}}\int_\Gamma \delta(s)\,|f^e(p+\varepsilon s\nu(p))|^2\,|1+Z(\varepsilon s,p)|\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{ds} + C_{\mathrm{sup}}\|f^{Ec}\|^2_{L^2(\Omega)} \\ &\le \int_{\frac{-\eta}{\varepsilon}}^{\frac{\eta}{\varepsilon}}\int_\Gamma \delta(s)(1+\tilde{c}\varepsilon|s|)\,|f(p)|^2\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{ds} + C_{\mathrm{sup}}\|f^{Ec}\|^2_{L^2(\Omega)} \\ &\le (1+\tilde{c}\eta)\|f\|^2_{L^2(\Gamma)} + C_{\mathrm{sup}}\|f^{Ec}\|^2_{L^2(\Omega)} \le C\|f\|^2_{L^2(\Gamma)}. \end{aligned}$$

Hence, $f^{Ec} \in L^2(\Omega,\delta_\varepsilon)$. For the gradient, using (3.31), (3.6), (2.9), and (3.33) with $q = 2$ yield

$$\begin{aligned} \|\nabla f^{Ec}\|^2_{0,\delta_\varepsilon} &\le C_{\mathrm{sup}}\|\nabla f^{Ec}\|^2_{L^2(\Omega\setminus\mathrm{Tub}^\eta(\Gamma))} \\ &\quad + C(1+\|d\|)\int_{\frac{-\eta}{\varepsilon}}^{\frac{\eta}{\varepsilon}}\int_\Gamma \delta(s)(1+\tilde{c}\varepsilon|s|)\,|\nabla_\Gamma f(p)|^2\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{ds} \\ &\le C_{\mathrm{sup}}\|\nabla f^{Ec}\|^2_{L^2(\Omega)} + C(d,\eta,\tilde{c})\|\nabla_\Gamma f\|^2_{L^2(\Gamma)} \le C\|\nabla_\Gamma f\|^2_{L^2(\Gamma)}, \end{aligned}$$

which then implies that $f^{Ec} \in H^1(\Omega, \delta_\varepsilon)$.

Next, we have by the definition of b^e, (2.6), (3.5) with a change of variables $t \mapsto \varepsilon s$, and (2.8),

$$\int_{\mathrm{Tub}^\eta(\Gamma)} \delta_\varepsilon b^e |f^e|^2 \,\mathrm{dx} = \int_{\frac{-\eta}{\varepsilon}}^{\frac{\eta}{\varepsilon}} \int_\Gamma \delta(s) b(p) |f(p)|^2 |1 + Z(\varepsilon s, p)| \,\mathrm{d}\mathcal{H}^{n-1} \,\mathrm{ds},$$
$$\int_\Gamma b |f|^2 \,\mathrm{d}\mathcal{H}^{n-1} = \int_{\mathbb{R}} \int_\Gamma \delta(s) b(p) |f(p)|^2 \,\mathrm{d}\mathcal{H}^{n-1} \,\mathrm{ds}.$$

Thanks to (3.37), we have that

$$\int_{\Omega \setminus \mathrm{Tub}^\eta(\Gamma)} \delta_\varepsilon b^{Ec} \left|f^{Ec}\right|^2 \mathrm{dx} \le \|\delta_\varepsilon\|_{L^\infty(\Omega \setminus \mathrm{Tub}^\eta(\Gamma))} \|b^{Ec}\|_{L^\infty(\Omega)} \|f^{Ec}\|^2_{L^2(\Omega)} \to 0, \text{ as } \varepsilon \to 0.$$

The first statement of (3.35) follows from

$$\left| \int_{\mathrm{Tub}^\eta(\Gamma)} \delta_\varepsilon b^e |f^e|^2 \,\mathrm{dx} - \int_\Gamma b |f|^2 \,\mathrm{d}\mathcal{H}^{n-1} \right|$$
$$\le \left| \int_{\mathbb{R} \setminus (\frac{-\eta}{\varepsilon}, \frac{\eta}{\varepsilon})} \delta(s) \,\mathrm{ds} \right| \|b\|_{L^\infty(\Gamma)} \|f\|^2_{L^2(\Gamma)} + \left| \int_{\frac{-\eta}{\varepsilon}}^{\frac{\eta}{\varepsilon}} \int_\Gamma \delta(s) b(p) |f(p)|^2 |Z(\varepsilon s, p)| \,\mathrm{d}\mathcal{H}^{n-1} \,\mathrm{ds} \right|$$
$$\le \left(\int_{\mathbb{R} \setminus (\frac{-\eta}{\varepsilon}, \frac{\eta}{\varepsilon})} \delta(s) \,\mathrm{ds} + \varepsilon \tilde{c} C_{\delta,\mathrm{int}} \right) \|b\|_{L^\infty(\Gamma)} \|f\|^2_{L^2(\Gamma)} \to 0 \text{ as } \varepsilon \to 0,$$

where we have used (3.6) and (2.9).

The second statement of (3.35) for the gradient follows from a similar calculation, and hence we omit the details. ∎

Remark 3.1. *For the double-obstacle regularization, we use the fact that $\delta_\varepsilon = 0$ on $\Omega \setminus \mathrm{Tub}^\eta(\Gamma)$ for $\eta \ge \varepsilon \frac{\pi}{2}$ to deduce the same results.*

3.4. On the regularized indicator functions. Due to the boundedness of ξ_ε, (2.7), and Lebesgue's dominated convergence theorem, we have the next lemma.

Lemma 3.4. *Suppose that Assumption 2.5 is satisfied; then for any $g \in L^1(\Omega)$,*

$$\int_{\Omega^*} \xi_\varepsilon g \,\mathrm{dx} \to \int_{\Omega^*} g\,|_{\Omega^*} \,\mathrm{dx}, \quad \int_{\Omega^*} (1 - \xi_\varepsilon) g \,\mathrm{dx} \to 0, \quad \int_{\Omega \setminus \Omega^*} \xi_\varepsilon g \,\mathrm{dx} \to 0 \textit{ as } \varepsilon \to 0.$$

The next lemma allows us to consider $H^1(\Omega)$ functions as a test function in the weak formulation of (CDD).

Lemma 3.5. *Suppose that Assumptions 2.1 and 2.6 are satisfied. Then, there exists a constant $C > 0$ such that, for all $f \in W^{1,1}(\Omega)$ and all $\varepsilon \in (0, 1]$,*

$$\int_\Omega \delta_\varepsilon |f| \,\mathrm{dx} \le C \|f\|_{W^{1,1}(\Omega)}.$$

In particular, there exists a constant $C > 0$ such that, for all $f \in H^1(\Omega)$ and all $\varepsilon \in (0, 1]$,

$$\|f\|_{0,\delta_\varepsilon} \le C \|f\|_{H^1(\Omega)}.$$

Proof. Choose $\eta > 0$ and the diffeomorphism Θ^η as defined in (2.3). Let $f \in W^{1,1}(\Omega)$. Then, by (3.38),

$$\int_{\Omega \setminus \mathrm{Tub}^\eta(\Gamma)} \delta_\varepsilon |f| \,\mathrm{dx} \le C_{\sup} \|f\|_{L^1(\Omega \setminus \mathrm{Tub}^\eta(\Gamma))} \le C_{\sup} \|f\|_{W^{1,1}(\Omega)}. \tag{3.39}$$

Using the diffeomorphism Θ^η, $f\,|_{\mathrm{Tub}^\eta(\Gamma)} \in W^{1,1}(\Gamma\times(-\eta,\eta))$. By absolute continuity on lines for $W^{1,1}$ functions (see [37, Theorem 2.1.4] or [29, Theorem 1, §1.1.3]) there exists a version of f, which we denote by the same symbol, such that for a.e. $p\in\Gamma$, it is absolutely continuous as a function of $t\in(-\eta,\eta)$. With absolute continuity with respect to t, we have for $t\in(-\eta,\eta)$,

$$f(p+t\nu(p)) = f(p) + \int_0^t \frac{\mathrm{d}}{\mathrm{d}\zeta} f(p+\zeta\nu(p))\,\mathrm{d}\zeta\,.$$

Then, by (3.5), a change of coordinates $t=\varepsilon s$, (2.8), and (3.7),

$$\begin{aligned}
&\int_{\mathrm{Tub}^\eta(\Gamma)} \delta_\varepsilon |f|\,\mathrm{dx} = \int_{-\eta}^{\eta}\int_\Gamma \frac{1}{\varepsilon}\delta\left(\frac{t}{\varepsilon}\right)|f(p+t\nu(p))|\,|1+Z(t,p)|\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{dt}\\
&= \int_{-\eta}^{\eta}\int_\Gamma \frac{1}{\varepsilon}\delta\left(\frac{t}{\varepsilon}\right)\left[|f(p)| + \int_0^t \left|\frac{\mathrm{d}}{\mathrm{d}\zeta} f(p,\zeta)\right|\mathrm{d}\zeta\right]|1+Z(t,p)|\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{dt}\\
&\le \left(\|\gamma_0(f)\|^2_{L^1(\Gamma)} + \int_\Gamma\int_{-\eta}^{\eta} |\nabla f(p+\zeta\nu(p))|\,\mathrm{d}\zeta\,\mathrm{d}\mathcal{H}^{n-1}\right)\int_{\frac{-\eta}{\varepsilon}}^{\frac{\eta}{\varepsilon}} \delta(s)(1+\tilde{c}\varepsilon|s|)\,\mathrm{ds}\\
&\le (1+\varepsilon\tilde{c}C_{\delta,\mathrm{int}})\left(\|\gamma_0(f)\|_{L^1(\Gamma)} + C(\tilde{c}\eta)\|\nabla f\|_{L^1(\mathrm{Tub}^\eta(\Gamma))}\right)\\
&\le (1+\tilde{c}C_{\delta,\mathrm{int}})\left(\|\gamma_0(f)\|_{L^1(\Gamma)} + C(\tilde{c}\eta)\|f\|_{W^{1,1}(\mathrm{Tub}^\eta(\Gamma))}\right).
\end{aligned}$$

Using the trace theorem [16, Theorem 1, p. 272] and (3.39), we have

$$\int_\Omega \delta_\varepsilon|f|\,\mathrm{dx} \le ((1+\tilde{c}C_{\delta,\mathrm{int}})(C_{\mathrm{tr}}+C(\tilde{c}\eta)) + C_{\sup})\|f\|_{W^{1,1}(\Omega)},$$

where C_{tr} is the constant from the trace theorem. Thanks to the fact that

$$\|\nabla(|f|^2)\|_{L^1(\Omega)} \le 2\|f\|_{L^2(\Omega)}\|\nabla f\|_{L^2(\Omega)} \le 2\|f\|^2_{H^1(\Omega)},$$

the second assertion follows immediately. ☐

Remark 3.2. *For the double-obstacle regularization, we directly obtain*

$$\int_{\mathrm{Tub}^{\varepsilon\frac{\pi}{2}}(\Gamma)} \delta_\varepsilon|f|\,\mathrm{dx} \le (1+\tilde{c}C_{\delta,\mathrm{int}})(C_{\mathrm{tr}}+C(\tilde{c}\eta))\|f\|_{W^{1,1}(\Omega)}.$$

Lemma 3.6. *Suppose that Assumptions 2.1 and 2.6 are satisfied. For $f\in W^{1,1}(\Omega)$, it holds that*

$$\int_\Omega \delta_\varepsilon f\,\mathrm{dx} \to \int_\Gamma \gamma_0(f)\,\mathrm{d}\mathcal{H}^{n-1} \quad \text{as } \varepsilon\to 0. \tag{3.40}$$

Moreover, for $\eta>0$ sufficiently small, if $f\in W^{1,q}(\Omega)$, $1\le q<\infty$, or if $f=C^1(\overline{\Omega})$ and $q=\infty$, then there exists a constant $C>0$, independent of f and ε, such that

$$\left|\int_{\mathrm{Tub}^\eta(\Gamma)} \delta_\varepsilon f\,\mathrm{dx} - \int_\Gamma \gamma_0(f)\,\mathrm{d}\mathcal{H}^{n-1}\right| \le C\varepsilon^{1-\frac{1}{q}}\|f\|_{W^{1,q}(\Omega)}. \tag{3.41}$$

Proof. By the trace theorem and Lemma 3.5, the integrals in (3.40) are well-defined. Choose $0<\eta<(\tilde{c})^{-1}$ and the diffeomorphism Θ^η as defined in (2.3), where $\tilde{c}$ is the constant in (3.6).

By (2.10) with $q = 2$, we see that $\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\delta\left(\frac{\eta}{\varepsilon}\right) = 0$. Then, for all $\varepsilon \in (0,1]$, we have

$$\delta\left(\frac{\eta}{\varepsilon}\right) \le C_{\sup,\eta}\varepsilon^2, \text{ where } C_{\sup,\eta} := \sup_{\varepsilon\in(0,1]} \frac{1}{\varepsilon^2}\delta\left(\frac{\eta}{\varepsilon}\right) < \infty$$

and so, from the monotonicity of $\delta(|s|)$, the symmetry of $\delta(s)$ about $s = 0$, and (2.9),

$$\begin{aligned}\int_{\mathbb{R}\setminus(\frac{-\eta}{\varepsilon},\frac{\eta}{\varepsilon})} \delta(s)\,\mathrm{d}s &= 2\int_{\frac{\eta}{\varepsilon}}^{\infty} \delta(s)\,\mathrm{d}s \le 2\sqrt{\delta\left(\frac{\eta}{\varepsilon}\right)}\left(\int_{\frac{\eta}{\varepsilon}}^{\infty}\sqrt{\delta(s)}\,\mathrm{d}s\right) \\ &\le 2\sqrt{C_{\sup,\eta}}C_{\delta,\mathrm{int}}\varepsilon \le C\varepsilon,\end{aligned} \tag{3.42}$$

for some constant $C > 0$.

