# On a Discontinuous Galerkin Method for Surface PDEs

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(joint work with Andreas Dedner and Björn Stinner)

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Mathematics and Statistics
Centre for Doctoral Training

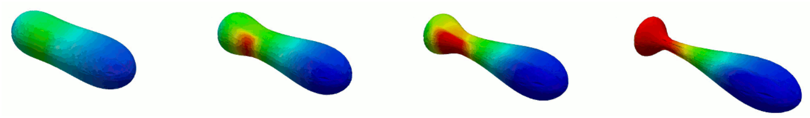
THE UNIVERSITY OF WARWICK

Centre for Scientific Computing

# Motivation - PDEs on Surfaces

PDEs on surfaces arise in various areas, for instance

- materials science: enhanced species diffusion along grain boundaries,
- cell biology: phase separation on biomembranes, diffusion processes on plasma membranes,
- fluid dynamics: surface active agents.



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FEM methods:
[ Dziuk 1988 ]: elliptic problem,
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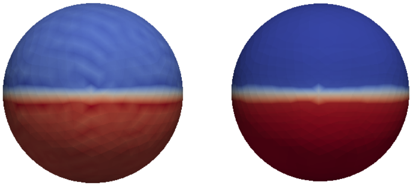

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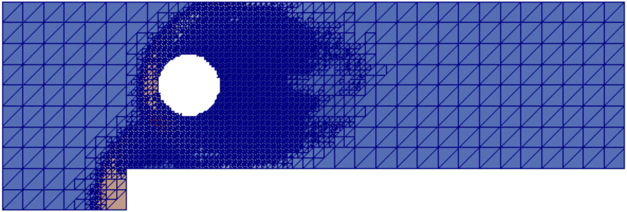

# Outline

1. Notation and Setting
2. DG Approximation
3. Convergence Proof
4. Numerical Tests

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# Some Notation

- *Hypersurface:* $\Gamma \subset \mathbb{R}^3$ compact, smooth, simply connected, oriented, no boundary.
- *Signed distance function:* $d : U \to \mathbb{R}$ with $U$ a thin tube around $\Gamma$.
- *Unit normal:* $\nu(\xi) = \nabla d(\xi),\ \xi \in \Gamma$.
- *Projection* of $\mathbb{R}^3$ onto the tangent space $T_\xi\Gamma$, $\xi \in \Gamma$:

$$P(\xi) := I - \nu(\xi) \otimes \nu(\xi),\ \xi \in \Gamma.$$

- *Surface gradient:* For any function $\eta : U \to \mathbb{R}$,

$$\nabla_\Gamma \eta := \nabla \eta - \nabla \eta \cdot \nu\nu = P\nabla\eta = (D_1\eta, D_2\eta, D_3\eta).$$

# Strong Problem Formulation

*Laplace-Beltrami* operator:

$$\Delta_\Gamma \eta := \nabla_\Gamma \cdot (\nabla_\Gamma \eta) = \sum_{i=1}^{3} D_i D_i \eta.$$

Strong problem: For a given function $f : \Gamma \to \mathbb{R}$, find $u : \Gamma \to \mathbb{R}$ such that

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For weak formulation: *Integration by parts formula on surfaces:*

$$\int_\Gamma \eta \nabla_\Gamma \cdot v = -\int_\Gamma \left( v \cdot \nabla_\Gamma \eta + \eta v \cdot \kappa \right) + \int_{\partial\Gamma} \eta v \cdot \mu$$

where $\mu$: *outer co-normal* of $\Gamma$ on $\partial\Gamma$, $\kappa$: *mean curvature vector*.

