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Bayesian Geometric Inverse Problems in Groundwater Flow

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MASDOC Mathematics and Statistics Centre for Doctoral Training

Aim

- We are interested in determining the permeability of a porous medium given noisy pressure measurements, using a Bayesian approach to the inverse problem.
- The permeability may have jump discontinuities (interfaces), the shapes and positions of which are of interest.
- How can we characterise the modes of the posterior distribution?
- How can we produce samples from the posterior distribution?

Darcy model for groundwater flow (steady state)

- Pressure p
- Permeability κ
- Given $\kappa \in L^\infty(D)$ and $f \in H^{-1}(D)$, $p \in H_0^1(D)$ satisfies the PDE

$$\begin{cases} -\nabla \cdot (\kappa \nabla p) = f & \text{in } D \\ p = 0 & \text{on } \partial D \end{cases}$$

- A unique solution exists if, for example, κ is bounded below by a positive constant

Defining the permeability

- We consider the case when κ is piecewise continuous
- Let $\Lambda \subseteq \mathbb{R}^k$ be the finite-dimensional geometric parameter space, and $X = L^\infty(D)^N$ the space of fields
- Take a collection of maps $A_i : \Lambda \to \mathcal{B}(D)$, $i = 1, \dots, N$ such that

$$\bigcup_{i=1}^N A_i(a) = D, \quad A_i(a) \cap A_j(a) = \emptyset \text{ if } i \neq j$$

  and $|a - b| \to 0 \Rightarrow |A_i(a) \Delta A_i(b)| \to 0$, where Δ denotes the symmetric difference
- Given fields $u = (u_1, \dots, u_N) \in X$ and $a \in \Lambda$, define the log-permeability $u^a \in L^\infty(D)$ by

$$u^a = \log(\kappa) = \sum_{i=1}^N u_i \mathbb{1}_{A_i(a)}$$

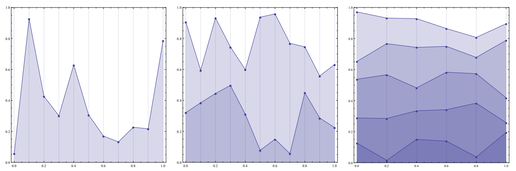


Figure : Some examples of the maps A_i. Here we have $\Lambda = [0,1]^k$ for $k = 11, 22, 30$ respectively

The forward operator

Define the forward operator $\mathcal{G} : X \times \Lambda \to \mathbb{R}^J$ via the composition

$$\begin{array}{ccccccccc} (u,a) & \mapsto & u^a & \mapsto & \kappa = e^{u^a} & \mapsto & p & \mapsto & (\ell_1(p), \dots, \ell_J(p)) \\ X \times \Lambda & \to & L^\infty(D) & \to & L^\infty_+(D) & \to & H_0^1(D) & \to & \mathbb{R}^J \end{array}$$

$\mathcal{G}$ is continuous, despite the discontinuities in κ. This is important for its measurability later, as well as the existence of minimisers of a related functional.

The inverse problem

Let $\eta \sim \mathbb{Q}_0 := N(0, \Gamma)$ be some Gaussian noise on $\mathbb{R}^J$. We observe data y,

$$y = \mathcal{G}(u, a) + \eta$$

and wish to find (u, a) (and hence $\kappa = \exp(u^a)$)

- Problem is severely underdetermined: y is finite dimensional, but (u, a) is infinite dimensional
- y may not lie in the image of $\mathcal{G}$, so we cannot simply invert the map.

The Bayesian approach

- We look at the problem probabilistically, using a Bayesian approach
- Incorporate prior beliefs about (u, a) via **prior measures** μ_0 on X and ν_0 on Λ
- Easy to see that the conditional distribution of $y|(u,a)$ is $N(\mathcal{G}(u,a), \Gamma)$. We have **negative log-likelihood** (or potential)

$$\Phi(u, a; y) = \frac{1}{2}|\mathcal{G}(u,a) - y|_\Gamma^2$$

  where $|\cdot|_\Gamma = |\Gamma^{-1/2} \cdot|$
- Our solution to the problem will be the **posterior distribution** of $(u,a)|y$

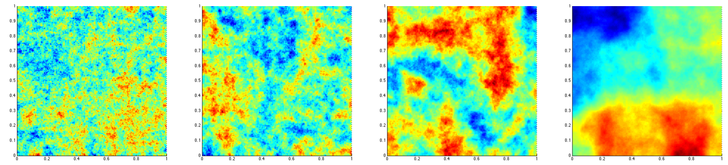
$$\mu^y(\mathrm{d}u, \mathrm{d}a) = (\mu_0 \times \nu_0 \,|\, y)(\mathrm{d}u, \mathrm{d}a)$$