Let $q < \infty$ and $f \in W^{1,q}(\Omega)$. Then, by the diffeomorphism Θ^η, absolute continuity on lines, and Hölder's inequality, we have

$$\begin{aligned}|f(p+\varepsilon s\nu(p)) - f(p)| &= \left|\int_0^s \frac{\mathrm{d}}{\mathrm{d}\zeta} f(p+\varepsilon\zeta\nu(p))\,\mathrm{d}\zeta\right| \\ &= \varepsilon\left|\int_0^s \nabla f(p+\varepsilon\zeta\nu(p))\cdot\nu(p)\,\mathrm{d}\zeta\right| \le \varepsilon|s|^{\frac{q-1}{q}}\left(\int_0^s |\nabla f(p+\varepsilon\zeta\nu(p))|^q\,\mathrm{d}\zeta\right)^{\frac{1}{q}}.\end{aligned} \tag{3.43}$$

Then, by (3.43), (3.5), a change of variables $t \mapsto \varepsilon s$, Hölder's inequality, and (3.6),

$$\begin{aligned}&\left|\int_{\mathrm{Tub}^\eta(\Gamma)} \delta_\varepsilon f\,\mathrm{d}x - \int_\Gamma \gamma_0(f)\,\mathrm{d}\mathcal{H}^{n-1}\right| \\ &\le \int_{\mathbb{R}}\int_\Gamma \delta(s)\left|\chi_{(\frac{-\eta}{\varepsilon},\frac{\eta}{\varepsilon})}(s)f(p+\varepsilon s\nu(p))\,|1+Z(\varepsilon s,p)| - f(p)\right|\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s \\ &\le \int_{\mathbb{R}}\int_\Gamma \delta(s)\,|f(p)|\left|1-\chi_{(\frac{-\eta}{\varepsilon},\frac{\eta}{\varepsilon})}(s)\,|1+Z(\varepsilon s,p)|\right|\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s \\ &+ \int_{\mathbb{R}}\int_\Gamma \delta(s)\chi_{(\frac{-\eta}{\varepsilon},\frac{\eta}{\varepsilon})}(s)\,|f(p+\varepsilon s\nu(p)) - f(p)|\,|1+Z(\varepsilon s,p)|\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s \\ &\le \|\gamma_0(f)\|_{L^1(\Gamma)}\int_{\mathbb{R}}\delta(s)\left(\chi_{\mathbb{R}\setminus(\frac{-\eta}{\varepsilon},\frac{\eta}{\varepsilon})}(s) + \varepsilon\tilde{c}|s|\right)\mathrm{d}s \\ &+ \varepsilon C(\tilde{c}\eta)\int_{\mathbb{R}}\int_\Gamma \chi_{(\frac{-\eta}{\varepsilon},\frac{\eta}{\varepsilon})}(s)\delta(s)\,|s|^{\frac{q-1}{q}}\left(\int_0^s|\nabla f(p+\varepsilon\zeta\nu(p))|^q\,\mathrm{d}\zeta\right)^{\frac{1}{q}}\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s \\ &\le \|\gamma_0(f)\|_{L^1(\Gamma)}\int_{\mathbb{R}}\delta(s)\left(\chi_{\mathbb{R}\setminus(\frac{-\eta}{\varepsilon},\frac{\eta}{\varepsilon})}(s) + \varepsilon\tilde{c}|s|\right)\mathrm{d}s \\ &+ \varepsilon C(\tilde{c}\eta)\,|\Gamma|^{\frac{q-1}{q}}\int_{\mathbb{R}}\delta(s)(1+|s|)\left(\int_\Gamma\int_0^{\frac{\eta}{\varepsilon}}|\nabla f(p+\varepsilon\zeta\nu(p))|^q\,\mathrm{d}\zeta\,\mathrm{d}\mathcal{H}^{n-1}\right)^{\frac{1}{q}}\mathrm{d}s.\end{aligned} \tag{3.44}$$

Here, we have also used that $|s|^{1-\frac{1}{r}} \le 1+|s|$ for all $s \in \mathbb{R}$ and $1 \le r \le \infty$. By a change of variables $\varepsilon\zeta \mapsto t$, and (3.7), we observe that

$$\begin{aligned}&\left(\int_\Gamma\int_0^{\frac{\eta}{\varepsilon}}|\nabla f(p+\varepsilon\zeta\nu(p))|^q\,\mathrm{d}\zeta\,\mathrm{d}\mathcal{H}^{n-1}\right)^{\frac{1}{q}} \\ &= \varepsilon^{-\frac{1}{q}}\left(\int_\Gamma\int_0^\eta|\nabla f(p+t\nu(p))|^q\,\mathrm{d}t\,\mathrm{d}\mathcal{H}^{n-1}\right)^{\frac{1}{q}} \le C(\tilde{c}\eta)\varepsilon^{-\frac{1}{q}}\|\nabla f\|_{L^q(\mathrm{Tub}^\eta(\Gamma))},\end{aligned} \tag{3.45}$$

Together with (3.42), and (2.9), we see from (3.44) that

$$\begin{aligned}\left|\int_{\mathrm{Tub}^\eta(\Gamma)}\delta_\varepsilon f\,\mathrm{d}x - \int_\Gamma\gamma_0(f)\,\mathrm{d}\mathcal{H}^{n-1}\right| &\le \varepsilon C\|\gamma_0(f)\|_{L^1(\Gamma)} + \varepsilon^{1-\frac{1}{q}}C\|\nabla f\|_{L^q(\mathrm{Tub}^\eta(\Gamma))} \\ &\le C\varepsilon^{1-\frac{1}{q}}\|f\|_{W^{1,q}(\Omega)},\end{aligned}$$

which is (3.41) for $q < \infty$.

For the case $q = \infty$, let $f \in C^1(\overline{\Omega})$. Then, by the fundamental theorem of calculus, we have

$$
\begin{aligned}
|f(p+\varepsilon s\nu(p)) - f(p)| &= \varepsilon \left| \int_0^s \nabla f(p+\varepsilon\zeta\nu(p)) \cdot \nu(p) \, \mathrm{d}\zeta \right| \\
&\leq \varepsilon\, |s| \sup_{\zeta\in[0,s]} |\nabla f(p+\varepsilon\zeta\nu(p))| \leq \varepsilon\, |s|\, \|\nabla f\|_{C^0(\mathrm{Tub}^\eta(\Gamma))},
\end{aligned} \tag{3.46}
$$

and hence, a similar calculation to (3.44) yields

$$
\begin{aligned}
&\left| \int_{\mathrm{Tub}^\eta(\Gamma)} \delta_\varepsilon f \, \mathrm{dx} - \int_\Gamma \gamma_0(f) \, \mathrm{d}\mathcal{H}^{n-1} \right| \\
&\leq \varepsilon C \|\gamma_0(f)\|_{L^1(\Gamma)} + \varepsilon(1+\tilde{c}\eta)\, |\Gamma|\, \|\nabla f\|_{C^0(\mathrm{Tub}^\eta(\Gamma))} \int_{\mathbb{R}} \delta(s)\, |s| \, \mathrm{ds} \leq C\varepsilon \|f\|_{C^1(\Omega)}.
\end{aligned}
$$

Next, by (3.37), we see that

$$
\int_{\Omega\setminus\mathrm{Tub}^\eta(\Gamma)} \delta_\varepsilon \,|f| \, \mathrm{dx} \leq \|\delta_\varepsilon\|_{L^\infty(\Omega\setminus\mathrm{Tub}^\eta(\Gamma))} \|f\|_{L^1(\Omega)} \to 0 \text{ as } \varepsilon \to 0.
$$

Together with (3.41) for $q = \infty$, we have that (3.40) holds true for all $C^1(\overline{\Omega})$ functions. Let $\zeta > 0$ be arbitrary; then by the density of smooth functions, for any $f \in W^{1,1}(\Omega)$, there exists $g \in C^1(\overline{\Omega})$ such that

$$
\|f - g\|_{W^{1,1}(\Omega)} < \zeta.
$$

Then, by (3.40) for $C^1(\overline{\Omega})$ functions, there exists $\varepsilon_0(\zeta)$ such that for $\varepsilon < \varepsilon_0(\zeta)$,

$$
\left| \int_\Omega \delta_\varepsilon g \, \mathrm{dx} - \int_\Gamma \gamma_0(g) \, \mathrm{d}\mathcal{H}^{n-1} \right| < \zeta.
$$

By Lemma 3.5, and the trace theorem, we find that for $\varepsilon < \varepsilon_0(\zeta)$,

$$
\begin{aligned}
&\left| \int_\Omega \delta_\varepsilon f \, \mathrm{dx} - \int_\Gamma \gamma_0(f) \, \mathrm{d}\mathcal{H}^{n-1} \right| \\
&\leq \left| \int_\Omega \delta_\varepsilon (f-g) \, \mathrm{dx} - \int_\Gamma \gamma_0(f-g) \, \mathrm{d}\mathcal{H}^{n-1} \right| + \left| \int_\Omega \delta_\varepsilon g \, \mathrm{dx} - \int_\Gamma \gamma_0(g) \, \mathrm{d}\mathcal{H}^{n-1} \right| \\
&\leq C\|f-g\|_{W^{1,1}(\Omega)} + \|f-g\|_{L^1(\Gamma)} + \zeta \leq (C + C_{\mathrm{tr}} + 1)\zeta.
\end{aligned} \tag{3.47}
$$

By the arbitrariness of ζ, we conclude that (3.40) holds for any $f \in W^{1,1}(\Omega)$. ◻

REMARK 3.3. *The analogous assertions for the double-obstacle regularization with* $\eta = \varepsilon\frac{\pi}{2}$ *follows along a similar argument. For this case, we point out that* (3.42) *is not needed.*

4. Proof of the main results.

4.1. Well-posedness for (CSI). We consider the product Hilbert space and associated inner product

$$
\begin{aligned}
\mathcal{X} &:= H^1(\Omega^*) \times H^1(\Gamma), \\
\langle (u_1, v_1), (u_2, v_2) \rangle_{\mathcal{X}} &:= \int_{\Omega^*} u_1 u_2 + \nabla u_1 \cdot \nabla u_2 \, \mathrm{dx} + \int_\Gamma v_1 v_2 + \nabla_\Gamma v_1 \cdot \nabla_\Gamma v_2 \, \mathrm{d}\mathcal{H}^{n-1}.
\end{aligned}
$$

Let

$$
\begin{aligned}
a_{\mathrm{CSI}}((u,v),(\varphi,\psi)) &\coloneqq a_B(u,\varphi) + a_S(v,\psi) + K l_S(v-\gamma_0(u), \psi - \gamma_0(\varphi)),\\
l_{\mathrm{CSI}}((\varphi,\psi)) &\coloneqq l_B(f,\varphi) + \beta l_S(g,\psi),
\end{aligned}
$$

where $a_B(\cdot,\cdot)$, $a_S(\cdot,\cdot)$, $l_B(\cdot,\cdot)$, and $l_S(\cdot,\cdot)$ are as defined in (2.1) and (2.2). The weak formulation for (CSI) is as follows: Find $(u,v)\in\mathcal{X}$ such that for all $(\varphi,\psi)\in\mathcal{X}$,

$$
a_{\mathrm{CSI}}((u,v),(\varphi,\psi)) = l_{\mathrm{CSI}}((\varphi,\psi)).
$$

By the root mean square inequality, i.e.,

$$
a + b \le \sqrt{2}\sqrt{a^2+b^2}, \tag{4.1}
$$

one can show that

$$
\begin{aligned}
|l_{\mathrm{CSI}}((\varphi,\psi))| &\le \|f\|_{L^2(\Omega^*)}\|\varphi\|_{L^2(\Omega^*)} + \|g\|_{L^2(\Gamma)}\|\psi\|_{L^2(\Gamma)}\\
&\le (\|f\|_{L^2(\Omega^*)} + \|g\|_{L^2(\Gamma)})(\|\varphi\|_{L^2(\Omega^*)} + \|\psi\|_{L^2(\Gamma)})\\
&\le \sqrt{2}(\|f\|_{L^2(\Omega^*)} + \|g\|_{L^2(\Gamma)})\|(\varphi,\psi)\|_{\mathcal{X}}
\end{aligned} \tag{4.2}
$$

and

$$
\begin{aligned}
\|(v-\gamma_0(u))(\psi-\gamma_0(\varphi))\|_{L^1(\Gamma)} &\le \|v-\gamma_0(u)\|_{L^2(\Gamma)}\|\psi-\gamma_0(\varphi)\|_{L^2(\Gamma)}\\
&\le (\|v\|_{L^2(\Gamma)} + C_{\mathrm{tr}}\|u\|_{H^1(\Omega^*)})(\|\psi\|_{L^2(\Gamma)} + C_{\mathrm{tr}}\|\varphi\|_{H^1(\Omega^*)})\\
&\le 2(1+C_{\mathrm{tr}}^2)\|(u,v)\|_{\mathcal{X}}\|(\varphi,\psi)\|_{\mathcal{X}}.
\end{aligned}
$$

This implies that

$$
|a_{\mathrm{CSI}}((u,v),(\varphi,\psi))| \le (C_{\mathcal{A},\mathcal{B},a,b} + 2K(1+C_{\mathrm{tr}}^2))\|(u,v)\|_{\mathcal{X}}\|(\varphi,\psi)\|_{\mathcal{X}}.
$$

Moreover,

$$
\begin{aligned}
a_{\mathrm{CSI}}((u,v),(u,v)) &\ge \min(\theta_0,\theta_2)\|u\|_{H^1(\Omega^*)}^2 + \min(\theta_1,\theta_3)\|v\|_{H^1(\Gamma)}^2 + K\|v-\gamma_0(u)\|_{L^2(\Gamma)}^2\\
&\ge (\min_i \theta_i)\|(u,v)\|_{\mathcal{X}}^2.
\end{aligned}
$$

By the Lax–Milgram theorem, there exists a unique weak solution $(u,v)\in\mathcal{X}$ to (CSI) such that

$$
\|(u,v)\|_{\mathcal{X}} \le C(\theta_i)(\|f\|_{L^2(\Omega^*)} + \|g\|_{L^2(\Gamma)}).
$$

