Example Based Mini-Course on Mathematical Modelling: Brake Squeal

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1 Introduction

Driven by the desire for safety and automobile regulations, there has been a movement from the drum brake system, where braking force is not as predictable, to the disc brake system that still continues to this day. This shift is perhaps most noticeable in the heavy vehicle industry. However, despite the improvements in stopping distance, the open nature of the disc brake system makes it more prone to contamination and brake squeal; caused by unstable resonance of the brake and its components. On account of this, many attempts have been made to understand the dynamics of the brake and the thermoelasticity that underpins this. We shall focus on and attempt to model an experiment by J. R. Barber [1].

In an initial experiment, it was apparent that a large part of the vibration was due to only small areas of the surface being in contact with the pad at any one time. This was partly because of the initial rough nature of the surface at a microscopic level and partly because of the accentuation of the roughness of the surface due to thermal expansion. Thus, in order to model this more simply and to understand the effects of this non-uniform distribution of load and even the subsequent transfer of load, Barber [1] devised a simpler experiment. In this experiment, three cast iron pins of diameter 843 mm were attached to the loading arm to represent roughness of a surface in its most manageable form. The loading arm was then able to press the pins against a wheel that was moving at speeds of between 5 m/s and 35 m/s to simulate braking. The arm itself was able to apply forces of up to 1.5kN to the wheel. In order to measure how the temperature changed in each pin upon contact with the wheel a small device called a thermocouple, which produces a voltage related to the temperature of the device, was placed near the surface of each pin.

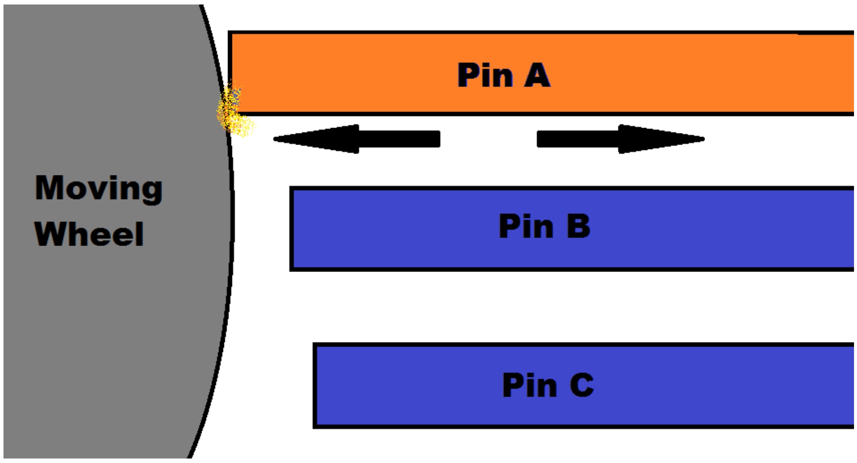


Figure 1: Experimental Setup

After conducting the experiment, Barber made the following explanation to accompany the observations. Initially, the wheel makes contact with one pin, which is at a slightly higher height than the other two pins - pin A. Upon making contact with the pin the temperature observed from the thermocouple associate with this pin rises and as it does so the pin expands. See Figure 1 for an illustration of what is happening. As the temperature at this point is still reltaively low, the thermal expansion vastly outweighs the effects of the wear and the initial irregularity of the surface of a brake pad will be exaggerated; viewed in our case as an increase in the gap in the apparent height of the pins. The thermal expansion of the pin then begins to slow as the temperature of the pin approaches an equilibrium where the heat generated due to friction is balanced by the heat lost by the pin to the surrounding environment. At this point the thermal expansion stops. Perhaps of even greater importance is what happens to the rate of wear on the pin caused by the contact between the wheel and the pin. At a constant force and wheel speed the rate of wear of the pin increases with temperature. This suggests that a time will be reached at which the increase in height of pin A due to thermal expansion is superceded by the decrease in height of the pin due to wear. This provides a sense of stability to the system. However, because of this wear, a change in contact point is inevitable and the load is transferred to a more unstable contact area, pin B. This pin has a lower wear rate and a higher rate of expansion and thus the cycle is started again. At this point, it should also be noted that pin A cools and shrinks making it the smallest of the three pins and implying that there is an ordered nature to the contact areas of the pad. Barber also notes that at high temperatures and velocities the dynamics of this system are even visible to the naked eye.