# Weak Problem Formulation

Sobolev spaces:

$$H^m(\Gamma) := \{u \in L^2(\Gamma) \ : \nabla_\Gamma^\alpha u \in L^2(\Gamma) \ \forall |\alpha| \le m\}$$

with corresponding norm

$$\|u\|_{H^m(\Gamma)} := \left( \sum_{|\alpha| \le m} \|\nabla_\Gamma^\alpha u\|^2_{L^2(\Gamma)} \right)^{1/2} .$$

Problem $(\mathbf{P}_\Gamma)$: Find $u \in V := H^1(\Gamma)$ such that

$$\int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma v + uv \ dA = \int_\Gamma fv \ dA, \ \forall v \in V.$$

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## Theorem (Aubin 1982)

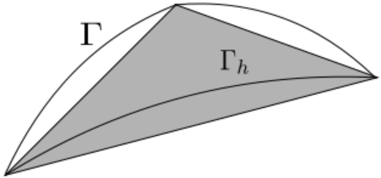
*If $f \in L^2(\Gamma)$ then there is a unique weak solution $u \in V$ to $(\mathbf{P}_\Gamma)$ which satisfies*

$$\|u\|_{H^2(\Gamma)} \leq C\|f\|_{L^2(\Gamma)}.$$

# Outline

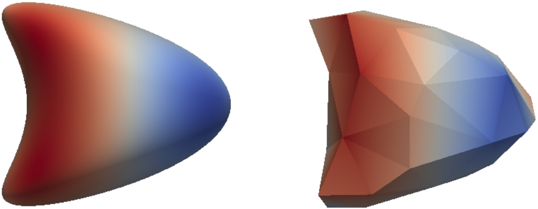
# Triangulated Surfaces

- $\Gamma$ is approximated by a polyhedral surface $\Gamma_h$ composed of planar triangles $K_h$.




- The vertices sit on $\Gamma \quad \Rightarrow \quad \Gamma_h$ is its linear interpolation.
- $\mathcal{T}_h$: Associated regular, conforming triangulation i.e.

$$\Gamma_h = \bigcup_{K_h \in \mathcal{T}_h} K_h.$$



# DG Setting

*DG space:*

$$V_h := \left\{ v_h \in L^2(\Gamma_h) \ : \ v_h\big|_{K_h} \in P^1(K_h)\ \forall K_h \in \mathcal{T}_h \right\}.$$

This space allows for jumps across edges, to be penalised in the DG method.

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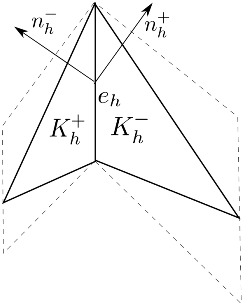
- *Set of edges:* $\mathcal{E}_h$.
- *Unit conormals:* $n_h^+$, $n_h^-$ to $K_h^+$, $K_h^-$ on $e_h \in \mathcal{E}_h$.
- *Trace values:* $v_h^\pm := v_h|_{\partial K_h^\pm}$ for $v_h \in V_h$.



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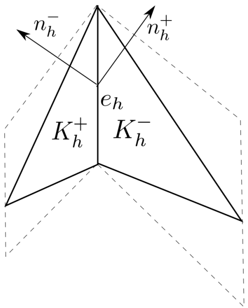


*DG norm:*

$$|u_h|_{1,h}^2 := \sum_{K_h \in \mathcal{T}_h} \|u_h\|_{H^1(K_h)}^2, \quad |u_h|_{*,h}^2 := \sum_{e_h \in \mathcal{E}_h} h_{e_h}^{-1} \|u_h^+ - u_h^-\|_{L^2(e_h)}^2,$$

$$\|u_h\|_{DG(\Gamma_h)}^2 := |u_h|_{1,h}^2 + |u_h|_{*,h}^2.$$

# DG Problem

Problem ($\mathbf{P}^{DG}_{\Gamma_h}$): Find $u_h \in V_h$ such that

$$a^{DG}_{\Gamma_h}(u_h, v_h) = \int_{\Gamma_h} f_h v_h \; dA_h \; \forall v_h \in V_h$$