4.2. Well-posedness for (CDD). We consider the product Hilbert space and associated inner product

$$
\begin{aligned}
\mathcal{X}_\varepsilon &\coloneqq \mathcal{V}_\varepsilon \times H^1(\Omega,\delta_\varepsilon),\\
\langle (u_1,v_1),(u_2,v_2)\rangle_{\mathcal{X}_\varepsilon} &\coloneqq \int_\Omega (\xi_\varepsilon+\delta_\varepsilon)u_1u_2 + \xi_\varepsilon\nabla u_1\cdot\nabla u_2 + \delta_\varepsilon v_1 v_2 + \delta_\varepsilon \nabla v_1\cdot\nabla v_2 \,\mathrm{dx}\,.
\end{aligned}
$$

Let

$$
a_{\mathrm{CDD}}((u,v),(\varphi,\psi)) \coloneqq a_B^\varepsilon(u,\varphi) + a_S^\varepsilon(v,\psi) + K l_S^\varepsilon(v-u,\psi-\varphi), \tag{4.3}
$$

$$
l_{\mathrm{CDD}}((\varphi,\psi)) \coloneqq l_B^\varepsilon(f^{Ea},\varphi) + \beta l_S^\varepsilon(g^{Ec},\psi), \tag{4.4}
$$

where $a_B^\varepsilon(\cdot,\cdot)$, $a_S^\varepsilon(\cdot,\cdot)$, $l_B^\varepsilon(\cdot,\cdot)$, and $l_S^\varepsilon(\cdot,\cdot)$ are as defined in (2.15) and (2.16). The weak formulation for (CDD) is as follows: Find $(u^\varepsilon, v^\varepsilon) \in \mathcal{X}_\varepsilon$ such that for all $(\varphi,\psi) \in \mathcal{X}_\varepsilon$,

$$a_{\mathrm{CDD}}((u^\varepsilon, v^\varepsilon),(\varphi,\psi)) = l_{\mathrm{CDD}}((\varphi,\psi)).$$

A similar calculation to (4.2) involving the root mean square inequality (4.1) applied to l_{CDD} yields

$$|l_{\mathrm{CDD}}((\varphi,\psi))| \leq \sqrt{2}\left(\|f^{Ea}\|_{0,\xi_\varepsilon} + \|g^{Ec}\|_{0,\delta_\varepsilon}\right)\|(\varphi,\psi)\|_{\mathcal{X}_\varepsilon}.$$

Similarly, we have

$$|a_{\mathrm{CDD}}((u,v),(\varphi,\psi))| \leq (C(\mathcal{A}^{Ea}, a^{Ea}, \mathcal{B}^{Ec}, b^{Ec}) + K)\|(u^\varepsilon, v^\varepsilon)\|_{\mathcal{X}_\varepsilon}\|(\varphi,\psi)\|_{\mathcal{X}_\varepsilon}.$$

By Young's inequality with constant $\mu \in (1,2)$, we have

$$\begin{aligned}\int_\Omega \delta_\varepsilon |v^\varepsilon - u^\varepsilon|^2 \,\mathrm{dx} &\geq \int_\Omega \delta_\varepsilon(|v^\varepsilon|^2 - 2|v^\varepsilon||u^\varepsilon| + |u^\varepsilon|^2)\,\mathrm{dx} \\ &\geq (1-\mu)\|v^\varepsilon\|_{0,\delta_\varepsilon}^2 + (1-\mu^{-1})\|u^\varepsilon\|_{0,\delta_\varepsilon}^2.\end{aligned}$$

Then, by the assumption specifically for (CDD), we have $\theta_3 \geq K$, and so

$$\begin{aligned}a_{\mathrm{CDD}}((u^\varepsilon, v^\varepsilon),(u^\varepsilon, v^\varepsilon)) &\geq C(\theta_0,\theta_2)\|u^\varepsilon\|_{1,\xi_\varepsilon}^2 + \theta_1\|\nabla v^\varepsilon\|_{0,\delta_\varepsilon}^2 \\ &\quad + (\theta_3 + K(1-\mu))\|v^\varepsilon\|_{0,\delta_\varepsilon}^2 + K(1-\mu^{-1})\|u^\varepsilon\|_{0,\delta_\varepsilon}^2 \\ &\geq C(\theta_i, K, \mu)\|(u^\varepsilon, v^\varepsilon)\|_{\mathcal{X}_\varepsilon}^2.\end{aligned} \tag{4.5}$$

Hence, by the Lax–Milgram theorem, for every $\varepsilon > 0$ there exists a unique pair of functions $(u^\varepsilon, v^\varepsilon) \in \mathcal{X}_\varepsilon$ that is a weak solution to (CDD) and satisfies

$$\|(u^\varepsilon, v^\varepsilon)\|_{\mathcal{X}_\varepsilon} \leq C\left(\|f^{Ea}\|_{0,\xi_\varepsilon} + \|g^{Ec}\|_{0,\delta_\varepsilon}\right),$$

where the constant C is independent of ε. Next, by the assumption that $\xi_\varepsilon \leq 1$ and Lemma 3.3, there exists a constant C, independent of ε such that

$$\|(u^\varepsilon, v^\varepsilon)\|_{\mathcal{X}_\varepsilon} \leq C\left(\|f^{Ea}\|_{0,\xi_\varepsilon} + \|g^{Ec}\|_{0,\delta_\varepsilon}\right) \leq C\left(\|f^{Ea}\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma)}\right),$$

which is (2.18).

4.3. Compactness. As a consequence of (2.11), we have

$$\frac{1}{\varepsilon}\delta\left(\frac{d(x)}{\varepsilon}\right) \leq \frac{1}{\varepsilon}\frac{1}{C_\xi}\xi\left(\frac{d(x)}{\varepsilon}\right), \tag{4.6}$$

and thus, for all $\varepsilon \in (0,1]$, and any $f \in L^2(\Omega, \xi_\varepsilon)$, we have

$$\int_\Omega \delta_\varepsilon |f|^2 \,\mathrm{dx} \leq \frac{1}{\varepsilon}\frac{1}{C_\xi}\int_\Omega \xi_\varepsilon |f|^2 \,\mathrm{dx}. \tag{4.7}$$

We now introduce the following weighted Sobolev space.

DEFINITION 4.1.

$$L^2(\Gamma\times\mathbb{R},\delta) := \left\{f : \Gamma\times\mathbb{R}\to\mathbb{R} \textit{ measurable s.t. } \int_\mathbb{R}\int_\Gamma \delta(z)|f(p,z)|^2 \,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{dz} < \infty\right\},$$

with the inner product and induced norm:

$$\langle f,g\rangle_{L^2(\Gamma\times\mathbb{R},\delta)} := \int_\mathbb{R}\int_\Gamma \delta(z) f(p,z) g(p,z)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{dz}, \quad \|f\|_{L^2(\Gamma\times\mathbb{R},\delta)}^2 := \langle f,f\rangle_{L^2(\Gamma\times\mathbb{R},\delta)},$$

along with the identification,

$$f = g \Leftrightarrow f(p,z) = g(p,z) \textit{ for a.e. } p\in\Gamma \textit{ and a.e. } z \in \{t\in\mathbb{R} : \delta(t) > 0\}.$$

We will now prove the assertions of Theorem 2.3.

4.3.1. Compactness in $\mathcal{V}_\varepsilon$. By Assumption 2.5, $\xi_\varepsilon \geq \frac{1}{2}$ in $\overline{\Omega^*}$, and so

$$\|u^\varepsilon|_{\Omega^*}\|_{H^1(\Omega^*)} \leq 2\int_{\Omega^*} \xi_\varepsilon |u^\varepsilon|^2 \,\mathrm{dx} \leq 2\int_{\Omega} \xi_\varepsilon |u^\varepsilon|^2 \,\mathrm{dx} \leq 2C \text{ for all } \varepsilon \in (0,1]. \tag{4.8}$$

Hence, $\{u^\varepsilon|_{\Omega^*}\}_{\varepsilon\in(0,1]} \subset H^1(\Omega^*)$ is a bounded sequence. Then, by the reflexive weak compactness theorem [16, Theorem 3, p. 639], there exists a function $\tilde{u} \in H^1(\Omega^*)$ such that

$$u^\varepsilon|_{\Omega^*} \rightharpoonup \tilde{u} \text{ in } H^1(\Omega^*) \text{ as } \varepsilon \to 0,$$

along a subsequence.

Choose $0 < \eta < (\tilde{c})^{-1}$ and the diffeomorphism Θ^η as defined in (2.3), where $\tilde{c}$ is the constant in (3.6). We consider the scaled tubular neighborhood $X^\varepsilon = \mathrm{Tub}^{\varepsilon^k \eta}(\Gamma)$ for $0 < k < 1$. Furthermore, choosing $q = \frac{1}{1-k} > 1$ in (2.10), and using (2.8) and a rescaling, we obtain

$$\delta_\varepsilon(x) \leq \frac{1}{\varepsilon}\delta\left(\frac{\eta}{\varepsilon^{1-k}}\right) = \frac{1}{\tilde{\varepsilon}^{\frac{1}{1-k}}}\delta\left(\frac{\eta}{\tilde{\varepsilon}}\right) \to 0 \text{ as } \varepsilon \to 0$$

for all $x \in \Omega \setminus X^\varepsilon$. Here, we have used that if $x \in \Omega \setminus X^\varepsilon$, then $|d(x)| \geq \varepsilon^k \eta$. Thus, we deduce that

$$\|\delta_\varepsilon\|_{L^\infty(\Omega\setminus X^\varepsilon)} \to 0 \text{ as } \varepsilon \to 0. \tag{4.9}$$

Next, for any $\varphi \in H^1(\Omega)$, by the Cauchy–Schwarz inequality and the uniform boundedness of $\{u^\varepsilon\}_{\varepsilon\in(0,1]}$ in $\mathcal{V}_\varepsilon$, we have

$$\begin{aligned}\left|\int_{\Omega\setminus X^\varepsilon} \delta_\varepsilon u^\varepsilon \varphi \,\mathrm{dx}\right| &\leq \|u^\varepsilon\|_{L^2(\Omega\setminus X^\varepsilon,\delta_\varepsilon)} \|\varphi\|_{L^2(\Omega\setminus X^\varepsilon,\delta_\varepsilon)} \\ &\leq C\|\delta_\varepsilon\|^{\frac{1}{2}}_{L^\infty(\Omega\setminus X^\varepsilon)} \|\varphi\|_{L^2(\Omega)} \to 0 \text{ as } \varepsilon \to 0.\end{aligned} \tag{4.10}$$

Hence, it suffices to look at the integral over X^ε. Let $U^\varepsilon(p,z)$ and $\Phi_\varepsilon(p,z)$ denote the representation of u^ε and φ in the (p,z) coordinate system, respectively. For convenience, let us use the notation

$$\eta_{\varepsilon,k} := \frac{\eta}{\varepsilon^{1-k}}. \tag{4.11}$$

Then, as u^ε is bounded uniformly in $H^1(\Omega,\xi_\varepsilon)$, by (4.6), (3.29), and (3.6), we have

$$\begin{aligned} C \geq \|u^\varepsilon\|^2_{1,\xi_\varepsilon} &\geq \int_{X^\varepsilon} \xi_\varepsilon |\nabla u^\varepsilon|^2 \,\mathrm{dx} \geq \int_{X^\varepsilon} \varepsilon C_\xi \delta_\varepsilon |\nabla u^\varepsilon|^2 \,\mathrm{dx} \\ &\geq \int_{-\eta_{\varepsilon,k}}^{\eta_{\varepsilon,k}} \int_\Gamma \varepsilon C_\xi \delta(z) \frac{1}{\varepsilon^2} |\partial_z U^\varepsilon|^2 |1 + Z(\varepsilon z, p)| \,\mathrm{d}\mathcal{H}^{n-1} \,\mathrm{dz} \\ &\geq C_\xi(1-\tilde{c}\eta) \int_{\mathbb{R}} \int_\Gamma \chi_{(-\eta_{\varepsilon,k},\eta_{\varepsilon,k})}(z)\delta(z)\frac{1}{\varepsilon} |\partial_z U^\varepsilon|^2 (p,z) \,\mathrm{d}\mathcal{H}^{n-1} \,\mathrm{dz}. \end{aligned} \tag{4.12}$$

Multiplying by ε on both sides of the inequality allows us to deduce that

$$\chi_{(-\eta_{\varepsilon,k},\eta_{\varepsilon,k})}(z)\partial_z U^\varepsilon(p,z) \to 0 \text{ in } L^2(\Gamma\times\mathbb{R},\delta) \text{ as } \varepsilon \to 0. \tag{4.13}$$

By (2.8), $\delta(0) \geq \varepsilon\delta_\varepsilon(x)$ for all $x \in \Omega$. As $\varphi \in H^1(\Omega)$, using (3.29), and (2.8) we have

$$\begin{aligned}\|\varphi\|^2_{H^1(\Omega)} &\geq \int_{X^\varepsilon} |\nabla\varphi|^2 \,\mathrm{dx} \geq \frac{1}{\delta(0)} \int_{X^\varepsilon} \varepsilon\delta_\varepsilon |\nabla\varphi|^2 \,\mathrm{dx} \\ &\geq \frac{1-\tilde{c}\eta}{\delta(0)} \int_{\mathbb{R}} \int_\Gamma \chi_{(-\eta_{\varepsilon,k},\eta_{\varepsilon,k})}(z)\delta(z)\frac{1}{\varepsilon} |\partial_z \Phi_\varepsilon|^2 (p,z) \,\mathrm{d}\mathcal{H}^{n-1} \,\mathrm{dz}\,,\end{aligned} \tag{4.14}$$

and thus,

$$\chi_{(-\eta_{\varepsilon,k},\eta_{\varepsilon,k})}(z)\partial_z\Phi_\varepsilon(p,z) \to 0 \text{ in } L^2(\Gamma\times\mathbb{R},\delta) \text{ as } \varepsilon \to 0. \tag{4.15}$$

Meanwhile, by Lemma 3.5, we see that

$$\begin{aligned}\|\varphi\|^2_{H^1(\Omega)} &\geq C\|\varphi\|^2_{0,\delta_\varepsilon} \geq C\int_{X^\varepsilon} \delta_\varepsilon |\varphi|^2 \,\mathrm{dx} \\ &\geq C(1-\tilde{c}\eta) \int_{\mathbb{R}} \int_\Gamma \chi_{(-\eta_{\varepsilon,k},\eta_{\varepsilon,k})}(z)\delta(z) |\Phi_\varepsilon|^2 (p,z) \,\mathrm{d}\mathcal{H}^{n-1} \,\mathrm{dz}\,.\end{aligned} \tag{4.16}$$

