# Benjamin Lees - Research Statement

## Current Research

My thesis has focused on classical and quantum spin systems. I have been especially interested in investigating phase diagrams of spin systems using tools of mathematics, mathematical physics and probability. In spin systems the Hamiltonian consists, for example, of interactions of the form $J_{xy}\mathbf{S}_x \cdot \mathbf{S}_\mathbf{y}$, $J_{xy} \in \mathbb{R}$ and $\mathbf{S}_\mathbf{x} = (S^1_x, S^2_x, S^3_x)$ for particles $x$ and $y$. In the classical case $\mathbf{S}_x$ is a unit vector and in the quantum case it is given by *spin operators*. For quantum spin-$S$ particles they are matrices on $\mathbb{C}^{2S+1}$ satisfying certain relations. There have been many important results for both classical and quantum spin systems. For example the proof of a phase transition for the classical Heisenberg model [6]. For spins on a parallelepiped in $\mathbb{Z}^d$ ($d \geq 3$) and Hamiltonian $H(\{\mathbf{S}\}) = -J \sum_{|x-y|=1} \mathbf{S}_x \cdot \mathbf{S}_\mathbf{y}$ it was shown that there is a phase transition at low temperatures. The proof used the methods of reflection positivity and Gaussian domination, these have been very important in my own research. This result was extended to the quantum system soon after [5].

### The Spin-1 Quantum System

Much of my research has concerned the spin-1 system. Force carrying particles (photons, gluons) have spin-1, making the system of interest to physicists. Here a general Hamiltonian for pair interactions would be

$$H = -\sum_{|x-y|=1} J_1 \mathbf{S}_x \cdot \mathbf{S}_\mathbf{y} + J_2(\mathbf{S}_x \cdot \mathbf{S}_\mathbf{y})^2. \tag{1}$$

This seemingly simple model has a surprisingly rich behaviour, with several phases expected. These phases are ferromagnetic, antiferromagnetic, nematic and staggered nematic (see [11] for details). A wide array of tools (both theoretical and experimental) have been used to understand this system.

#### The Nematic Phase

In the classical case it is intuitively clear how the $J_2$ term affects the energy of the system. In the quantum case the same intuition leads us astray. I was able to prove the following theorem (for more detail see [12]).

**Theorem 1.** *For cubic $\Lambda \subset \mathbb{Z}^d$, $d \geq 6$ for $0 < -J_1 \ll J_2$ and $\beta$ large enough there exists $c = c(\alpha, d) > 0$ such that*

$$\lim_{|\Lambda| \to \mathbb{Z}^d} \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \langle ((S^3_0)^2 - 2/3)((S^3_x)^2 - 2/3) \rangle \geq c. \tag{2}$$

Here $\langle \cdot \rangle$ is the Gibbs state at inverse temperature $\beta$. It is known that such a bound on the correlations implies the occurrence of a phase transition. This inequality implies a nematic phase transition. It has been shown [17] that for $J_1 = 0$ there is also a stronger antiferromagnetic phase transition. This result did not extend to $J_1 < 0$ but interestingly the results of [5] show such an antiferromagnetic phase transition for $J_1 < 0 = J_2$ (and hence for some unspecified $J_2$'s with $-J_1 \gg J_2 > 0$).

#### The Antiferromagnetic Phase

The phase diagram in the remainder of the quadrant $J_1 < 0 < J_2$ was a focus of a lot of my research [11]. In order to study the system given by (1) in this quadrant I used and further developed a model introduced in [13]. The idea of the model is to represent a spin-1 system as a projection of two spin-$\frac{1}{2}$ systems. By attaching an interval $[0, \beta]$ to each lattice site in this new system one can introduce a probabilistic model of loops inspired by [1, 15, 17], built up by geometric events on edges $\{x, y\} \times [0, \beta]$. The events are laid down according to a Poisson point process and a configuration $\omega$ is then given a weighting of $2^{\#\text{loops in } \omega}$. The following theorem can be found in [11].