where $f_h$ is related to $f$ (later) and

$$\begin{aligned} a^{DG}_{\Gamma_h}(u_h, v_h) := & \sum_{K_h \in \mathcal{T}_h} \int_{K_h} \nabla_{\Gamma_h} u_h \cdot \nabla_{\Gamma_h} v_h + u_h v_h \; dA_h \\ & - \sum_{e_h \in \mathcal{E}_h} \int_{e_h} (u_h^+ - u_h^-) \frac{1}{2} (\nabla_{\Gamma_h} v_h^+ \cdot n_h^+ - \nabla_{\Gamma_h} v_h^- \cdot n_h^-) \; ds_h \\ & - \sum_{e_h \in \mathcal{E}_h} \int_{e_h} (v_h^+ - v_h^-) \frac{1}{2} (\nabla_{\Gamma_h} u_h^+ \cdot n_h^+ - \nabla_{\Gamma_h} u_h^- \cdot n_h^-) \; ds_h \\ & + \sum_{e_h \in \mathcal{E}_h} \int_{e_h} \beta_{e_h} (u_h^+ - u_h^-)(v_h^+ - v_h^-) \; ds_h \end{aligned}$$

with $\beta_{e_h} \sim h_{e_h}^{-1}$.

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*Interior penalty method* [ Arnold 1982 ]: If $\beta_{e_h} = \omega_{e_h} h_{e_h}^{-1}$ and $\omega_{e_h}$ big enough then $a^{DG}_{\Gamma_h}$ is coercive, and there is a unique solution $u_h \in V_h$ to problem ($\mathbf{P}^{DG}_{\Gamma_h}$) with

$$\|u_h\|_{DG(\Gamma_h)} \leq C \|f_h\|_{L^2(\Gamma_h)}.$$

# DG Problem: Remark

[ Arnold 1982 ] (classical) IP method:

$$
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a^{DG}_{\Gamma_h}(u_h, v_h) := &\sum_{K_h \in \mathcal{T}_h} \int_{K_h} \nabla_{\Gamma_h} u_h \cdot \nabla_{\Gamma_h} v_h + u_h v_h \; dA_h \\
&- \sum_{e_h \in \mathcal{E}_h} \int_{e_h} (u_h^+ - u_h^-) \frac{1}{2} (\nabla_{\Gamma_h} v_h^+ \cdot n_h^+ - \nabla_{\Gamma_h} v_h^- \cdot n_h^-) \; ds_h \\
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[ Arnold *et al.* 2002 ] (standard) IP method:

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# The Lift

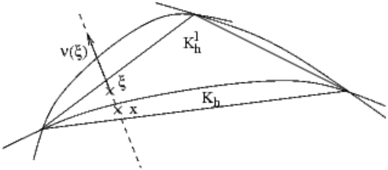
Goal: Compare $u \in H^2(\Gamma)$ solving $(\mathbf{P}_\Gamma)$ with $u_h \in V_h$ solving $(\mathbf{P}^{DG}_{\Gamma_h})$, but $\Gamma_h \not\subset \Gamma$.

Lift: For $\eta : \Gamma_h \to \mathbb{R}$ define

$$\eta^l(\xi) := \eta(x)$$

where

$$x = \xi + d(x)\nu(\xi)$$



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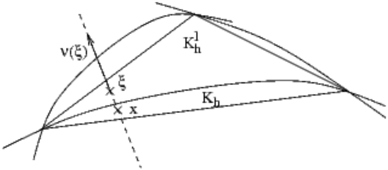



One-to-one relation between $\Gamma$ and $\Gamma_h$, write

$$x = x(\xi) \text{ or } \xi = \xi(x).$$

*Right hand side*:

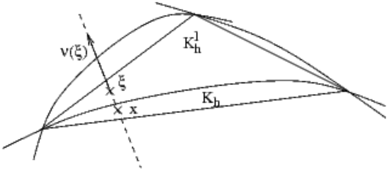
Define $f_h$ so that $f_h^l = f$ on $\Gamma$.

# Lifted Objects

- Lifted triangles: $K_h^l = \xi(K_h) \subset \Gamma$.
- Conforming triangulation $\mathcal{T}_h^l$,

$$\Gamma = \bigcup_{K_h^l \in \mathcal{T}_h^l} K_h^l.$$

- Lifted edges: $e_h^l := \xi(e_h) \in \mathcal{E}_h^l$.