Similarly, the uniform boundedness of u^ε in $L^2(\Omega,\delta_\varepsilon)$ gives

$$\begin{aligned}C &\geq \|u^\varepsilon\|^2_{0,\delta_\varepsilon} \geq \int_{X^\varepsilon} \delta_\varepsilon |u^\varepsilon|^2 \,\mathrm{dx} \\ &\geq (1-\tilde{c}\eta) \int_{\mathbb{R}} \int_\Gamma \chi_{(-\eta_{\varepsilon,k},\eta_{\varepsilon,k})}(z)\delta(z) |U^\varepsilon|^2 (p,z) \,\mathrm{d}\mathcal{H}^{n-1} \,\mathrm{dz}\,.\end{aligned} \tag{4.17}$$

Hence, by the reflexive weak compactness theorem, there exists a function $\overline{u} \in L^2(\Gamma \times \mathbb{R},\delta)$ such that

$$\chi_{(-\eta_{\varepsilon,k},\eta_{\varepsilon,k})}(z)U^\varepsilon(p,z) \rightharpoonup \overline{u}(p,z) \text{ in } L^2(\Gamma\times\mathbb{R},\delta) \text{ as } \varepsilon \to 0 \tag{4.18}$$

along a subsequence. By (4.13) we can deduce that

$$\partial_z \overline{u} = 0 \text{ on } M := \{z \in \mathbb{R} : \delta(z) > 0\}, \tag{4.19}$$

and so $\overline{u} = \overline{u}(p)$ in M. Indeed, for any $\Psi(p,z)$ that is smooth and compactly supported in the set M, we have from (4.13),

$$\int_{-\eta_{\varepsilon,k}}^{\eta_{\varepsilon,k}} \int_\Gamma \delta(z)(\partial_z U^\varepsilon \Psi)(p,z) \,\mathrm{d}\mathcal{H}^{n-1} \,\mathrm{dz} \to 0 \text{ as } \varepsilon \to 0. \tag{4.20}$$

Then, by (2.9), and the smoothness of Ψ, we have

$$\int_M \int_\Gamma \frac{|\delta'(z)|^2}{\delta(z)} |\Psi|^2 (p,z) \,\mathrm{d}\mathcal{H}^{n-1} \,\mathrm{dz} \leq C(|\Gamma|,\Psi) \int_M \frac{|\delta'(z)|^2}{\delta(z)} \,\mathrm{dx} < \infty.$$

This implies that $\chi_M(z)\frac{\delta'(z)}{\delta(z)}\Psi(p,z) \in L^2(\Gamma\times\mathbb{R},\delta)$, and so by (4.18), as $\varepsilon \to 0$,

$$\begin{aligned}&\int_{-\eta_{\varepsilon,k}}^{\eta_{\varepsilon,k}} \int_\Gamma \chi_M(z)\delta(z)\left(U^\varepsilon(p,z)\frac{\delta'(z)}{\delta(z)}\Psi(p,z)\right) \mathrm{d}\mathcal{H}^{n-1} \,\mathrm{dz} \\ &\to \int_M \int_\Gamma \delta'(z)(\overline{u}\Psi)(p,z) \,\mathrm{d}\mathcal{H}^{n-1} \,\mathrm{dz}\,.\end{aligned}$$

Thus, we see that, as $\varepsilon \to 0$,

$$\begin{aligned}
&\int_{-\eta_{\varepsilon,k}}^{\eta_{\varepsilon,k}} \int_\Gamma \delta(z)(\partial_z U^\varepsilon \Psi)(p,z)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{dz} \\
&= -\int_{-\eta_{\varepsilon,k}}^{\eta_{\varepsilon,k}} \int_\Gamma \chi_M(z)\delta'(z)(U^\varepsilon\Psi)(p,z) + \chi_M(z)\delta(z)(\partial_z \Psi U^\varepsilon)(p,z)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{dz} \\
&\to -\int_M \int_\Gamma \delta'(z)(\overline{u}\Psi)(p,z) + \delta(z)(\overline{u}\partial_z\Psi)(p,z)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{dz} \\
&= \int_M \int_\Gamma \delta(z)(\partial_z \overline{u}\Psi)(p,z)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{dz}\,.
\end{aligned}$$

But by (4.20), the left-hand side also converges to zero as $\varepsilon \to 0$. Hence, for arbitrary Ψ that is smooth and compactly supported in M, we have

$$\int_M \int_\Gamma \delta(z)(\partial_z \overline{u}\Psi)(p,z)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{dz} = 0, \tag{4.21}$$

which implies that $\partial_z \overline{u} = 0$ a.e. on M.

To finish the proof, we will show that for all $\varphi \in H^1(\Omega)$,

$$\int_{X^\varepsilon} \delta_\varepsilon u^\varepsilon \varphi\,\mathrm{dx} \to \int_\Gamma \overline{u}\gamma_0(\varphi)\,\mathrm{d}\mathcal{H}^{n-1} \text{ as } \varepsilon \to 0, \tag{4.22}$$

and then the identification

$$\overline{u} = \gamma_0(\tilde{u}) \text{ a.e. on } \Gamma. \tag{4.23}$$

First, we see that by (3.6) and the definition of $\Phi_\varepsilon(p,z)$ (see (3.13)), for a.e. $(p,z) \in \Gamma \times \mathbb{R}$,

$$\chi_{(-\eta_{\varepsilon,k},\eta_{\varepsilon,k})}(z)\Phi_\varepsilon(p,z)\,|1 + Z(\varepsilon z,p)|^{\frac{1}{2}} \to \gamma_0(\varphi)(p) \text{ as } \varepsilon \to 0. \tag{4.24}$$

By Lemma 3.5, we have the uniform boundedness of the norm:

$$\int_{-\eta_{\varepsilon,k}}^{\eta_{\varepsilon,k}} \int_\Gamma \delta(z)\,|\Phi_\varepsilon|^2\,|1 + Z(\varepsilon z,p)|\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{dz} = \int_{X^\varepsilon} \delta_\varepsilon\,|\varphi|^2\,\mathrm{dx} \le C\|\varphi\|^2_{H^1(\Omega)}. \tag{4.25}$$

By (3.40), we have

$$\int_\Omega \delta_\varepsilon\,\left|\varphi^2\right|\,\mathrm{dx} \to \int_\Gamma |\gamma_0(\varphi)|^2\,\mathrm{d}\mathcal{H}^{n-1} \text{ as } \varepsilon \to 0.$$

Furthermore, from (4.9) we see that

$$\int_{\Omega\setminus X^\varepsilon} \delta_\varepsilon\,|\varphi|^2\,\mathrm{dx} \le \|\delta_\varepsilon\|_{L^\infty(\Omega\setminus X^\varepsilon)}\|\varphi\|^2_{L^2(\Omega)} \to 0 \text{ as } \varepsilon \to 0.$$

Hence, we deduce that

$$\int_{X^\varepsilon} \delta_\varepsilon\,|\varphi|^2\,\mathrm{dx} \to \int_\Gamma |\gamma_0(\varphi)|^2\,\mathrm{d}\mathcal{H}^{n-1} \text{ as } \varepsilon \to 0.$$

In particular, we obtain the norm convergence:

$$\begin{aligned}
&\int_{-\eta_{\varepsilon,k}}^{\eta_{\varepsilon,k}} \int_\Gamma \delta(z)\,|\Phi_\varepsilon|^2(p,z)\,|1 + Z(\varepsilon z,p)|\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{dz} \\
&= \int_{X^\varepsilon} \delta_\varepsilon\,|\varphi|^2\,\mathrm{dx} \to \int_\Gamma |\gamma_0(\varphi)|^2\,\mathrm{d}\mathcal{H}^{n-1} = \int_\mathbb{R} \int_\Gamma \delta(z)\,|\gamma_0(\varphi)|^2(p)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{dz}
\end{aligned} \tag{4.26}$$

as $\varepsilon \to 0$.

Almost everywhere convergence (4.24) and uniform boundedness of the norm (4.25) imply weak convergence in $L^2(\Gamma\times\mathbb{R},\delta)$ [5, Proposition 4.7.12, p. 282]. Together with the norm convergence (4.26) yields strong convergence in $L^2(\Gamma\times\mathbb{R},\delta)$ [5, Corollary 4.7.16 p. 285]. That is, as $\varepsilon \to 0$,

$$\chi_{(-\eta_{\varepsilon,k},\eta_{\varepsilon,k})}(z)\Phi_\varepsilon(p,z)\,|1+Z(\varepsilon z,p)|^{\frac{1}{2}} \to \gamma_0(\varphi)(p) \text{ in } L^2(\Gamma\times\mathbb{R},\delta). \tag{4.27}$$

Recall that $\eta_{\varepsilon,k} = \frac{\eta}{\varepsilon^{1-k}}$. Then, by (3.6),

$$\sup_{(p,z)\in\Gamma\times(-\eta_{\varepsilon,k},\eta_{\varepsilon,k})} |Z(\varepsilon z,p)| \le \sup_{(p,z)\in\Gamma\times(-\eta_{\varepsilon,k},\eta_{\varepsilon,k})} \tilde{c}\varepsilon\,|z| \le \tilde{c}\varepsilon^k\eta, \tag{4.28}$$

and hence, using the identity $\sqrt{a}-\sqrt{b} = \frac{a-b}{\sqrt{a}+\sqrt{b}}$, we obtain, as $\varepsilon\to 0$,

$$\operatorname*{ess\,sup}_{(p,z)\in\Gamma\times(-\eta_{\varepsilon,k},\eta_{\varepsilon,k})} \Big||1+Z(\varepsilon z,p)|^{\frac{1}{2}}-1\Big| \le \frac{\tilde{c}\varepsilon^k\eta}{\sqrt{1+\tilde{c}\varepsilon^k\eta}+1} \le C\varepsilon^k\eta \to 0. \tag{4.29}$$

By the weak-strong product convergence, we have

$$\begin{aligned}\int_{X^\varepsilon}\delta_\varepsilon u^\varepsilon\varphi\,\mathrm{dx} &= \int_{\mathbb{R}}\int_\Gamma \delta(z)\chi_{(-\eta_{\varepsilon,k},\eta_{\varepsilon,k})}(z)(U^\varepsilon\Phi_\varepsilon)(p,z)\,|1+Z(\varepsilon z,p)|\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{dz}\\ &\to \int_{\mathbb{R}}\delta(z)\int_\Gamma \overline{u}(p)\gamma_0(\varphi)(p)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{dz} = \int_\Gamma \overline{u}\gamma_0(\varphi)\,\mathrm{d}\mathcal{H}^{n-1} \text{ as } \varepsilon\to 0.\end{aligned} \tag{4.30}$$

For the identification (4.23), we use that $\varepsilon<1 \Rightarrow \eta_{\varepsilon,k}>\eta$ to see that the weak convergence of U^ε to $\overline{u}$ also holds in the restricted space $L^2(\Gamma\times(-\eta,\eta),\delta)$, which is equivalent to restricting to $\mathrm{Tub}^{\varepsilon\eta}(\Gamma)$. Indeed, for $\Psi\in L^2(\Gamma\times\mathbb{R},\delta)$, we have

$$\begin{aligned}&\left|\int_{-\eta}^{\eta}\int_\Gamma \delta(z)(U^\varepsilon-\overline{u})\Psi\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{dz}\right|\\ &= \left|\int_{\mathbb{R}}\int_\Gamma \delta(z)(U^\varepsilon-\overline{u})(\chi_{(-\eta,\eta)}(z)\Psi)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{dz}\right| \to 0,\end{aligned} \tag{4.31}$$

as $\varepsilon\to 0$. We will show

$$\int_{-\eta}^{\eta}\int_\Gamma \delta(z)((U^\varepsilon\Phi_\varepsilon)(p,z)-(\gamma_0(\tilde{u})\gamma_0(\varphi))(p))\,|1+Z(\varepsilon z,p)|\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{dz} \to 0 \tag{4.32}$$

as $\varepsilon\to 0$. The second term in the integral is finite by the Cauchy–Schwarz inequality, (3.6), and (2.9):

$$\begin{aligned}&\int_{-\eta}^{\eta}\int_\Gamma \delta(z)\,|\gamma_0(\tilde{u})(p)\gamma_0(\varphi)(p)|\,|1+Z(\varepsilon z,p)|\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{dz}\\ &\le (1+\tilde{c}\varepsilon C_{\delta,\mathrm{int}})\|\gamma_0(\tilde{u})\|_{L^2(\Gamma)}\|\gamma_0(\varphi)\|_{L^2(\Gamma)} < \infty.\end{aligned}$$

Recalling that, by definition (see (3.13)), $\Phi_\varepsilon(p,0)=\gamma_0(\varphi)$, we can now compute

$$\begin{aligned}&\left|\int_{-\eta}^{\eta}\int_\Gamma \delta(z)((U^\varepsilon\Phi_\varepsilon)(p,z)-(\gamma_0(\tilde{u})\gamma_0(\varphi))(p))\,|1+Z(\varepsilon z,p)|\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{dz}\right|\\ &\le \left|\int_{-\eta}^{\eta}\int_\Gamma \delta(z)(U^\varepsilon(p,0)-\gamma_0(\tilde{u})(p))\gamma_0(\varphi)(p)\,|1+Z(\varepsilon z,p)|\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{dz}\right|\\ &+ \left|\int_{-\eta}^{\eta}\int_\Gamma \delta(z)\left(\int_0^z \frac{\mathrm{d}}{\mathrm{d}\zeta}(U^\varepsilon\Phi_\varepsilon)(p,\zeta)\,\mathrm{d}\zeta\right)|1+Z(\varepsilon z,p)|\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{dz}\right|\\ &\le (1+\tilde{c}C_{\delta,\mathrm{int}})\int_\Gamma |\gamma_0(u^\varepsilon)-\gamma_0(\tilde{u})|\,|\gamma_0(\varphi)|\,\mathrm{d}\mathcal{H}^{n-1}\\ &+ (1+\tilde{c}\eta)\int_{-\eta}^{\eta}\delta(z)\int_\Gamma\int_{-\eta}^{\eta}\left|\frac{\mathrm{d}}{\mathrm{d}\zeta}(U^\varepsilon\Phi_\varepsilon)(p,\zeta)\right|\mathrm{d}\zeta\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{dz}.\end{aligned} \tag{4.33}$$

