Discontinuous Galerkin Methods

- Discontinuous Galerkin (DG) methods are a class of numerical methods that have been successfully applied to hyperbolic, elliptic and parabolic PDEs arising from a wide range of applications. See Arnold et al. [2002].

Some of its main advantages compared to standard finite element methods include:
- Capturing solution discontinuities (namely those arising in advection driven equations) sharply in a given mesh. 
- Less restriction on grid structure and refinement (i.e. works with non-conforming grids). 
- Less restriction on choice of basis functions. 
- Easily parallelisable.

Surface FEM Approximation

- For FEM approximations we consider the finite-dimensional space

\[ V_h = \{ v : v \in C^0(T_h) : v|_{T} \in P^k(T) \forall T \in T_h \} \]

- \((P_h)\) Find \(v_h \in V_h\) such that

\[ a_h(u_h, v_h) = \int_{T_h} f \cdot v_h \, dA_h \quad \forall v_h \in V_h \]

\[ a_h(u_h, v_h) = \int_{T_h} (\nabla u_h \cdot \nabla v_h + a \cdot u_h v_h) \, dA_h \quad \forall v_h \in V_h \]

- Want to compare \(u\) satisfying \((P_h)\) with \(u_h\) satisfying \((P)\) but they do not live on the same space.

- For any function \(f\) defined on \(T_h\) we define the lift onto \(T\) by

\[ \tilde{f}(x) = f(x), \quad x \in f \subset c \in T_h \]

- This lift allows us to define the lifted approximation \(u_h^f\) on \(T\).

- The lifted finite element space

\[ V^f_h = \{ v_h : v_h \in C^0(T) : \tilde{v_h}(x) \in (P)_{\nabla v_h} \forall x \in V_h \} \]

Theorem (Surface FEM A-priori Error Estimate)

Let \(u\) and \(v_h\) denote the solutions to \((P)\) and \((P_h)\), respectively. Denote by \(e_h^u\) the lift of \(u\) onto \(T_h\). Then

\[ \| u - e_h^u \|_{H^1} + \| u - u_h \|_{(P)} \leq C \| f \|_{L^2(T)} \]

- Would like to derive error estimates for the surface FEM approximation on hypersurfaces in a similar way.

Surface DG Approximation I

- For DG approximations we consider the finite-dimensional space

\[ V_h = \{ v : v \in C^0(T_h) : v|_{T} \in P^k(T) \forall T \in T_h \} \]

DG space has no continuity requirement across elements. 
- Let \(K\) and \(K'\) be two adjacent elements sharing the surface \(\Gamma_{K,K'}\) with normal \(\nu_{K,K'}\) and \(\nu_{K',K}\) the corresponding normals.

\[ \| v_h \|_{DG,h} = \sum_{T \in T_h} \left( \sum_{x \in \partial T} \int_T | \nu_{K,K'} \cdot \nabla v_h |^2 \right)^{1/2} \]

Theorem (Surface DG A-priori Error Estimate)

Let \(u\) and \(v_h\) denote the solutions to \((P^{DG})\) and \((P^{DG}_h)\), respectively. Denote by \(e_h^u\) the lift of \(u\) onto \(T_h\). Then

\[ \| u - e_h^u \|_{H^1} + \| u - u_h \|_{(P)} \leq C \| f \|_{L^2(T)} \]

where \(C\) is the DG norm.