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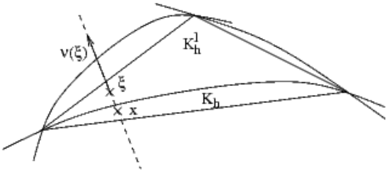


Lifted DG space:

$$V_h^l := \{v_h^l \in L^2(\Gamma) \ : \ v_h^l(\xi) = v_h(x(\xi)) \text{ with some } v_h \in V_h\},$$

norm:

$$\|v_h^l\|_{DG(\Gamma)}^2 := \sum_{K_h^l \in \mathcal{T}_h^l} \|v_h^l\|_{H^1(K_h^l)}^2 + \sum_{e_h^l \in \mathcal{E}_h^l} h_{e_h^l}^{-1} \|v_h^{l,+} - v_h^{l,-}\|_{L^2(e_h^l)}^2$$

# DG Bilinear Form on $\Gamma$

Consider

$$a_\Gamma^{DG}(u,v) := \sum_{K_h^l \in \mathcal{T}_h^l} \int_{K_h^l} \nabla_\Gamma u \cdot \nabla_\Gamma v + uv \; dA$$
$$- \sum_{e_h^l \in \mathcal{E}_h^l} \int_{e_h^l} (u^+ - u^-) \frac{1}{2} (\nabla_\Gamma v^+ \cdot n^+ - \nabla_\Gamma v^- \cdot n^-) \; ds$$
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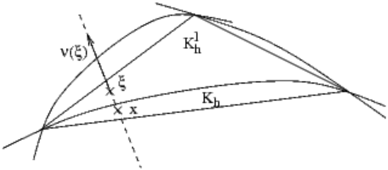
$$+ \sum_{e_h^l \in \mathcal{E}_h^l} \int_{e_h^l} \beta_{e_h^l} (u^+ - u^-)(v^+ - v^-) \; ds$$

- *Unit conormals* to $K_h^{l+}$ and $K_h^{l-}$ on $e_h^l \in \mathcal{E}_h^l$: $n^+ = -n^- \in T_\xi \Gamma$.
- *Penalty parameters* $\beta_{e_h^l} := \frac{\beta_{e_h}}{\delta_{e_h}}$.



# Convergence Statement

## Theorem (Dedner, M., Stinner 2012)

*Let $u \in H^2(\Gamma)$ and $u_h \in V_h$ denote the solutions to $(\mathbf{P}_\Gamma)$ and $(\mathbf{P}_{\Gamma_h}^{DG})$, respectively. Denote by $u_h^l \in V_h^l$ the lift of $u_h$ onto $\Gamma$. Then*

$$\|u - u_h^l\|_{L^2(\Gamma)} + h\|u - u_h^l\|_{DG(\Gamma)} \leq Ch^2\|f\|_{L^2(\Gamma)}.$$

# Outline

## Outline of the Proof

1. Starting:

$$\|u - u_h^l\|_{DG(\Gamma)} \leq \|u - \phi_h^l\|_{DG(\Gamma)} + \|\phi_h^l - u_h^l\|_{DG(\Gamma)}, \quad \phi_h^l = I_h^l u \text{ interpolate.}$$

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2. Interpolation estimate [ Dziuk 1988 ]:

$$\|u - \phi_h^l\|_{DG(\Gamma)} = \|u - \phi_h^l\|_{H^1(\Gamma)} \leq Ch\|u\|_{H^2(\Gamma)} \quad \big(\leq Ch\|f\|_{L^2(\Gamma)}\big).$$

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3. Using coercivity in $V_h^l$:

$$\begin{aligned} C_s^l\|\phi_h^l - u_h^l\|^2_{DG(\Gamma)} &\leq a_\Gamma^{DG}(\phi_h^l - u_h^l, \phi_h^l - u_h^l) \\ &= a_\Gamma^{DG}(\phi_h^l - u, \phi_h^l - u_h^l) + a_\Gamma^{DG}(u - u_h^l, \phi_h^l - u_h^l). \end{aligned}$$

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4. Using boundedness in $H^2(\Gamma) + V_h^l$:

$$a_\Gamma^{DG}(\phi_h^l - u, \phi_h^l - u_h^l) \leq C_b^l\big(\|\phi_h^l - u\|_{DG(\Gamma)} + h^2\|u\|_{H^2(\Gamma)}\big)\|\phi_h^l - u_h^l\|_{DG(\Gamma)}.$$