By the compactness of the trace operator ([12, Theorem 3.8.5, p. 167], [32, Theorem 6.2, p. 103]), as $u^\varepsilon \rightharpoonup \tilde{u}$ in $H^1(\Omega^*)$, we have $\gamma_0(u^\varepsilon)$ converges strongly to $\gamma_0(\tilde{u})$ in $L^2(\Gamma)$. Hence, the first term on the right-hand side converges to zero. For the second term, let us use the notation

$$\|f\|_{\eta,\delta} := \|f\|_{L^2(\Gamma\times(-\eta,\eta),\delta)}, \quad \|f\|_{\mathbb{R},\delta} := \|f\|_{L^2(\Gamma\times\mathbb{R},\delta)}.$$

Then, using the monotonicity of $\delta(|s|)$, and letting $C_\eta := \int_{-\eta}^{\eta}\delta(s)\,\mathrm{ds}$, we find that

$$\begin{aligned}
&\int_{-\eta}^{\eta}\delta(z)\int_\Gamma\int_{-\eta}^{\eta}\frac{\delta(\zeta)}{\delta(\zeta)}\left|\frac{\mathrm{d}}{\mathrm{d}\zeta}(U^\varepsilon\Phi_\varepsilon)(p,\zeta)\right|\mathrm{d}\zeta\ \mathrm{d}\mathcal{H}^{n-1}\ \mathrm{dz}\\
&\le \frac{C_\eta}{\delta(\eta)}\int_\Gamma\int_{-\eta}^{\eta}\delta(\zeta)(|\partial_\zeta U^\varepsilon(p,\zeta)\Phi_\varepsilon(p,\zeta)| + |U^\varepsilon(p,\zeta)\partial_\zeta\Phi_\varepsilon(p,\zeta)|)\,\mathrm{d}\zeta\ \mathrm{d}\mathcal{H}^{n-1}\\
&\le C\|\partial_z U^\varepsilon\|_{\eta,\delta}\|\Phi_\varepsilon\|_{\eta,\delta} + C\|U^\varepsilon\|_{\eta,\delta}\|\partial_z\Phi_\varepsilon\|_{\eta,\delta}\\
&\le C\|\varphi\|_{H^1(\Omega)}\|\chi_{(-\eta_{\varepsilon,k},\eta_{\varepsilon,k})}(z)\partial_z U^\varepsilon\|_{\mathbb{R},\delta} + C\|u^\varepsilon\|_{0,\delta_\varepsilon}\|\chi_{(-\eta_{\varepsilon,k},\eta_{\varepsilon,k})}(z)\partial_z\Phi_\varepsilon\|_{\mathbb{R},\delta}\\
&\to 0 \text{ as } \varepsilon\to 0,
\end{aligned}\tag{4.34}$$

where we used (4.13), (4.15), (4.16), and (4.17). This establishes (4.32).

We note that

$$\begin{aligned}
&\int_{-\eta}^{\eta}\int_\Gamma\delta(z)\gamma_0(\tilde{u})(p)\gamma_0(\varphi)(p)\,|1+Z(\varepsilon z,p)|\,\mathrm{d}\mathcal{H}^{n-1}\ \mathrm{dz}\\
\to &\int_{-\eta}^{\eta}\int_\Gamma\delta(z)\gamma_0(\tilde{u})(p)\gamma_0(\varphi)(p)\,\mathrm{d}\mathcal{H}^{n-1}\ \mathrm{dz} = C_\eta\int_\Gamma\gamma_0(\tilde{u})\gamma_0(\varphi)\,\mathrm{d}\mathcal{H}^{n-1}\ \text{ as } \varepsilon\to 0.
\end{aligned}$$

Together with the weak convergence of U^ε to $\overline{u}$ in $L^2(\Gamma\times(-\eta,\eta),\delta)$ (see (4.31)), we have that the left-hand side of (4.32) also converges to

$$\int_{-\eta}^{\eta}\int_\Gamma\delta(z)(\overline{u}(p)-\gamma_0(\tilde{u})(p))\gamma_0(\varphi)(p)\,\mathrm{d}\mathcal{H}^{n-1}\ \mathrm{dz} = C_\eta\int_\Gamma(\overline{u}-\gamma_0(\tilde{u}))\gamma_0(\varphi)\,\mathrm{d}\mathcal{H}^{n-1},$$

as $\varepsilon\to 0$. Hence, by (4.32), we have

$$0 = C_\eta\int_\Gamma(\overline{u}-\gamma_0(\tilde{u}))\gamma_0(\varphi)\,\mathrm{d}\mathcal{H}^{n-1}\ \text{ for all } \varphi\in H^1(\Omega),$$

which implies that $\overline{u} = \gamma_0(\tilde{u})$ a.e. on Γ.

REMARK 4.1. *For the identification* (4.23)*, we restricted ourselves to the tubular neighborhood* $\mathrm{Tub}^{\varepsilon\eta}(\Gamma)$*, so that in the* (p,z) *coordinate system, we have* $z\in(-\eta,\eta)$*. This is needed in* (4.34)*, where we estimate the fraction* $\frac{1}{\delta(\zeta)}$ *from above. Otherwise, we cannot deduce that the right-hand side of* (4.33) *converges to zero as* $\varepsilon\to 0$*.*

REMARK 4.2. *For the double-obstacle regularization, we choose* $X^\varepsilon = \mathrm{Tub}^{\varepsilon\frac{\pi}{2}}(\Gamma)$*, i.e.,* $k = 1$ *and* $\eta = \frac{\pi}{2}$*, and we will have* U^ε *converging weakly to* $\overline{u}(p)$ *in* $L^2(\Gamma\times(-\frac{\pi}{2},\frac{\pi}{2}),\delta)$*. A similar argument with the above elements will show* (4.22)*, and we restrict to* $\mathrm{Tub}^{\varepsilon\frac{\pi}{4}}(\Gamma)$ *in order to show* (4.23)*.*

4.3.2. Compactness in $H^1(\Omega,\delta_\varepsilon)$. Choose $0<\eta<(\tilde{c})^{-1}$ and the diffeomorphism Θ^η as defined in (2.3). Let $X^\varepsilon = \mathrm{Tub}^{\varepsilon^k\eta}(\Gamma)$ for $0<k<1$. By (4.9), and the uniform boundedness of $\{v^\varepsilon\}_{\varepsilon\in(0,1]}$ in $H^1(\Omega,\delta_\varepsilon)$,

$$\left|\int_{\Omega\setminus X^\varepsilon}\delta_\varepsilon v^\varepsilon g^{Ec}\,\mathrm{dx}\right| \le \|v^\varepsilon\|_{0,\delta_\varepsilon}\|\delta_\varepsilon\|^{\frac{1}{2}}_{L^\infty(\Omega\setminus X^\varepsilon)}\|g^{Ec}\|_{L^2(\Omega)} \to 0 \text{ as } \varepsilon\to 0$$

and

$$\left|\int_{\Omega\setminus X^\varepsilon} \delta_\varepsilon b^{Ec} v^\varepsilon \varphi \,\mathrm{dx}\right| \leq \|b^{Ec}\|_{L^\infty(\Omega)} \|v^\varepsilon\|_{0,\delta_\varepsilon} \|\delta_\varepsilon\|^{\frac{1}{2}}_{L^\infty(\Omega\setminus X^\varepsilon)} \|\varphi\|_{L^2(\Omega)} \to 0 \text{ as } \varepsilon \to 0$$

for any $\varphi \in H^1(\Omega)$. Similarly, by (4.9), and the uniform boundedness of $\{v^\varepsilon\}_{\varepsilon\in(0,1]}$ in $H^1(\Omega, \delta_\varepsilon)$, for any $\psi^{Ec} \in H^1(\Omega)$ constructed from $\psi \in H^1(\Gamma)$ as in Corollary 3.1,

$$\begin{aligned}&\left|\int_{\Omega\setminus X^\varepsilon} \delta_\varepsilon \mathcal{B}^{Ec} \nabla v^\varepsilon \cdot \nabla \psi^{Ec} \,\mathrm{dx}\right| \\ &\leq \|\mathcal{B}^{Ec}\|_{L^\infty(\Omega)} \|\nabla v^\varepsilon\|_{0,\delta_\varepsilon} \|\delta_\varepsilon\|^{\frac{1}{2}}_{L^\infty(\Omega\setminus X^\varepsilon)} \|\nabla \psi^{Ec}\|_{L^2(\Omega)} \to 0 \text{ as } \varepsilon \to 0.\end{aligned}$$

Hence, it is sufficient to restrict our attention to X^ε. Then, invoking the (p, z) coordinate system, and using (3.29) and (3.30),

$$\begin{aligned} C \geq \|v^\varepsilon\|^2_{1,\delta_\varepsilon} &\geq \int_{X^\varepsilon} \delta_\varepsilon (|v^\varepsilon|^2 + |\nabla v^\varepsilon|^2) \,\mathrm{dx} \\ &\geq (1 - \tilde{c}\eta) \int_{-\eta_{\varepsilon,k}}^{\eta_{\varepsilon,k}} \int_\Gamma \delta(z) \left(|V^\varepsilon|^2 + \frac{1}{\varepsilon^2} |\partial_z V^\varepsilon|^2 + |\nabla_{\Gamma^{\varepsilon z}} V^\varepsilon|^2 \right) (p, z) \,\mathrm{d}\mathcal{H}^{n-1} \,\mathrm{dz}. \end{aligned} \tag{4.35}$$

Hence, there exists a function $\overline{v} \in L^2(\Gamma \times \mathbb{R}, \delta)$ and a vector-valued function $\boldsymbol{Q} \in (L^2(\Gamma \times \mathbb{R}, \delta))^n$ such that, as $\varepsilon \to 0$,

$$\begin{aligned} \chi_{(-\eta_{\varepsilon,k},\eta_{\varepsilon,k})}(z) \partial_z V^\varepsilon(p, z) &\to 0 && \text{in } L^2(\Gamma \times \mathbb{R}, \delta), \\ \chi_{(-\eta_{\varepsilon,k},\eta_{\varepsilon,k})}(z) V^\varepsilon(p, z) &\rightharpoonup \overline{v}(p, z) && \text{in } L^2(\Gamma \times \mathbb{R}, \delta), \\ \chi_{(-\eta_{\varepsilon,k},\eta_{\varepsilon,k})}(z) \nabla_{\Gamma^{\varepsilon z}} V^\varepsilon(p, z) &\rightharpoonup \boldsymbol{Q}(p, z) && \text{in } (L^2(\Gamma \times \mathbb{R}, \delta))^n. \end{aligned}$$

Moreover, we can deduce, via a similar argument to the derivation of (4.21), that $\partial_z \overline{v} = 0$ and $\overline{v} = \overline{v}(p)$ a.e. on M.

We claim that $\boldsymbol{Q} = \nabla_\Gamma \overline{v}(p)$. As Γ is a C^3 compact hypersurface, and so for a finite open cover $\{W_i \cap \Gamma\}_{i=1}^N, W_i \subset \mathbb{R}^n$ of Γ, we have local regular parameterizations $\alpha_i : \mathcal{S}_i \to W_i \cap \Gamma$. Let $\{\mu_i\}_{i=1}^N$ denote a partition of unity subordinate to $\{W_i\}_{i=1}^N$. Take $Y \in C_c^\infty(\Gamma \times \mathbb{R})$, and for any $1 \leq r \leq n$, let $\tilde{Q}_r(s, z)$, $\tilde{V}^\varepsilon(s, z)$, $\tilde{Y}(s, z)$, $\tilde{\nabla}_{\Gamma^{\varepsilon z}} \tilde{V}^\varepsilon(s, z)$, $\tilde{v}$ denote the representation of Q_r, V^ε, Y, $\nabla_{\Gamma^{\varepsilon z}} V^\varepsilon$, and $\overline{v}$ in the (s, z) coordinate system. Then,

$$\int_\mathbb{R} \int_\Gamma \delta(z) Y(p, z) \,\mathrm{d}\mathcal{H}^{n-1} \,\mathrm{dz} = \sum_{i=1}^N \int_\mathbb{R} \int_{\mathcal{S}_i} \mu_i(s) \delta(z) \tilde{Y}(s, z) \,|\det J_{i,0}(s)| \,\mathrm{ds} \,\mathrm{dz}. \tag{4.36}$$