**Theorem 2.** *For cubic $\Lambda \subset \mathbb{Z}^d$, $d \geq 3$, there exists $\alpha(d) > 0$ such that $\alpha(d) \to 0$ as $d \to \infty$ and for $J_1 \leq 0 \leq J_2$ if $-J_1/J_2 > \alpha(d)$ then there exists $c = c(\alpha, d) > 0$ such that*

$$\lim_{\beta \to \infty} \lim_{|\Lambda| \to \infty} \frac{1}{|\Lambda|} \sum_{x \in \Lambda} (-1)^x \langle S^3_0 S^3_x \rangle_{\Lambda, \beta} \geq c. \tag{3}$$

This is equivalent to the existence of infinite loops in the probabilistic model when $|\Lambda| \to \infty$ for $\beta$ large. Here $\alpha(3) = 0.46$ (corresponding to $\approx 72\%$ of the quadrant). This result extends the proof of [5]. There are unexpected difficulties in the proof. One difficulty is bounding expectations of double commutators of spin operators (coming from the application of the Falk-Bruch inequality). These expectations can be given bounded using the loop model but careful consideration of the loop structure is required.

## Correlation Inequalities for Quantum Spin Systems

My co-authors C. Benassi, D. Ueltschi and I also obtained a new correlation inequality for quantum spin systems [2]. Correlation inequalities were proposed by Griffiths for classical spin systems [8]. They have been very useful for showing (for example) infinite volume limits of correlation functions and monotonicity of spontaneous magnetisation. There are few corresponding results for quantum systems. Gallavotti proved such inequalities for the quantum XY model in spin-$\frac{1}{2}$ with pair interactions. The results are as follows.

**Theorem 3.** *Let* $J^i_A \geq 0$ *for* $A \subset \Lambda$, $i = 1, 2$. *For* $S = \frac{1}{2}$ *or in the limit* $\beta \to \infty$ *for* $S = 1$ *we have for all* $A, B \subset \Lambda$

$$\Big\langle \prod_{x\in A} S^1_x ; \prod_{x\in B} S^1_x \Big\rangle - \Big\langle \prod_{x\in A} S^1_x \Big\rangle \Big\langle \prod_{x\in B} S^1_x \Big\rangle \geq 0, \tag{4}$$

## Annealed Spin Systems

R. Kotecký and I have investigated the phases of an annealed quantum Heisenberg model. We proved the occurrence of *staggered order* for a certain region of the parameters and intermediate inverse temperatures [9]. This order is characterised by preferential occupation of the even or odd sublattice. This complements the work of Chayes, Kotecký and Shlosman [4] on the classical models. The quantum case is entirely new. It is given by quantum particles on sites having a classical occupation number (0 or 1). The Hamiltonian is

$$H_L = -\frac{1}{S^2} \sum_{<x,y>} n_x n_y \Big( S^1_x S^1_y + u S^2_x S^2_y + S^3_x S^3_y - S(S+1) \Big) - \mu \sum_{x\in\Lambda_L} n_x - \kappa \sum_{<x,y>} n_x n_y. \tag{5}$$

For $S \in \frac{1}{2}\mathbb{N}$, $u, \mu, \kappa \in \mathbb{R}$ and $|u| \leq 1$. This mixture of quantum and classical added some extra challenges, for example reflection positivity does not hold for every model (famously the Heisenberg ferromagnet is not reflection positive). However for the antiferromagnet and XY model reflection positivity can be extended to the annealed case and we can prove the following theorem.