Prior modelling

- We take a compactly supported measure ν_0 with Lebesgue density ρ for the geometric prior
- For each field u_i we take independent Gaussian priors μ_0^i:

$$\mu_0^i = N(m_i, \mathcal{C}_i), \quad i = 1, \dots, N$$

- $\mu_0 := \mu_0^1 \dots \mu_0^N$ is hence Gaussian on X
- Examples of Gaussian samples from $\mu_0^i = N(0, (-\Delta)^{-\alpha})$, $\alpha = 1, 1.3, 1.6, 1.9$:



Theorem (Bayes' Theorem)

- Bayes' theorem allows us to characterise the posterior on $(u,a)|y$ in terms of Φ:

  Assume that $\Phi : X \times \Lambda \times \mathbb{R}^J \to \mathbb{R}$ is $\mu_0 \times \nu_0 \times \mathbb{Q}_0$ measurable, and

$$Z := \int_{X \times \Lambda} e^{-\Phi(u,a;y)} \mu_0(\mathrm{d}u)\nu_0(\mathrm{d}a) > 0$$

  Then the conditional distribution μ^y of $(u,a)|y$ exists, $\mu^y \ll \mu_0 \times \nu_0$ and

$$\mu^y(\mathrm{d}u, \mathrm{d}a) = \frac{1}{Z} e^{-\Phi(u,a;y)} \mu_0(\mathrm{d}u)\nu_0(\mathrm{d}a)$$

MAP estimators and Onsager-Machlup functionals

- We define the modes (maximum a-posteriori estimators) of the posterior distribution μ^y as follows:

  *Given $(u,a) \in X \times \Lambda$, let $(z^\delta, a^\delta) = \underset{(z,a) \in X \times \Lambda}{argmax} \mu^y(B^\delta(z,a))$. A point $(\bar{z}, \bar{a}) \in X \times \Lambda$ is called a **MAP estimator** for the measure μ^y if it satisfies*

$$\lim_{\delta \downarrow 0} \frac{\mu^y(B^\delta(\bar{z}, \bar{a}))}{\mu^y(B^\delta(z^\delta, a^\delta))} = 1$$

- We can often characterise the above limit in terms of the Onsager-Machlup functional:

  *A functional $I : X \times \Lambda \to \mathbb{R}$ is called the **Onsager-Machlup functional** for μ^y if, for each $(z_1, a_1), (z_2, a_2) \in X \times \Lambda$,*

$$\lim_{\delta \downarrow 0} \frac{\mu^y(B^\delta(z_1, a_1))}{\mu^y(B^\delta(z_2, a_2))} = \exp\left(I(z_2, a_2) - I(z_1, a_1)\right)$$

Theorem (Onsager-Machlup functional for μ^y)

Let E denote the Cameron-Martin space of μ_0 and let S denote the support of ν_0. Then on $E \times S$, the Onsager-Machlup functional for μ^y is given by

$$I(u,a) = \frac{1}{2}|\mathcal{G}(u,a) - y|_\Gamma^2 + \frac{1}{2}\|u\|_E^2 - \log \rho(a)$$

Outside of $E \times S$ it is infinite.

Theorem (Equivalence of MAP estimators and minimisers of I)

- It is standard to check that minimisers of I exist in $E \times S$
- These minimisers turn out to be modes of the posterior:

  Under certain conditions on the potential Φ, we have the following:
  1. *any MAP estimator minimises the Onsager-Machlup functional I;*
  2. *any $(z^*, a^*) \in E \times S$ which minimises the Onsager-Machlup functional I is a MAP estimator for μ.*

Numerical Simulations

- We can sample from the posterior directly by using MCMC methods (specifically, a Metropolis-within-Gibbs approach using the preconditioned Crank-Nicolson (pCN) method for the fields and a random-walk Metropolis (RWM) method for the geometric parameters)
- Owing to the theorem above, we could also seek to minimise I to obtain Dirac approximations to the posterior

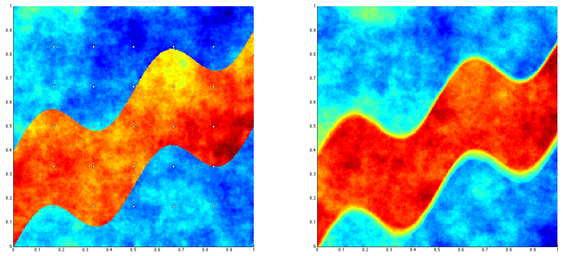


Figure : An example of a true log-permeability u^a and 25 observation points (left), and the approximate mean of the posterior arising from these observations(right)

- By changing the model we are using for the data to include multiplicative noise,

$$y = (1 + \eta_m)\mathcal{G}(u,a) + \eta_a$$

  we get a better-behaved potential Φ
- This allows us to make larger jumps in the MCMC proposals, leading to better convergence/mixing properties
- This model may also be more physically realistic

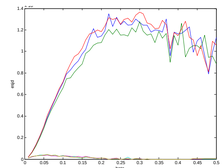


Figure : The expected squared jump distance (ESJD) of some observations of the fields as the jump size β varies. The upper curves correspond to the modified noise model, and the lower ones to the original additive model.

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