Moreover, since Γ is C^3, the normal ν and the components g_0^{ij} of the inverse of the metric tensor $\mathcal{G}_0$ on Γ both belong to the class C^2. From (3.17), we infer that the components g_ε^{ij} of the inverse of the metric tensor $\mathcal{G}_{\varepsilon z}$ are C^1 functions. In particular,

$$\partial_{s_j} \left(g_\varepsilon^{jr} \mu_i \tilde{Y} \,|\det J_{i,0}|\, \partial_{s_r} G_\varepsilon \right) \to \partial_{s_j} \left(g_0^{jr} \mu_i \tilde{Y} \,|\det J_{i,0}|\, \partial_{s_r} \alpha_i \right) \text{ strongly in } L^2(\mathcal{S}_i \times \mathbb{R}, \delta)$$

as $\varepsilon \to 0$ for $1 \leq i \leq N$. So, for any $1 \leq r \leq n$, we have by the representation (3.18), (3.14), and integration by parts (suppressing the dependence on s and z), and the weak convergence of $\tilde{V}^\varepsilon$ to $\tilde{v}$,

$$
\begin{aligned}
&\int_{-\eta_{\varepsilon,k}}^{\eta_{\varepsilon,k}} \int_{\mathcal{S}_i} \mu_i \delta(z) (\tilde{\nabla}_{\Gamma^{\varepsilon z}} \tilde{V}^\varepsilon)_r \tilde{Y} \left|\det J_{i,0}\right| \mathrm{ds}\,\mathrm{dz} \\
&= - \int_{-\eta_{\varepsilon,k}}^{\eta_{\varepsilon,k}} \int_{\mathcal{S}_i} \delta \sum_{j=1}^{n-1} \partial_{s_j} \left(g_\varepsilon^{jr} \mu_i \tilde{Y} \left|\det J_{i,0}\right| \partial_{s_r} G_\varepsilon \right) \tilde{V}^\varepsilon \,\mathrm{ds}\,\mathrm{dz} \\
&\to \int_{\mathbb{R}} \int_{\mathcal{S}_i} \delta \sum_{j=1}^{n-1} \partial_{s_j} \left(g_0^{jr} \mu_i \tilde{Y} \left|\det J_{i,0}\right| \partial_{s_r} \alpha_i \right) \tilde{v} \,\mathrm{ds}\,\mathrm{dz} \\
&= \int_{\mathbb{R}} \int_{\mathcal{S}_i} \mu_i \delta \sum_{j=1}^{n} g_0^{jr} \partial_{s_j} \tilde{v} \partial_{s_r} \alpha_i \tilde{Y} \left|\det J_{i,0}\right| \mathrm{ds}\,\mathrm{dz} \text{ as } \varepsilon \to 0.
\end{aligned} \tag{4.37}
$$

Summing from $i = 1$ to N and transforming back to the (p, z) coordinate system using (4.36) leads to

$$
\begin{aligned}
&\int_{\mathbb{R}} \int_{\Gamma} \chi_{(-\eta_{\varepsilon,k},\eta_{\varepsilon,k})}(z) \delta(z) (\nabla_{\Gamma^{\varepsilon z}} V^\varepsilon)_r(p,z) Y(p,z)\, \mathrm{d}\mathcal{H}^{n-1}\, \mathrm{dz} \\
&\to \int_{\mathbb{R}} \int_{\Gamma} \delta(z) (\nabla_\Gamma \overline{v})_r(p) Y(p,z)\, \mathrm{d}\mathcal{H}^{n-1}\, \mathrm{dz} \text{ as } \varepsilon \to 0.
\end{aligned}
$$

On the other hand, by the weak convergence of $\chi_{(-\eta_{\varepsilon,k},\eta_{\varepsilon,k})}(z)(\nabla_{\Gamma^{\varepsilon z}} V^\varepsilon)_r(p,z)$ to $Q_r(p,z)$ in $L^2(\Gamma \times \mathbb{R}, \delta)$ we have

$$
\int_{\mathbb{R}} \int_{\Gamma} \delta(z) (Q_r Y)(p,z)\, \mathrm{d}\mathcal{H}^{n-1}\, \mathrm{dz} = \int_{\mathbb{R}} \int_{\Gamma} \delta(z) (\nabla_\Gamma \overline{v})_r(p) Y(p,z)\, \mathrm{d}\mathcal{H}^{n-1}\, \mathrm{dz},
$$

and the claim follows from the arbitrariness of Y.

Let g^{Ec}, b^{Ec}, and $\mathcal{B}^{Ec}$ denote the extensions of the data g, b, and $\mathcal{B}$ as mentioned in Assumption 2.4. We note that by the definition of the extensions,

$$
g^{Ec}(x) = g^e(x) = g(p(x)), \quad b^{Ec}(x) = b^e(x) = b(p(x)), \quad \mathcal{B}^{Ec}(x) = \mathcal{B}^e(x) = \mathcal{B}(p(x)),
$$

for any $x \in X^\varepsilon$. To show (2.20) it suffices to show the strong convergence in $L^2(\Gamma \times \mathbb{R}, \delta)$:

$$
\chi_{(-\eta_{\varepsilon,k},\eta_{\varepsilon,k})}(z) g(p) \left|1 + Z(\varepsilon z, p)\right|^{\frac{1}{2}} \to g(p) \text{ as } \varepsilon \to 0. \tag{4.38}
$$

We use the method in Section 4.3.1. Note that (4.38) holds pointwise for a.e. $p \in \Gamma$ and $z \in \mathbb{R}$. Moreover, by Corollary 3.1, we have uniform boundedness of the norm:

$$
\int_{X^\varepsilon} \delta_\varepsilon \left|g^e\right|^2 \mathrm{dx} \le C \|g\|_{L^2(\Gamma)}^2.
$$

Furthermore, we may appeal to $(3.36)_1$ to show norm convergence, since the proof also works for the extension of a function in $L^2(\Gamma)$. Then, arguing as in Section 4.3.1 yields the required strong convergence. Using the weak convergence $\chi_{(-\eta_{\varepsilon,k},\eta_{\varepsilon,k})}(z)V^\varepsilon$ to $\overline{v}$ in $L^2(\Gamma \times \mathbb{R}, \delta)$, we obtain by the weak-strong product convergence,

$$
\begin{aligned}
\int_{X^\varepsilon} \delta_\varepsilon v^\varepsilon g^e \,\mathrm{dx} &= \int_{\mathbb{R}} \int_{\Gamma} \delta(z) \chi_{(-\eta_{\varepsilon,k},\eta_{\varepsilon,k})}(z) V^\varepsilon(p,z) g(p) \left|1 + Z(\varepsilon z, p)\right| \mathrm{d}\mathcal{H}^{n-1}\, \mathrm{dz} \\
&\to \int_{\mathbb{R}} \int_{\Gamma} \delta(z) \overline{v}(p) g(p)\, \mathrm{d}\mathcal{H}^{n-1}\, \mathrm{dz} = \int_\Gamma \overline{v} g \,\mathrm{d}\mathcal{H}^{n-1} \text{ as } \varepsilon \to 0.
\end{aligned}
$$

We note that the proof of (2.21) is similar to that of (4.22). More precisely, we replace u^ε with v^ε and $\overline{u}$ with $\overline{v}$. Using that $b^{Ec}(x) = b(p(x))$ for all $x \in X^\varepsilon$, and (4.29), we can deduce the strong convergence

$$b^{Ec}(p + \varepsilon z \nu(p)) \,|1 + Z(\varepsilon z, p)|^{\frac{1}{2}} = b(p)\,|1 + Z(\varepsilon z, p)|^{\frac{1}{2}} \to b(p) \text{ in } L^\infty(\Gamma \times \mathbb{R}, \delta).$$

Together with (4.27) and the weak convergence of $\chi_{(-\eta_{\varepsilon,k},\eta_{\varepsilon,k})}(z)V^\varepsilon$ to $\overline{v}$ in $L^2(\Gamma \times \mathbb{R}, \delta)$, we obtain by the weak-strong product convergence,

$$\begin{aligned}\int_{X^\varepsilon} \delta_\varepsilon b^{Ec} v^\varepsilon \varphi \,\mathrm{dx} &= \int_{\mathbb{R}} \int_\Gamma \delta(z)\chi_{(-\eta_{\varepsilon,k},\eta_{\varepsilon,k})}(z)(V^\varepsilon \Phi_\varepsilon)(p,z) b(p)\,|1 + Z(\varepsilon z, p)| \,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{dz} \\ &\to \int_{\mathbb{R}} \int_\Gamma \delta(z)\overline{v}(p)\gamma_0(\varphi)(p) b(p)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{dz} = \int_\Gamma b\overline{v}\gamma_0(\varphi)\,\mathrm{d}\mathcal{H}^{n-1}\end{aligned}$$

as $\varepsilon \to 0$.

Next, recall that for a given $\psi \in H^1(\Gamma)$ with extension $\psi^{Ec} \in H^1(\Omega)$ as constructed in Corollary 3.1, within the scaled tubular neighborhood X^ε, ψ^{Ec} is simply the constant extension ψ^e in the normal direction off Γ, as defined in (2.6). The same is true for the extension of the data $\mathcal{B}$, i.e., $\mathcal{B}^{Ec}(x) = \mathcal{B}(p(x))$ for all $x \in X^\varepsilon$. By Corollary 3.1, we have that $\psi^e \in H^1(X^\varepsilon)$, and by (3.31) we have $\nabla\psi^e(x)\cdot\nu(p(x)) = 0$. Moreover, let Ψ_ε denote the representation of ψ in the (p,z) coordinate system. Then, by (3.23) and (3.24), we see that, for $x = p + \varepsilon z\nu(p) \in X^\varepsilon$,

$$0 = \nabla\psi^e(x)\cdot\nu(p(x)) = \frac{1}{\varepsilon}\partial_z \Psi_\varepsilon(p,z) + \nabla_{\Gamma^{\varepsilon z}}\Psi_\varepsilon(p,z)\cdot\nu(p) = \frac{1}{\varepsilon}\partial_z\Psi_\varepsilon(p,z), \tag{4.39}$$

i.e., $\partial_z\Psi_\varepsilon = 0$. Together with (3.24), we compute that

$$\begin{aligned}&\int_{X^\varepsilon} \delta_\varepsilon \mathcal{B}^e \nabla v^\varepsilon \cdot \nabla\psi^e \,\mathrm{dx} \\ &= \int_{-\eta_{\varepsilon,k}}^{\eta_{\varepsilon,k}} \int_\Gamma \delta(z)\nabla_{\Gamma^{\varepsilon z}} V^\varepsilon \cdot \mathcal{B}(p)^T \nabla_{\Gamma^{\varepsilon z}}\Psi_\varepsilon\,|1 + Z(\varepsilon z, p)| \,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{dz}\,,\end{aligned} \tag{4.40}$$

where $\mathcal{B}^T$ denotes the transpose of $\mathcal{B}$.

With the weak convergence $\chi_{(-\eta_{\varepsilon,k},\eta_{\varepsilon,k})}(z)\nabla_{\Gamma^{\varepsilon z}}V^\varepsilon(p,z)$ to $\nabla_\Gamma\overline{v}(p)$ in $L^2(\Gamma\times\mathbb{R},\delta)$, in order to show

$$\int_{X^\varepsilon} \delta_\varepsilon \mathcal{B}^e \nabla v^\varepsilon \cdot \nabla\psi^e \,\mathrm{dx} \to \int_\Gamma \mathcal{B}\nabla_\Gamma \overline{v}\cdot\nabla_\Gamma\psi\,\mathrm{d}\mathcal{H}^{n-1} \text{ as } \varepsilon \to 0,$$

it is sufficient to show the following strong convergence result in $L^2(\Gamma \times \mathbb{R}, \delta)$: As $\varepsilon \to 0$,

$$\chi_{(-\eta_{\varepsilon,k},\eta_{\varepsilon,k})}(z)\mathcal{B}^T(p)\nabla_{\Gamma^{\varepsilon z}}\Psi_\varepsilon(p,z)\,|1 + Z(\varepsilon z,p)|^{\frac{1}{2}} \to \mathcal{B}^T(p)\nabla_\Gamma\psi(p). \tag{4.41}$$

By (3.13), (3.6), and (3.25), we have that (4.41) holds pointwise for almost every $p \in \Gamma$, $z \in \mathbb{R}$. By (4.39), (3.23), and a slight modification to the proof of Corollary 3.1, we have uniform boundedness of the norm:

$$\begin{aligned}&\int_{\mathbb{R}}\int_\Gamma \delta(z)\chi_{(-\eta_{\varepsilon,k},\eta_{\varepsilon,k})}(z)\left|\mathcal{B}^T\nabla_{\Gamma^{\varepsilon z}}\Psi_\varepsilon\right|^2 |1 + Z(\varepsilon z,p)|\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{dz} \\ &= \int_{X^\varepsilon} \delta_\varepsilon \left|(\mathcal{B}^T)^e\nabla\psi^e\right|^2 \mathrm{dx} \le C\|\mathcal{B}\|^2_{L^\infty(\Gamma)}\|\nabla_\Gamma\psi\|^2_{L^2(\Gamma)}.\end{aligned}$$

Last, by a slight modification of the proof of $(3.35)_2$ and using that

$$\int_{\Omega\setminus X^\varepsilon} \delta_\varepsilon \left|(\mathcal{B}^{Ec})^T \nabla\psi^{Ec}\right|^2 \mathrm{dx} \le \|\delta_\varepsilon\|_{L^\infty(\Omega\setminus X^\varepsilon)} \|\mathcal{B}^{Ec}\|^2_{L^\infty(\Omega)} \|\nabla\psi^{Ec}\|^2_{L^2(\Omega)} \to 0 \text{ as } \varepsilon\to 0,$$

we can deduce the norm convergence: As $\varepsilon \to 0$,

$$\int_{-\eta_{\varepsilon,k}}^{\eta_{\varepsilon,k}} \int_\Gamma \delta(z) \left|\mathcal{B}^T(p)\nabla_{\Gamma^{\varepsilon z}} \Psi_\varepsilon(p,z)\right|^2 |1+Z(\varepsilon z,p)| \,\mathrm{d}\mathcal{H}^{n-1} \,\mathrm{dz} = \int_{X^\varepsilon} \delta_\varepsilon \left|\mathcal{B}^T \nabla\psi^e\right|^2 \mathrm{dx}$$
$$\to \int_\Gamma \left|\mathcal{B}^T \nabla_\Gamma \psi\right|^2 \mathrm{d}\mathcal{H}^{n-1} = \int_{\mathbb{R}} \int_\Gamma \delta(z) \left|\mathcal{B}^T(p)(\nabla_\Gamma\psi)(p)\right|^2 \mathrm{d}\mathcal{H}^{n-1} \,\mathrm{dz}.$$

Then, arguing as in the proof of (4.27), the required strong convergence (4.41) follows.