**Theorem 4.** *Let* $u = -1$ *and* $S \geq \frac{1}{2}$ *or* $u = 0$ *and* $S = \frac{1}{2}$. *For each case there exists* $\mu_0 > 0$ *and a function* $\kappa_0$ *(both depending on* $u, S$*, and* $d$*) that is positive on* $(0, \mu_0)$ *and such that for any* $\mu > 0$, $\kappa < \max(\kappa_0(\mu), 0)$, *and any* $0 < \varepsilon < \frac{1}{2}$, *there exists* $\beta_0 = \beta_0(\mu, \kappa, \epsilon)$ *such that for any* $\beta > \beta_0$ *there exist two distinct KMS states,* $\langle\cdot\rangle^{\mathrm{e}}_\beta$ *and* $\langle\cdot\rangle^{\mathrm{o}}_\beta$, *that are staggered,*

$$\langle \mathcal{G}^{\mathrm{e}} \rangle^{\mathrm{e}}_\beta \geq 1 - \varepsilon \text{ and } \langle \mathcal{G}^{\mathrm{o}} \rangle^{\mathrm{o}}_\beta \geq 1 - \varepsilon. \tag{6}$$

# Further Research

In the future I very much hope for and look forward to working on some of the many beautiful problems in my and related fields. I wish to obtain postdoctoral positions in the area of mathematical physics with the aim of obtaining a permanent position involving both research and teaching.

## Poisson Dirichlet Structure of Loop Models

My future research would concern the underlying loop models used for study of quantum spin systems. These loop models find their origin in the works of Tóth [15] and Aizenman and Nachtergaele [1] for the spin-$\frac{1}{2}$ ferromagnet and antiferromagnet respectively. These models were combined and extended by Ueltschi [16] where it was shown they can also be used for higher spins. The work of Nachtergaele [13] has more recently been used by myself [11] to show occurrence of Néel order (see theorem 1) as well as some correlation inequalities. It has been shown that similar representations to the one presented in [13] can be used for any values of $(J_1, J_2)$. Loop models can be exploited to obtain results for quantum spin systems such as emptiness formation probability, classification of gapped ground states and the nature of pure Gibbs states.

These loops models are expected to exhibit interesting behaviour in the thermodynamic limit for $\beta$ large. Namely they are expected to obey a Poisson Dirichlet law. There are only initial results in this directions [3, 14]. Proof of such a structure would add much to current understanding. For example the calculation of certain two-point correlation functions at long range and understanding of the nature of pure Gibbs states of the related quantum model. One could also gain insight into the relation between Néel and nematic order for quantum systems. The first step towards this conjecture would be to consider the loop model in the region $J_1 < 0 < J_2$ without the factor $2^{\#\text{loops in }\omega}$. There has been some success in similar areas. In particular the random interchange model (equivalent to the loop model for $J_1 > 0 = J_2$ without the factor $2^{\#\text{loops in }\omega}$) was studied by Schramm [14]. It was shown that in this case there is the Poisson Dirichlet structure. The first step towards the conjecture would be to understand the methods used in these papers. For the model with the factor $2^{\#\text{loops in } \omega}$ there has been work on the complete graph by Björnberg [3], understanding the method of this paper could be of much use. The recent work of Kotecký, Miłoś and Ueltschi [10] is also of interest.

### Correlation inequalities for Quantum Spin Systems

Correlation inequalities were proposed by Griffiths for classical spin systems [8]. A more general framework was laid out by Ginibre [7] which included quantum systems, however it is not easy to check a given quantum systems satisfies the conditions required. It was proved [2] that XY models with general interactions for all temperatures in spin-$\frac{1}{2}$ and for the ground state in spin-1 fits this framework. The extension of this result to positive temperatures in higher spins would be well received by the community. Such results are useful for showing, for example, infinite volume limits of correlation functions and monotonicity of spontaneous magnetisation. A conjecture for this result would be the same as theorem 3 for higher spins. The model of Nachtergaele [13] was for the spin-1 result, this model can also be used for higher spins. Careful consideration of the framework of Ginibre could yield ideas or results allowing progress. Currently it seems the full Heisenberg model is out of reach however a general result for XY models would be a useful first step.

## References


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