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5. Estimating variational crime error:

$$a_\Gamma^{DG}(u - u_h^l, \phi_h^l - u_h^l) \leq Ch^2\|f\|_{L^2(\Gamma)}\|\phi_h^l - u_h^l\|_{DG(\Gamma)}.$$

# Outline of the Proof

1. Starting:

$$\|u - u_h^l\|_{DG(\Gamma)} \leq \|u - \phi_h^l\|_{DG(\Gamma)} + \|\phi_h^l - u_h^l\|_{DG(\Gamma)}, \quad \phi_h^l = I_h^l u \text{ interpolate.}$$

2. Interpolation estimate [ Dziuk 1988 ]:

$$\|u - \phi_h^l\|_{DG(\Gamma)} = \|u - \phi_h^l\|_{H^1(\Gamma)} \leq Ch\|u\|_{H^2(\Gamma)} \quad \big(\leq Ch\|f\|_{L^2(\Gamma)}\big).$$

3. Using coercivity in $V_h^l$:

$$\begin{aligned} C_s^l\|\phi_h^l - u_h^l\|_{DG(\Gamma)}^2 &\leq a_\Gamma^{DG}(\phi_h^l - u_h^l, \phi_h^l - u_h^l) \\ &= a_\Gamma^{DG}(\phi_h^l - u, \phi_h^l - u_h^l) + a_\Gamma^{DG}(u - u_h^l, \phi_h^l - u_h^l). \end{aligned}$$

4. Using boundedness in $H^2(\Gamma) + V_h^l$:

$$a_\Gamma^{DG}(\phi_h^l - u, \phi_h^l - u_h^l) \leq C_b^l\big(\|\phi_h^l - u\|_{DG(\Gamma)} + h^2\|u\|_{H^2(\Gamma)}\big)\|\phi_h^l - u_h^l\|_{DG(\Gamma)}.$$

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6. Concluding:

$$\|u - u_h^l\|_{DG(\Gamma)} \leq (1 + C)\|\phi_h^l - u\|_{H^1(\Gamma)} + Ch^2\|u\|_{H^2(\Gamma)} + Ch^2\|f\|_{L^2(\Gamma)} \leq Ch\|f\|_{L^2(\Gamma)}.$$

## Coercivity and Boundedness: Inverse Estimate

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### Lemma (Inverse Estimate)

*Let $w \in H^2(\Gamma)$ and $w_h^l \in V_h^l$. Let $K_h^l \in \mathcal{T}_h^l$. Then for sufficiently small $h$,*

$$\|\nabla_\Gamma (w + w_h^l)\|_{L^2(\partial K_h^l)}^2 \leq C \left( \frac{1}{h} \|\nabla_\Gamma (w + w_h^l)\|_{L^2(K_h^l)}^2 + h \|w\|_{H^2(K_h^l)}^2 \right).$$

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## Proof.

Trace theorem and a standard scaling argument on $K_h \in \mathcal{T}_h$, lift estimate onto $K_h^l \in \mathcal{T}_h^l$ using result in [ Demlow 2009 ] and apply geometric estimates. □

# Variational Crime Error: Geometric Estimates

5. Estimating variational crime error:

$$a_\Gamma^{DG}(u - u_h^l, \phi_h^l - u_h^l) \leq Ch^2 \|f\|_{L^2(\Gamma)} \|\phi_h^l - u_h^l\|_{DG(\Gamma)}.$$