COROLLARY 4.1. *Suppose that Assumptions 2.1 and 2.6 are satisfied. Let* $\varphi \in H^1(\Omega)$ *and let* $g^{Ec} \in L^2(\Omega)$ *denote the extension of the data* $g \in L^2(\Gamma)$ *as mentioned in Assumption 2.4. Then, as* $\varepsilon \to 0$,

$$\int_\Omega \delta_\varepsilon g^{Ec} \varphi \,\mathrm{dx} \to \int_\Gamma g\gamma_0(\varphi) \,\mathrm{d}\mathcal{H}^{n-1}. \tag{4.42}$$

Proof. Choose $0 < \eta < (\tilde{c})^{-1}$ and the diffeomorphism Θ^η as defined in (2.3). Let $X^\varepsilon = \mathrm{Tub}^{\varepsilon^k \eta}(\Gamma)$ for $0 < k < 1$. By (4.9),

$$\left|\int_{\Omega\setminus X^\varepsilon} \delta_\varepsilon \varphi g^{Ec} \,\mathrm{dx}\right| \le \|\delta_\varepsilon\|_{L^\infty(\Omega\setminus X^\varepsilon)} \|g^{Ec}\|_{L^2(\Omega)} \|\varphi\|_{L^2(\Omega)} \to 0 \text{ as } \varepsilon \to 0.$$

Hence, it is sufficient to restrict our attention to X^ε. Invoking the (p,z) coordinate system, we note that by definition $g^{Ec}(x) = g^e(x) = g(p(x))$ for all $x \in X^\varepsilon$. Moreover, (4.24), (4.25), (4.26), and (4.27) are still valid. Hence, by (4.27) and (4.29), we have

$$\int_{X^\varepsilon} \delta_\varepsilon g\varphi \,\mathrm{dx} = \int_{-\eta_{\varepsilon,k}}^{\eta_{\varepsilon,k}} \int_\Gamma \delta(z) g(p) \Phi_\varepsilon(p,z) \,|1+Z(\varepsilon z,p)| \,\mathrm{d}\mathcal{H}^{n-1} \,\mathrm{dz}$$
$$\to \int_{\mathbb{R}} \int_\Gamma \delta(z) g(p) \gamma_0(\varphi)(p) \,\mathrm{d}\mathcal{H}^{n-1} \,\mathrm{dz} = \int_\Gamma g\gamma_0(\varphi) \,\mathrm{d}\mathcal{H}^{n-1} \text{ as } \varepsilon \to 0.$$

∎

4.4. Weak convergence. Let $\varphi \in H^1(\Omega^*)$ and $\psi \in H^1(\Gamma)$ be arbitrary. Let $\varphi^{Er} \in H^1(\Omega)$ denote the extension of φ to Ω by the extension theorem [16, Theorem 1, p. 254], and let $\psi^{Ec} \in H^1(\Omega)$ denote the extension of ψ to Ω as outlined in Corollary 3.1.

By $\xi_\varepsilon \le 1$, Lemma 3.5, and Lemma 3.3, we see that $\varphi^{Er} \in \mathcal{V}_\varepsilon$ and $\psi^{Ec} \in H^1(\Omega, \delta_\varepsilon)$ for all $\varepsilon \in (0,1]$. Moreover, there exists a constant $C > 0$, independent of ε, such that

$$\|\varphi^{Er}\|_{\mathcal{V}_\varepsilon} \le C\|\varphi\|_{H^1(\Omega^*)}, \quad \|\psi^{Ec}\|_{1,\delta_\varepsilon} \le C\|\psi\|_{H^1(\Gamma)}.$$

Thus, we may test with φ^{Er} and ψ^{Ec} in the weak formulation for (CDD).

For $\varepsilon \in (0,1]$, let $(u^\varepsilon, v^\varepsilon) \in \mathcal{X}_\varepsilon$ denote the unique weak solution to (CDD). Then, they satisfy

$$\begin{aligned}&\int_\Omega \xi_\varepsilon \mathcal{A}^{Ea} \nabla u^\varepsilon \cdot \nabla\varphi^{Er} + \xi_\varepsilon a^{Ea} u^\varepsilon \varphi^{Er} + \delta_\varepsilon \mathcal{B}^{Ec} \nabla v^\varepsilon \cdot \nabla\psi^{Ec} + \delta_\varepsilon b^{Ec} v^\varepsilon \psi^{Ec} \,\mathrm{dx} \\ &+ \int_\Omega K\delta_\varepsilon (v^\varepsilon - u^\varepsilon)(\psi^{Ec} - \varphi^{Er}) \,\mathrm{dx} - \int_\Omega \xi_\varepsilon f^{Ea} \varphi^{Er} + \delta_\varepsilon \beta g^{Ec} \psi^{Ec} \,\mathrm{dx} = 0.\end{aligned} \tag{4.43}$$

We analyze the bulk and surface terms separately. From (2.18), we have

$$\|u^\varepsilon\|_{1,\xi_\varepsilon}^2 + \|u^\varepsilon\|_{0,\delta_\varepsilon}^2 \le \|(u^\varepsilon, v^\varepsilon)\|_{\mathcal{X}_\varepsilon}^2 \le C(\|f^{Ea}\|_{L^2(\Omega)}^2 + \|g\|_{L^2(\Gamma)}^2) =: C_{f,g}, \tag{4.44}$$

where $C_{f,g}$ is independent of ε. Then, by (2.19) there exists $\tilde{u} \in H^1(\Omega^*)$ such that along a subsequence

$$u^\varepsilon|_{\Omega^*} \rightharpoonup \tilde{u} \text{ in } H^1(\Omega^*) \text{ as } \varepsilon \to 0, \tag{4.45}$$

$$\int_\Omega \delta_\varepsilon u^\varepsilon (\psi^{Ec} - \varphi^{Er})\,\mathrm{dx} \to \int_\Gamma \gamma_0(\tilde{u})(\psi - \gamma_0(\varphi))\,\mathrm{d}\mathcal{H}^{n-1} \text{ as } \varepsilon \to 0. \tag{4.46}$$

By Hölder's inequality, $f^{Ea}\varphi^{Er} \in L^1(\Omega)$, and so by Lemma 3.4, we see that

$$\int_\Omega \xi_\varepsilon f^{Ea}\varphi^{Er}\,\mathrm{dx} = \int_{\Omega^*} \xi_\varepsilon f\varphi\,\mathrm{dx} + \int_{\Omega\setminus\Omega^*} \xi_\varepsilon f^{Ea}\varphi^{Er}\,\mathrm{dx} \to \int_{\Omega^*} f\varphi\,\mathrm{dx} \text{ as } \varepsilon \to 0. \tag{4.47}$$

By the Cauchy–Schwarz inequality, Lemma 3.4, (4.44), and the fact that $(1-\xi_\varepsilon) \le \frac{1}{2} \le \xi_\varepsilon$ in Ω^*, we have

$$\begin{aligned} &\int_{\Omega^*} (1-\xi_\varepsilon)\,|\nabla u^\varepsilon|\,|\nabla\varphi|\,\mathrm{dx} \le \left(\int_{\Omega^*} \tfrac{1}{2}|\nabla u^\varepsilon|^2\,\mathrm{dx}\right)^{\frac{1}{2}} \left(\int_{\Omega^*} (1-\xi_\varepsilon)\,|\nabla\varphi|^2\,\mathrm{dx}\right)^{\frac{1}{2}} \\ &\le \|\nabla u^\varepsilon\|_{0,\xi_\varepsilon} \left(\int_{\Omega^*} (1-\xi_\varepsilon)\,|\nabla\varphi|^2\,\mathrm{dx}\right)^{\frac{1}{2}} \le C_{f,g}\left(\int_{\Omega^*} (1-\xi_\varepsilon)\,|\nabla\varphi|^2\,\mathrm{dx}\right)^{\frac{1}{2}} \to 0 \end{aligned} \tag{4.48}$$

and

$$\begin{aligned} \left|\int_{\Omega\setminus\Omega^*} \xi_\varepsilon \nabla u^\varepsilon \cdot (\mathcal{A}^{Ea})^T \nabla\varphi^{Er}\,\mathrm{dx}\right| &\le \|\nabla u^\varepsilon\|_{L^2(\Omega\setminus\Omega^*,\xi_\varepsilon)} C(\mathcal{A}^{Ea}) \|\nabla\varphi^{Er}\|_{L^2(\Omega\setminus\Omega^*,\xi_\varepsilon)} \\ &\le C_{f,g} C(\mathcal{A}^{Ea}) \|\nabla\varphi^{Er}\|_{L^2(\Omega\setminus\Omega^*,\xi_\varepsilon)} \to 0 \end{aligned}$$

as $\varepsilon \to 0$. Thus, together with (4.45), we obtain

$$\begin{aligned} &\left|\int_\Omega \xi_\varepsilon \mathcal{A}^{Ea}\nabla u^\varepsilon \cdot \nabla\varphi^{Er}\,\mathrm{dx} - \int_{\Omega^*} \mathcal{A}\nabla\tilde{u}\cdot\nabla\varphi\,\mathrm{dx}\right| \\ &\le \left|\int_{\Omega^*} \nabla(u^\varepsilon - \tilde{u})\cdot\mathcal{A}^T\nabla\varphi\,\mathrm{dx}\right| + \|\mathcal{A}\|_{L^\infty(\Omega^*)} \int_{\Omega^*} (1-\xi_\varepsilon)\,|\nabla u^\varepsilon|\,|\nabla\varphi|\,\mathrm{dx} \\ &+ \left|\int_{\Omega\setminus\Omega^*} \xi_\varepsilon \nabla u^\varepsilon \cdot (\mathcal{A}^{Ea})^T\nabla\varphi^{Er}\,\mathrm{dx}\right| \to 0 \text{ as } \varepsilon \to 0. \end{aligned} \tag{4.49}$$

An argument similar to (4.49), using (4.48), and (4.45) will yield that

$$\int_\Omega \xi_\varepsilon a^{Ea} u^\varepsilon \varphi^{Er}\,\mathrm{dx} \to \int_{\Omega^*} a\tilde{u}\varphi\,\mathrm{dx}. \tag{4.50}$$

Next, by (3.36), we have

$$\begin{aligned} \int_\Omega \delta_\varepsilon g^{Ec}\psi^{Ec}\,\mathrm{dx} &= \frac{1}{2}\int_\Omega \delta_\varepsilon \left(\left|g^{Ec} + \psi^{Ec}\right|^2 - \left|g^{Ec}\right|^2 - \left|\psi^{Ec}\right|^2\right)\mathrm{dx} \\ &\to \frac{1}{2}\int_\Gamma (|g+\psi|^2 - |g|^2 - |\psi|^2)\,\mathrm{d}\mathcal{H}^{n-1} = \int_\Gamma g\psi\,\mathrm{d}\mathcal{H}^{n-1} \text{ as } \varepsilon \to 0. \end{aligned} \tag{4.51}$$

By (2.18), we have

$$\|v^\varepsilon\|_{1,\delta_\varepsilon}^2 \le \|(u^\varepsilon, v^\varepsilon)\|_{\mathcal{X}_\varepsilon}^2 \le C_{f,g},$$

and by Theorem 2.3, (2.21) and (2.22) hold for some function $\overline{v} \in H^1(\Gamma)$. Moreover, using the fact that $\psi^{Ec} - \varphi^{Er} \in H^1(\Omega)$, we obtain from (2.21) with $b^{Ec} = b = 1$,

$$\int_\Omega \delta_\varepsilon v^\varepsilon (\psi^{Ec} - \varphi^{Er}) \, \mathrm{dx} \to \int_\Gamma \overline{v}(\psi - \gamma_0(\varphi)) \, \mathrm{d}\mathcal{H}^{n-1} \text{ as } \varepsilon \to 0. \tag{4.52}$$

So, passing to the limit $\varepsilon \to 0$ in (4.43) leads to

$$\begin{aligned} &\int_{\Omega^*} \mathcal{A}\nabla\tilde{u} \cdot \nabla\varphi + a\tilde{u}\varphi \, \mathrm{dx} + \int_\Gamma \mathcal{B}\nabla_\Gamma \overline{v} \cdot \nabla_\Gamma \psi + b\overline{v}\psi \, \mathrm{d}\mathcal{H}^{n-1} \\ &+ \int_\Gamma K(\overline{v} - \gamma_0(\tilde{u}))(\psi - \gamma_0(\varphi)) \, \mathrm{d}\mathcal{H}^{n-1} - \int_{\Omega^*} f\varphi \, \mathrm{dx} - \int_\Gamma \beta g \psi \, \mathrm{d}\mathcal{H}^{n-1} = 0. \end{aligned}$$

In particular $(\tilde{u}, \overline{v})$ is a weak solution of (CSI). But by the well-posedness of (CSI), we must have that $\tilde{u} = u$ and $\overline{v} = v$ and that the whole sequence converges as $\varepsilon \to 0$.