2 Model

2.1 Derivation

The pin is modelled as a 1-dimensional elastic material and is thus subject to the physical laws of elasticity. We define the displacement $u(\mathbf{X})$ of a material point $\mathbf{X}$ of the undeformed/reference pin (also known as *Lagrangian coordinates*) to be given by

$$u(\mathbf{X}) = x(\mathbf{X}) - \mathbf{X}$$

where $x = x(\mathbf{X})$ is the spatial location of the material point $\mathbf{X}$ once the pin has been deformed (also known as *Eulerian* coordinates). As such we define $l_0 := (\mathbf{X} + \delta\mathbf{X}) - \mathbf{X} = \delta\mathbf{X}$ to be the length of a small portion of the undeformed pin, and $l := x(\mathbf{X} + \delta\mathbf{X}) - x(\mathbf{X})$ to be the length of this portion once the pin has been deformed. Figure 2 illustrates our setting.

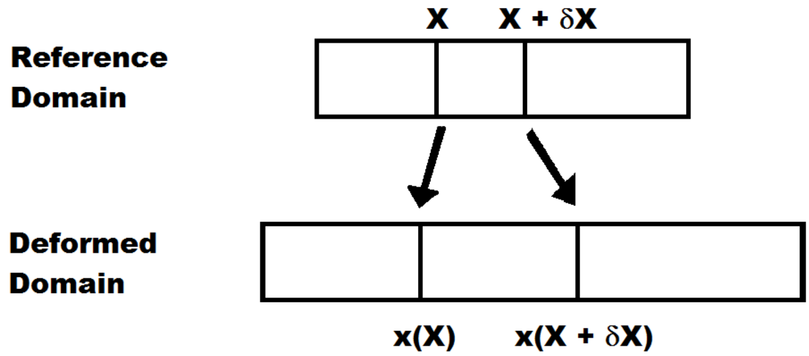


Figure 2: Deformation of a pin

In the *isothermal* case, elastic materials satisfy Hooke's law which states that the stress τ is proportional to strain. At a macroscopic level, the law is given as follows:

$$\tau = k\frac{l - l_0}{l_0}.$$

We may localise the above macroscopic law by taking $\delta\mathbf{X} \to 0$ which corresponds to considering an infinitesimally small portion of the bar. Plugging in the above definitions of l and l_0, we have

$$\begin{aligned}\tau(\mathbf{X}) &= k \lim_{\delta\mathbf{X}\to 0} \frac{x(\mathbf{X} + \delta\mathbf{X}) - x(\mathbf{X}) - (\mathbf{X} + \delta\mathbf{X} - \mathbf{X})}{\delta\mathbf{X}} = k\frac{\partial u}{\partial \mathbf{X}} \\ &= k\left(\frac{\partial x}{\partial \mathbf{X}} - 1\right).\end{aligned}$$

Due to the nature of the material making up the pin, we will assume that the strain is small i.e. $\left|\frac{\partial u}{\partial \mathbf{X}}\right| \ll 1 \Rightarrow \frac{\partial x}{\partial \mathbf{X}} \approx 1$. Under the small strain assumption, we thus have that $\frac{\partial u}{\partial \mathbf{X}} = \frac{\partial u}{\partial \mathbf{x}}$ i.e. the Eulerian and Lagrangian formulations are approximately the same, and so we can derive our model in terms of the more familiar Eulerian framework as an approximation.

Remark. Note that this cannot be generally done in higher dimensional models. In 3D, x can be very different from $\mathbf{X}$ (e.g. by a rigid body rotation) yet τ can still be small. In such a situation, the Eulerian framework cannot be used as an approximation.

Given the above comments, the localised form of Hooke's law takes the form

$$\tau(x) = k\frac{\partial u}{\partial x}. \tag{1}$$

A simple model elastic materials subject to non-isothermal conditions *without stress* satisfy is the following localised equation

$$\frac{\partial u}{\partial x} = \alpha(T - T_0). \tag{2}$$

i.e. strain is proportional to the change in temperature. Here α is known as the thermal expansion coefficient. This model accounts for the fact that an increase in temperature yields expansion of the elastic material. As our problem involves both thermal forcing and stress, we propose to model strain as being a *linear combination of both stress and the change in temperature.* Making use of (1) and (2), such a linear model yields