$$\begin{aligned}
& a_\Gamma^{DG}(u - u_h^l, w_h^l) \\
&= \sum_{K_h^l \in \mathcal{T}_h^l} \int_{K_h^l} (R_h - P)\nabla_\Gamma u_h^l \cdot \nabla_\Gamma w_h^l + \left(\frac{1}{\delta_h} - 1\right) u_h^l w_h^l + \left(1 - \frac{1}{\delta_h}\right) f w_h^l \; dA \\
&\quad + \sum_{e_h^l \in \mathcal{E}_h^l} \int_{e_h^l} (u_h^{l+} - u_h^{l-}) \frac{1}{2} \Big( \nabla_\Gamma w_h^{l+} \cdot (n^+ - \frac{1}{\delta_{e_h}} P(I - dH) n_h^{l+}) \\
&\qquad\qquad - \nabla_\Gamma w_h^{l-} \cdot (n^- - \frac{1}{\delta_{e_h}} P(I - dH) n_h^{l-}) \Big) ds \\
&\quad + \sum_{e_h^l \in \mathcal{E}_h^l} \int_{e_h^l} (w_h^{l+} - w_h^{l-}) \frac{1}{2} \Big( \nabla_\Gamma u_h^{l+} \cdot (n^+ - \frac{1}{\delta_{e_h}} P(I - dH) n_h^{l+}) \\
&\qquad\qquad - \nabla_\Gamma u_h^{l-} \cdot (n^- - \frac{1}{\delta_{e_h}} P(I - dH) n_h^{l-}) \Big) ds.
\end{aligned}$$

## Variational Crime Error: Geometric Estimates

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### Lemma (Dziuk 1988 and Giesselman & Mueller 2012)

$$\|d\|_{L^\infty(\Gamma)} \leq Ch^2,$$

$$\|1 - \delta_h\|_{L^\infty(\Gamma)} \leq Ch^2, \qquad \delta_h : \text{ local area change, } \delta_h dA_h = dA,$$

$$\|\nu - \nu_h\|_{L^\infty(\Gamma)} \leq Ch, \qquad \nu, \nu_h : \text{ unit normals on } \Gamma \text{ and } \Gamma_h,$$

$$\|1 - \delta_{e_h}\|_{L^\infty(\mathcal{E}_h^l)} \leq Ch^2, \qquad \delta_{e_h} : \text{ local length change, } \delta_{e_h} ds_h = ds,$$

$$\|n - Pn_h^l\|_{L^\infty(\mathcal{E}_h^l)} \leq Ch^2,$$

$$\|P - R_h\|_{L^\infty(\Gamma)} \leq Ch^2 \qquad \text{where } R_h := \frac{1}{\delta_h} P(I - dH)P_h(I - dH)$$

$$\text{with } H = \nabla^2 d \text{ and } P_h = I - \nu_h \otimes \nu_h.$$

# Outline

1. Notation and Setting
2. DG Approximation
3. Convergence Proof
4. Numerical Tests

# Software


Distributed and Unified Numerics Environment

- All simulations have been performed using the Distributed and Unified Numerics Environment (DUNE).
- Initial mesh generation made use of 3D surface mesh generation module of the Computational Geometry Algorithms Library (CGAL).
- Further information about DUNE and CGAL can be found respectively on http://www.dune-project.org/ and http://www.cgal.org/

# Test Problem on Unit Sphere

Surface Helmholtz equation:

$$-\Delta_\Gamma u + u = f$$

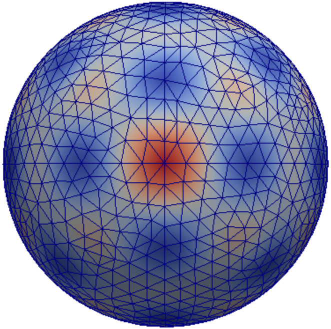
on the unit sphere

$$\Gamma = \{x \in \mathbb{R}^3 \ : \ |x| = 1\}.$$

The right-hand side $f$ is chosen such that

$$u(x_1, x_2, x_3) = \cos(2\pi x_1)\cos(2\pi x_2)\cos(2\pi x_3)$$

is the exact solution.