4.5. Strong convergence. We can choose $\varphi = u \in H^1(\Omega)$ and $\psi = v \in H^1(\Gamma)$, where $(u, v) \in H^1(\Omega^*) \times H^1(\Gamma)$ denote the unique weak solution to (CSI). Then, the extension $u^{Er} \in H^1(\Omega)$ by the extension theorem [16, Theorem 1, p. 254] and $\psi^{Ec} \in H^1(\Omega)$ as constructed in Corollary 3.1 are admissible test functions in the weak formulation of (CDD). Due to the coercivity of the bilinear form (4.3) (see (4.5)) we obtain

$$\begin{aligned} &l_{\mathrm{CDD}}((u^\varepsilon - u^{Er}, v^\varepsilon - v^{Ec})) - a_{\mathrm{CDD}}((u^{Er}, v^{Ec}), (u^\varepsilon - u^{Er}, v^\varepsilon - v^{Ec})) \\ &= a_{\mathrm{CDD}}((u^\varepsilon - u^{Er}, v^\varepsilon - v^{Ec}), (u^\varepsilon - u^{Er}, v^\varepsilon - v^{Ec})) \\ &\geq C(\theta_i, K)\left(\|u^\varepsilon - u^{Er}\|^2_{1,\xi_\varepsilon} + \|u^\varepsilon - u^{Er}\|^2_{0,\delta_\varepsilon} + \|v^\varepsilon - v^{Ec}\|^2_{1,\delta_\varepsilon} \right). \end{aligned} \tag{4.53}$$

We claim that the left-hand side converges to zero as $\varepsilon \to 0$. Indeed, by (4.47) with $\varphi^{Er} = u^{Er}$, (4.50) with $a^{Ea} = a = 1$, and φ^{Er} replaced with f^{Ea}, we have

$$\int_\Omega \xi_\varepsilon f^{Ea}(u^\varepsilon - u^{Er}) \, \mathrm{dx} \to \int_{\Omega^*} f(u - u) \, \mathrm{dx} = 0 \text{ as } \varepsilon \to 0. \tag{4.54}$$

Similarly, by (4.51) with $\psi^{Ec} = v^{Ec}$ and (2.20), we have

$$\int_\Omega \delta_\varepsilon g^{Ec}(v^\varepsilon - v^{Ec}) \, \mathrm{dx} \to \int_\Gamma g(v - v) \, \mathrm{d}\mathcal{H}^{n-1} = 0 \text{ as } \varepsilon \to 0. \tag{4.55}$$

Thus,

$$l_{\mathrm{CDD}}((u^\varepsilon - u^{Er}, v^\varepsilon - v^{Er})) \to 0 \text{ as } \varepsilon \to 0.$$

Meanwhile, using (4.47) with $f^{Ea}\varphi^{Er}$ replaced with $a^{Ea}\left|u^{Er}\right|^2$ and with $\mathcal{A}^{Ea}\nabla u^{Er} \cdot \nabla u^{Er}$, (4.50) with $\varphi^{Er} = u^{Er}$ and (4.49) with $\varphi^{Er} = u^{Er}$, we have

$$\int_\Omega \xi_\varepsilon (\mathcal{A}^{Ea}\nabla u^{Er} \cdot \nabla(u^\varepsilon - u^{Er}) + a^{Ea}u^{Er}(u^\varepsilon - u^{Er})) \, \mathrm{dx} \to 0 \text{ as } \varepsilon \to 0. \tag{4.56}$$

Next, by (3.35) with $f^{Ec} = v^{Ec}$, (2.21) and (2.22) with $\varphi = \psi^{Ec} = v^{Ec}$, we have

$$\int_\Omega \delta_\varepsilon (\mathcal{B}^{Ec}\nabla v^{Ec} \cdot \nabla(v^\varepsilon - v^{Ec}) + b^{Ec}v^{Ec}(v^\varepsilon - v^{Ec})) \, \mathrm{dx} \to 0 \text{ as } \varepsilon \to 0. \tag{4.57}$$

Finally, by (3.40) applied to $f = u^{Er}(2v^{Ec} - u^{Er}) \in W^{1,1}(\Omega)$, (2.19) with $\varphi = v^{Ec} - u^{Er} \in H^1(\Omega)$, $(3.36)_1$ with $f^{Ec} = v^{Ec}$, (2.20) with $g^{Ec} = v^{Ec}$, and (2.21) with $b^{Ec} = 1$ and $\varphi = u^{Er}$, we obtain

$$\begin{aligned}&\int_\Omega \delta_\varepsilon (v^{Ec} - u^{Er})(v^\varepsilon - u^\varepsilon - v^{Ec} + u^{Er}) \,\mathrm{dx} \\ = &\int_\Omega \delta_\varepsilon \Big(u^{Er}(2v^{Ec} - u^{Er}) - u^\varepsilon (v^{Ec} - u^{Er}) - \left|v^{Ec}\right|^2 + v^\varepsilon v^{Ec} - v^\varepsilon u^{Er} \Big) \,\mathrm{dx} \\ \to &\int_\Gamma \gamma_0(u)(2v - \gamma_0(u)) - \gamma_0(u)(v - \gamma_0(u)) - |v|^2 + |v|^2 - v\gamma_0(u) \,\mathrm{d}\mathcal{H}^{n-1} = 0 \text{ as } \varepsilon \to 0.\end{aligned}$$

Thus,

$$a_{\mathrm{CDD}}((u^{Er}, v^{Ec}), (u^\varepsilon - u^{Er}, v^\varepsilon - v^{Ec})) \to 0 \text{ as } \varepsilon \to 0.$$

From (4.53), this implies that

$$\|u^\varepsilon - u^{Er}\|_{1,\xi_\varepsilon}^2 + \|u^\varepsilon - u^{Er}\|_{0,\delta_\varepsilon}^2 + \|v^\varepsilon - v^{Ec}\|_{1,\delta_\varepsilon}^2 \to 0 \text{ as } \varepsilon \to 0, \tag{4.58}$$

which is the first assertion of Theorem 2.5. By assumption, $\xi_\varepsilon \geq \frac{1}{2}$ in Ω^*, and so we see that

$$\|u^\varepsilon\,|_{\Omega^*} - u\|_{H^1(\Omega^*)} \leq 2\|u^\varepsilon - u^{Er}\|_{1,\xi_\varepsilon} \to 0 \text{ as } \varepsilon \to 0.$$

Furthermore, by the triangle inequality, (4.58), and (3.36) with $f^{Ec} = v^{Ec}$, we obtain

$$\begin{aligned}\left|\|v^\varepsilon\|_{1,\delta_\varepsilon} - \|v\|_{H^1(\Gamma)}\right| &\leq \left|\|v^\varepsilon\|_{1,\delta_\varepsilon} - \|v^{Ec}\|_{1,\delta_\varepsilon}\right| + \left|\|v^{Ec}\|_{1,\delta_\varepsilon} - \|v\|_{H^1(\Gamma)}\right| \\ &\leq \|v^\varepsilon - v^{Ec}\|_{1,\delta_\varepsilon} + \left|\|v^{Ec}\|_{1,\delta_\varepsilon} - \|v\|_{H^1(\Gamma)}\right| \to 0 \text{ as } \varepsilon \to 0.\end{aligned}$$

Hence, we obtain the norm convergence

$$\|v^\varepsilon\|_{1,\delta_\varepsilon} \to \|v\|_{H^1(\Gamma)} \text{ as } \varepsilon \to 0.$$

Similarly, by (3.40) with $f = \left|u^{Er}\right|^2$, we have $\|u^{Er}\|_{0,\delta_\varepsilon} \to \|\gamma_0(u)\|_{L^2(\Gamma)}$ as $\varepsilon \to 0$. Together with the triangle inequality and (4.58), we have

$$\|u^\varepsilon\|_{0,\delta_\varepsilon} \to \|\gamma_0(u)\|_{L^2(\Gamma)} \text{ as } \varepsilon \to 0.$$

5. Discussion.

5.1. Regularity of Γ. In the proof of (2.22), we used that the components g_ε^{ij}, $1 \leq i, j \leq n$, of the inverse of the metric tensor $\mathcal{G}_\varepsilon^{-1}$ are C^1 functions. Then, from the definition, we require that Γ is of class C^3.

We point out that using (3.26) and (3.14), we can consider the following splitting in (4.37):

$$(\tilde{\nabla}_{\Gamma^{\varepsilon z}} \tilde{V}^\varepsilon)_r = \sum_{j=1}^{n-1} g_0^{jr} \partial_{s_j} \tilde{V}^\varepsilon \partial_{s_r} \alpha + \varepsilon z g_0^{jr} \partial_{s_j} \tilde{V}^\varepsilon \partial_{s_r} \nu(\alpha) + \mathcal{E}_{jr} \partial_{s_j} \tilde{V}^\varepsilon \partial_{s_r} G_\varepsilon. \tag{5.1}$$

If we have some control over the components $(\partial_{s_j} \tilde{V}^\varepsilon)_{1 \leq j \leq n-1}$, then it is sufficient to apply integration by parts only on the first term on the right-hand side. This in turn implies that we can potentially drop the required regularity of Γ from C^3 to C^2. However, the hypothesis that $\|v^\varepsilon\|_{1,\delta_\varepsilon}$ is bounded uniformly in ε seems not to be sufficient to give any control over the components $(\partial_{s_j} \tilde{V}^\varepsilon)_{1 \leq j \leq n-1}$.

5.2. Comparison with the results in [14]. In a time-independent setting, by choosing $q = 1$, $\rho_\varepsilon(s,z) = \overline{\rho}(z) = \delta(z)$, the function spaces B and X employed in [14] are equivalent to $L^2(\mathrm{Tub}^\varepsilon(\Gamma), \delta_\varepsilon)$ and $H^1(\mathrm{Tub}^\varepsilon(\Gamma), \delta_\varepsilon)$, respectively. Moreover, by comparing with the notation and results in Section 3.2, the results of [14] in our notation are

$$c_\varepsilon(s,z) \to c(s) \in B \iff \|c_\varepsilon - c\|_{L^2(\mathrm{Tub}^\varepsilon(\Gamma),\delta_\varepsilon)} \to 0$$

and

$$c_\varepsilon(s,z) \rightharpoonup c(s) \in X \iff \int_{\mathrm{Tub}^\varepsilon(\Gamma)} \delta_\varepsilon(c_\varepsilon - c)\psi \,\mathrm{dx} \to 0, \int_{\mathrm{Tub}^\varepsilon(\Gamma)} \delta_\varepsilon \nabla(c_\varepsilon - c)\cdot\nabla\psi \,\mathrm{dx} \to 0$$

for $\psi \in H^1(\mathrm{Tub}^\varepsilon(\Gamma), \delta_\varepsilon)$. Thus, in the limit $\varepsilon \to 0$, c_ε converges weakly to a function c defined only on the surface Γ.

We point out that, in the proof of (2.22), it is crucial that (4.39) holds, i.e., the test function ψ is extended constantly in the normal direction. Otherwise, for an arbitrary test function $\lambda \in H^1(\Omega, \delta_\varepsilon)$ with representation Λ_ε in the (p,z) coordinate system, a calculation similar to (4.35) yields that

$$\chi_{(-\eta_{\varepsilon,k},\eta_{\varepsilon,k})}(z)\frac{1}{\varepsilon}\partial_z\Lambda_\varepsilon \text{ is bounded in } L^2(\Gamma\times\mathbb{R},\delta) \text{ for all } \varepsilon \in (0,1].$$

When computing (4.40), we have an additional term of the form

$$\int_{\eta_{\varepsilon,k}}^{\eta_{\varepsilon,k}} \int_\Gamma \delta(z)\frac{1}{\varepsilon^2}\partial_z V^\varepsilon \partial_z \Lambda_\varepsilon\,|1 + Z(\varepsilon z, p)|\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{dz}\,, \tag{5.2}$$

and by (4.35) and the Cauchy–Schwarz inequality, we obtain only the uniform boundedness of (5.2) in ε, and we are unable to show that (5.2) converges to zero as $\varepsilon \to 0$.

However, in [14, proof of Theorem 4.1], to show that the limit c satisfies the correct weak formulation of an advection diffusion surface PDE, the authors considered a test function $\chi \in X$ with $\partial_z\chi = 0$, which is similar to what we do Section 4.3.2.

5.3. Comparison with the results in [9]. In our notation, the object of study in [9] is

$$\begin{aligned} -\nabla\cdot(A\nabla u) + cu &= f \text{ in } \Omega^*,\\ A\nabla u\cdot\nu + bu &= g \text{ on } \Gamma. \end{aligned} \tag{5.3}$$

With the choice $\delta_\varepsilon = |\nabla\xi_\varepsilon|$, the diffuse domain approximation in weak formulation is given as

$$\int_{\Omega_\varepsilon} \xi_\varepsilon(A\nabla u^\varepsilon\cdot\nabla v + cu^\varepsilon v) + |\nabla\xi_\varepsilon| bu^\varepsilon v\,\mathrm{dx} = \int_{\Omega_\varepsilon} \xi_\varepsilon f v + |\nabla\xi_\varepsilon| g v\,\mathrm{dx}$$

for all $v \in H^1(\Omega_\varepsilon, \xi_\varepsilon)$, where Ω_ε is defined in (2.13). Under the assumption that ξ_ε satisfies the following behavior near the boundary $\partial\Omega_\varepsilon$: there exists $\zeta_1, \zeta_2 > 0$ and $\alpha > 0$ such that for all $x \in \Omega_\varepsilon$,

$$\zeta_1\left(\frac{\mathrm{dist}(x,\partial\Omega_\varepsilon)}{\varepsilon}\right)^\alpha \le \xi_\varepsilon(x) \le \zeta_2\left(\frac{\mathrm{dist}(x,\partial\Omega_\varepsilon)}{\varepsilon}\right)^\alpha, \tag{5.4}$$

the authors can show continuous and compact embeddings from $W^{1,p}(\Omega_\varepsilon, \xi_\varepsilon)$ into $L^q(\Omega_\varepsilon, \xi_\varepsilon)$, a trace type inequality as well as a Poincaré type inequality, which then lead to the existence of a unique weak solution $u^\varepsilon \in H^1(\Omega_\varepsilon, \xi_\varepsilon)$. The chief result of [9] is the following error estimate:

$$\|u - u^\varepsilon\|_{W^{1,2}(\Omega_\varepsilon,\xi_\varepsilon)} \le C\varepsilon^{\frac{1}{2}-\frac{1}{p}}, \quad 2 < p \le 2^*_\alpha := \begin{cases} \frac{2(n+\alpha)}{n+\alpha-2} & \text{if } 2 < n+\alpha, \\ \infty & \text{if } 2 \ge n+\alpha, \end{cases} \tag{5.5}$$

where $u \in W^{1,p}(\Omega^*)$, $2 < p \le 2^*_\alpha$ is the weak solution to (5.3), $u^\varepsilon \in H^1(\Omega_\varepsilon, \xi_\varepsilon)$ is the unique weak solution to the diffuse domain approximation, C is independent of ε, and α is the exponent in (5.4). We make the following observations:

- The work in [9] utilizes a double-obstacle regularization and the fact that $|\Gamma_\varepsilon| \le C\varepsilon$ for some constant C to deduce (5.5). In our work, we cover both the double-well and double-obstacle regularizations. Moreover, as we see in the proofs of the technical and main results, the double-well regularization requires more work than the double-obstacle regularization.
- A $W^{1,p}$-solution to the (SI) problem is required to deduce (5.5); this follows in the same spirit as (3.41), where for more regular functions, we are able to deduce a rate of convergence.
- In [9], the regularization δ_ε is chosen to be $\delta_\varepsilon = |\nabla \xi_\varepsilon|$. Our setting is more general, where δ_ε and ξ_ε can be unrelated, as long as (2.11) is satisfied.
- Our work focuses on equations with tangential derivatives on the boundary, whereas [9] focuses on a bulk equation.

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