$$\frac{\partial u}{\partial x} = \frac{1}{k}\tau + \alpha(T - T_0).$$

Note that such a model is consistent in the sense that we may recover (1) in the isothermal case and (2) in the absence of stress. Under the equilibrium conditions, conservation of momentum yields $\frac{\partial \tau}{\partial x} = 0$. As $\tau = -F_0$ at $x = 1$ we have that the stress takes the constant value $\tau = -F_0$ for all $x \in [0, 1]$. Re-scaled so that $T_0 = 0$, the *thermoelasticity equation* is thus given by

$$\frac{\partial u}{\partial x} = \alpha T - \frac{1}{k}F_0. \tag{3}$$

Along with the thermoelasticity equation, we also pose a thermal model on the brake pad which models the diffusion of the heat generated from the brake pad's contact with the plate at $x = 0$. The model is derived from pointwise conservation of energy on a time-dependent subdomain of the break pad, which yields

$$\frac{\partial T}{\partial t} + a\frac{\partial^2 u}{\partial x \partial t} = \kappa\frac{\partial^2 T}{\partial x^2} \tag{4}$$

where $a > 0$ is small. A very similar model was considered in [2]. The term $a\frac{\partial^2 u}{\partial x \partial t}$ takes into account the expansion of the elastic material as the heat gets diffused through the material.

2.2 Initial/Boundary Conditions

We need to now prescribe physically appropriate boundary conditions satisfied by the displacement u and the temperature T at the boundaries $x = 0, 1$. With no heat flow at $x = 1$, to lowest order we have that

$$\kappa\frac{\partial T}{\partial x} = 0 \quad \text{at } x = 1. \tag{5}$$

The frictional heating occuring at $x = 0$ is modelled to be proportional to stress $\times$ velocity i.e.

$$\kappa \frac{\partial T}{\partial x} = \gamma\tau = -\gamma F_0 \quad \text{at } x = 0. \tag{6}$$

The initial condition for the temperature is chosen to be

$$T(x, 0) = 0 \quad \text{for all } x \in [0, 1]. \tag{7}$$

Finally, we consider the wear of the break pad due to the friction with the plate at $x = 0$ to be linear in time and also proportional to stress $\times$ velocity, i.e.

$$u(0, t) = -\beta F_0 t \quad \text{at } x = 0. \tag{8}$$

2.3 Decoupled Problem

Problems (3)–(4) along with the above initial/boundary conditions are well-posed. In order to decouple problems (3) and (4), we differentiate (3) with respect to t to get

$$\frac{\partial^2 u}{\partial x \partial t} = \alpha \frac{\partial T}{\partial t}.$$

Plugging the above into (4), we obtain

$$\frac{\partial T}{\partial t} = \frac{\kappa}{(1 + a\alpha)} \frac{\partial^2 T}{\partial x^2}. \tag{9}$$

Our solution algorithm for the one pin problem can be summarised as follows:

- Find temperature T satisfying (9) along with boundary/initial conditions (5), (6) and (7).
- Construct right-hand side of (3).
- Find displacement u satisfying (3) along with the boundary condition (8).

3 One Pin Problem

When considering only the one pin problem, we may derive some interesting analytical results. Physical intuition suggests that, asymptotically in time, the temperature in the pin would increase indefinitely due to the friction between the pin and the plate. Plugging in an ansatz of the form

$$T(x, t) = Ct + T_1(x, t)$$

into (9) where C is a constant to be determined and $T_1(x, t)$ is a higher-order term (i.e. of order less than $O(t)$) and making use of the boundary conditions (5) and (6), we obtain

$$T \sim \gamma F_0(1 + a\alpha)t \quad \text{as } t \to \infty.$$

This expression suggests that, asymptotically, the temperature in the pin inceases at a rate proportional to the flux introduced in the system. Plugging this expression into the right-hand side of (3) and solving for the displacement along with the boundary conditions (8), we obtain

$$u \sim \alpha\gamma F_0(1 + a\alpha)tx - \frac{1}{k}F_0 x - \beta F_0 t \quad \text{as } t \to \infty.$$

Such an expression suggests that if $|\alpha| \ll |\beta|$ (i.e. thermal expansion is dominant compared to the wearing) the pin will lengthen indefinitely, whereas if $|\beta| \ll |\alpha|$ (i.e. wearing is dominant compared to thermal expansion) the pin would shrink in size. These differences in behaviour are illustrated in the figures below, where Figures 3(a)–(d) show the case when wearing is dominant and Figures 4(a)–(d) show the case when thermal expansion is dominant. All simulations have been done on Matlab using an explicit Euler finite-difference approximation of (9).