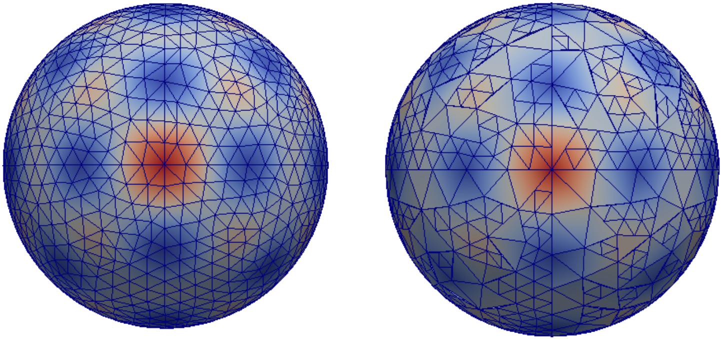


# EOCs for Sphere Test Problem

| Elements | $h$ | $L_2$-error | $L_2$-eoc | $DG$-error | $DG$-eoc |
|---|---|---|---|---|---|
| 623 | 0.223929 | 0.171459 | | 5.07662 | |
| 2528 | 0.112141 | 0.0528817 | 1.70 | 2.64273 | 0.94 |
| 10112 | 0.0560925 | 0.0146074 | 1.86 | 1.3151 | 1.01 |
| 40448 | 0.028049 | 0.00378277 | 1.95 | 0.653612 | 1.01 |
| 161792 | 0.0140249 | 0.000957472 | 1.98 | 0.325961 | 1.00 |
| 647168 | 0.00701247 | 0.000240483 | 1.99 | 0.162822 | 1.00 |

# Visualisation, Sphere Test Problem

Can easily work with non-conforming grid:



# Test Problem on Dziuk Surface

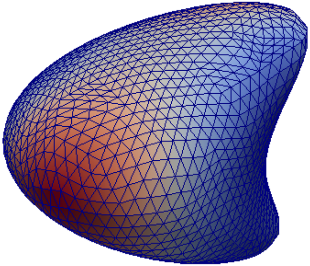
Solve the Helmholtz equation
on the *Dziuk surface*

$$\Gamma = \{x \in \mathbb{R}^3 \ : \ (x_1 - x_3^2)^2 + x_2^2 + x_3^2 = 1\}.$$

The right-hand side $f$ is chosen such that

$$u(x) = x_1 x_2$$

is the exact solution.



# EOCs for Dziuk Surface

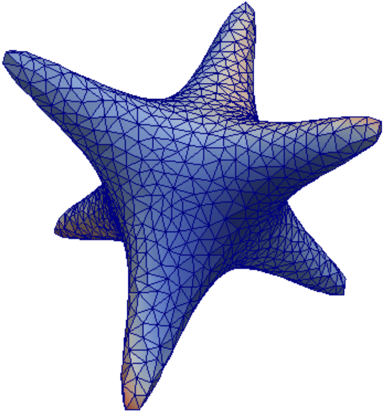
| Elements | $h$ | $L_2$-error | $L_2$-eoc | $DG$-error | $DG$-eoc |
|---|---|---|---|---|---|
| 92 | 0.704521 | 0.243493 | | 0.894504 | |
| 368 | 0.353599 | 0.0842372 | 1.53 | 0.490805 | 0.87 |
| 1472 | 0.176993 | 0.0268596 | 1.65 | 0.263808 | 0.90 |
| 5888 | 0.0885231 | 0.00637826 | 2.07 | 0.135162 | 0.97 |
| 23552 | 0.0442651 | 0.00171047 | 1.90 | 0.0685366 | 0.98 |
| 94208 | 0.022133 | 0.00041636 | 2.04 | 0.0343677 | 1.00 |
| 376832 | 0.0110666 | 0.00010427 | 2.00 | 0.0171891 | 1.00 |
| 1507328 | 0.0055333 | 2.60734e-05 | 2.00 | 0.0085934 | 1.00 |

# Test Problem on Enzensberger-Stern Surface

Solve the Helmholtz equation
on the *Enzensberger-Stern surface*

$$\Gamma = \{x \in \mathbb{R}^3 \; : \; 400(x^2y^2 + y^2z^2 + x^2z^2) - (1 - x^2 - y^2 - z^2)^3 - 40 = 0.\}$$

The right-hand side $f$ is again chosen such that
$u(x) = x_1 x_2$ is the exact solution.