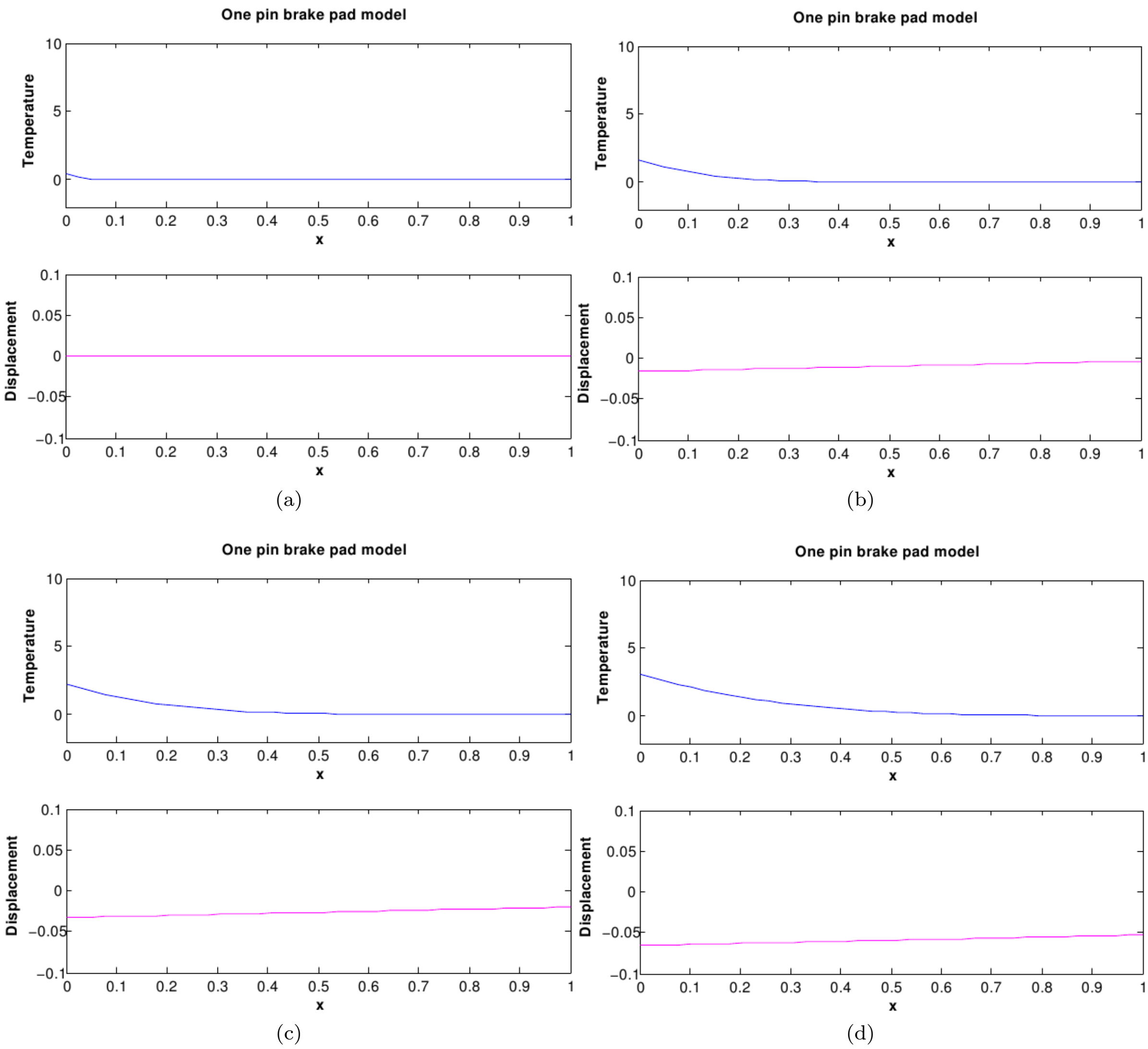
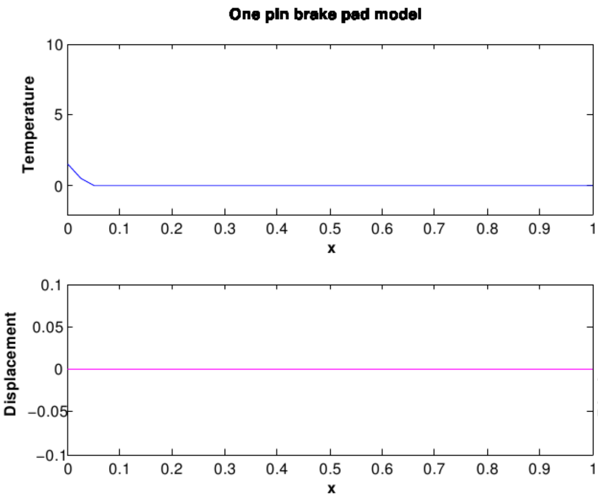
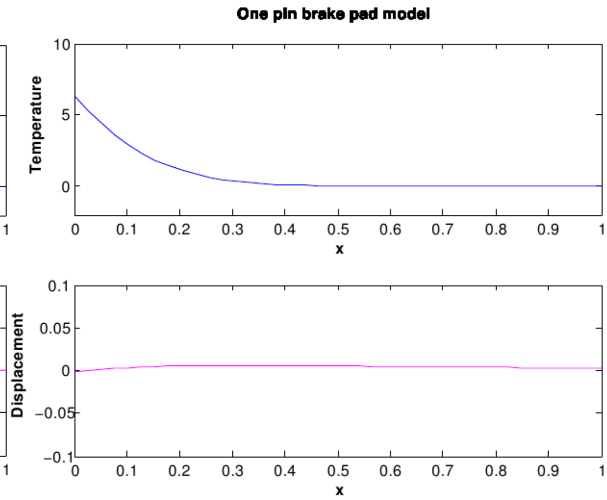


Figure 3: Wearing dominant.



(a)



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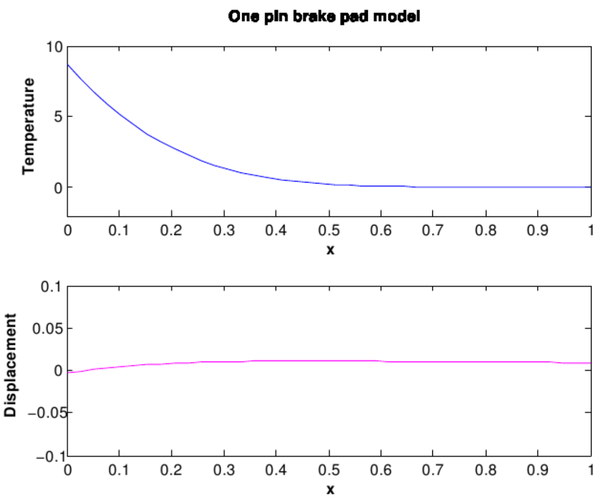
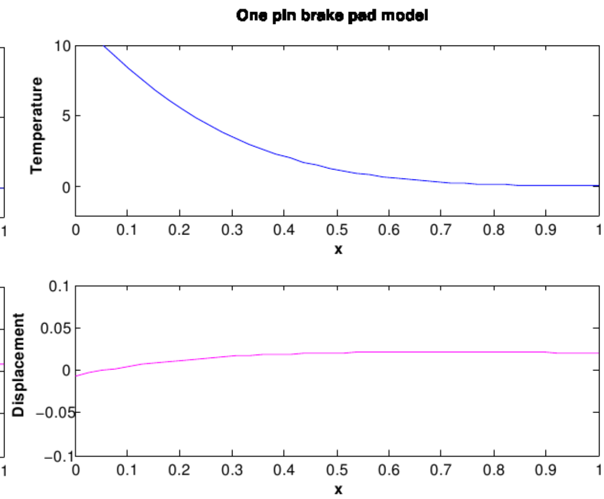


(c)



(d)

Figure 4: Thermal expansion dominant.

4 Two Pin Problem

We now apply the model to two pins rather than one, with one of the pins being slightly longer than the other. We assume that pin 1 is longer than pin 2 and only the former is in contact with the plate at time $t = 0$. In the numerical experiments that follow, the parameters are chosen so that the wearing effect dominates the overall dynamics of the system and, in particular, is stronger than the effects of thermal expansion/contraction. The wearing causes pin 1 to shrink in length and there will be a time $t = t^*$ at which the lengths of the two pins will be equal. When this happens, we "switch off" the heat source from pin 1 (i.e. $\kappa \frac{\partial T_1}{\partial x} = 0$ at $x = 0$ where T_1 is the temperature in pin 1) and simultaneously apply the heat source (5) to pin 2 (i.e. $\kappa \frac{\partial T_2}{\partial x} = -\gamma F_0$ at $x = 0$ where T_2 is the temperature in pin 2). Figures (5)(a)–(f) show the dynamics of the two pin model when performing the above mechanism. Notice how pin 1 briefly keeps shrinking past the time t_* at which the two pins are of the same length, which is due to thermal contraction. The roles of pin 1 and pin 2 have switched around, with pin 2 now being longer than pin 1 and heating up due to contact with the plate whilst pin 1 isn't in contact and is cooling down. Our model thus appears to capture the pin contact switching behaviour observed in [1].