# EOCs for Enzensberger-Stern Surface

| Elements | $h$ | $L_2$-error | $L_2$-eoc | $DG$-error | $DG$-eoc |
|---|---|---|---|---|---|
| 2358 | 0.163789 | 0.476777 | | 0.998066 | |
| 9432 | 0.0817973 | 0.175293 | 1.44 | 0.472241 | 1.08 |
| 37728 | 0.040885 | 0.0160606 | 3.45 | 0.150144 | 1.65 |
| 150912 | 0.0204411 | 0.00139698 | 3.52 | 0.0703901 | 1.09 |
| 603648 | 0.0102204 | 0.000338462 | 2.04 | 0.0347345 | 1.02 |
| 2414592 | 0.00511 | 7.86713e-05 | 2.10 | 0.0172348 | 1.01 |

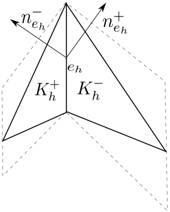
Tricky: The computation of the lifted points $\xi(x)$ when refining the surface.
The EOC rates thus are a bit more volatile.

# Other Choices for the Conormals

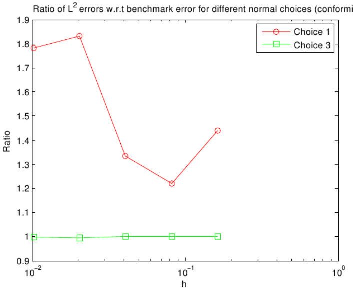
Generalise the bilinear form:

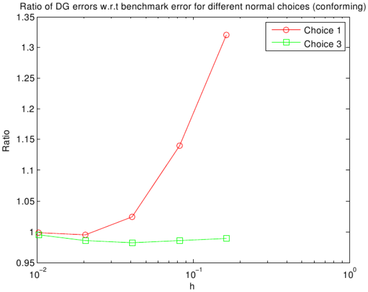
$$\tilde{a}^{DG}_{\Gamma_h}(u_h, v_h) := -\sum_{e_h \in \mathcal{E}_h} \int_{e_h} (u_h^+ - u_h^-) \frac{1}{2} (\nabla_{\Gamma_h} v_h^+ \cdot n_{e_h}^+ - \nabla_{\Gamma_h} v_h^- \cdot n_{e_h}^-) \, ds_h$$
$$- \sum_{e_h \in \mathcal{E}_h} \int_{e_h} (v_h^+ - v_h^-) \frac{1}{2} (\nabla_{\Gamma_h} u_h^+ \cdot n_{e_h}^+ - \nabla_{\Gamma_h} u_h^- \cdot n_{e_h}^-) \, ds_h \quad + \ldots$$

| Choice | $n_{e_h}^-$ | $n_{e_h}^+$ | Description |
|---|---|---|---|
| 1 | $n_h^-$ | $-n_h^-$ | planar (non-sym) |
| 2 | $n_h^-$ | $n_h^+$ | [ Arnold 1982 ] (sym pos-def) |
| 3 | $\frac{\frac{1}{2}(n_h^- - n_h^+)}{\lvert\frac{1}{2}(n_h^- - n_h^+)\rvert}$ | $\frac{\frac{1}{2}(n_h^+ - n_h^-)}{\lvert\frac{1}{2}(n_h^+ - n_h^-)\rvert}$ | averaged (sym pos-def) |
| 4 | $-n_h^+$ | $-n_h^-$ | [ Arnold *et al.* 2002 ] (sym pos-def) |



# Comparison of Choices




Ratio of $L^2$ and $DG$ errors for the test problem on the Enzensberger-Stern surface, benchmark is the analysed choice 2.

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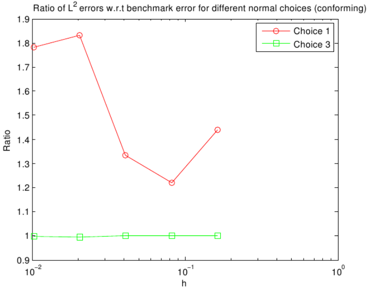


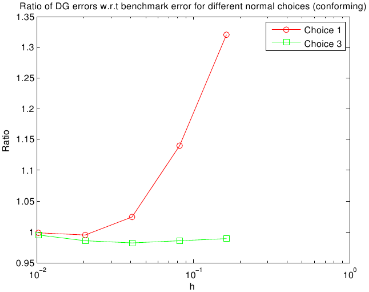


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The [ Arnold *et al.* 2002 ] (standard) IP method did not converge!

# Conclusion

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Thanks for your attention!

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