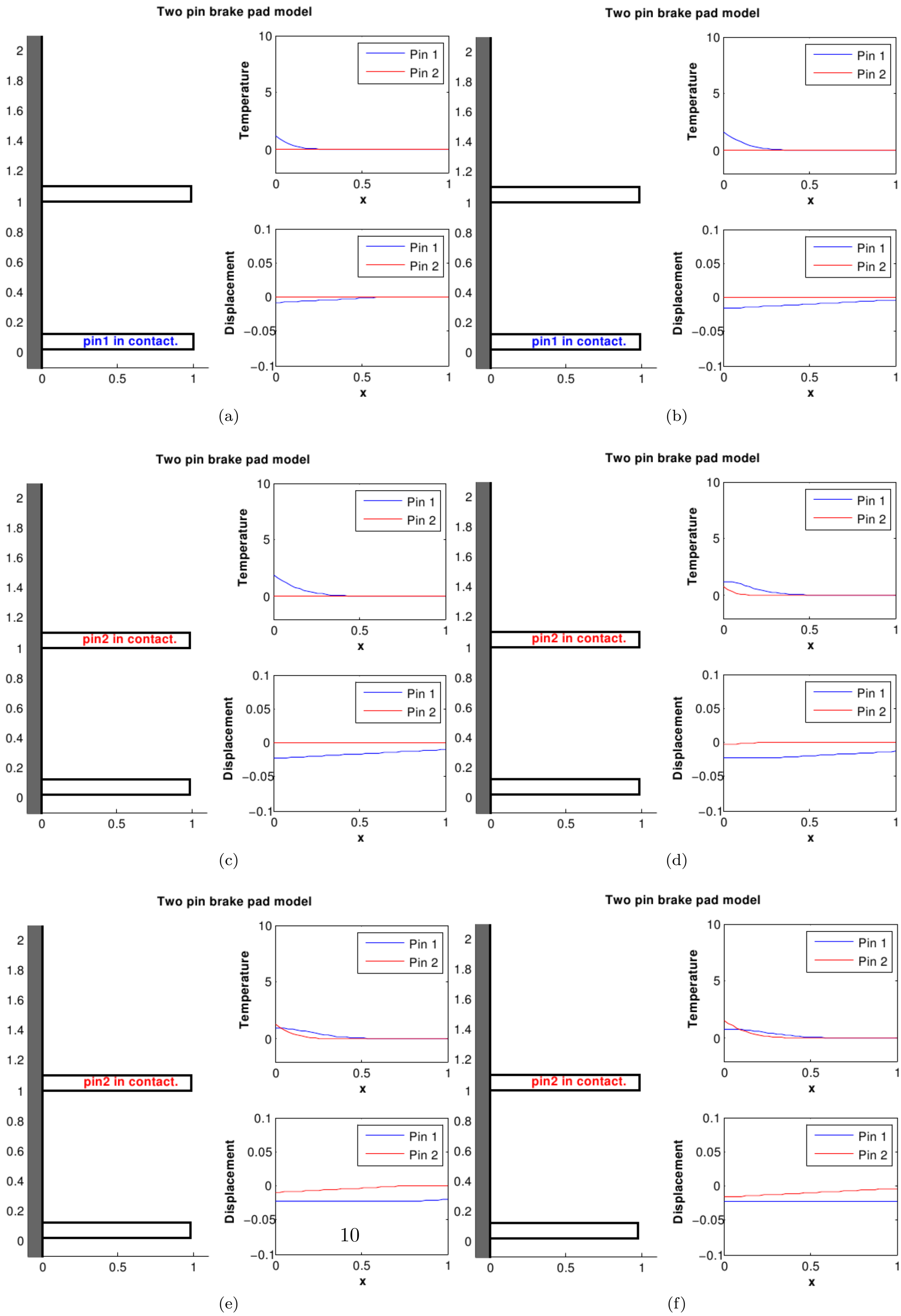


Figure 5: Two Pin Brake Pad Simulation

Acknowledgements

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References

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