ES441 Advanced Fluid Dynamics Support 3 – Turbulence Similarity Solutions

Reynolds Averaged Navier-Stokes

Idea: Split velocity into components

$$u^T = U + u$$

U: time averaged or mean velocity

u: fluctuations from the average velocity (turbulent part)

then re-write Navier-Stokes:

describe average momentum U and V. RANS

Stream-wise (x-component) (U)

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + \frac{\partial}{\partial y}(\overline{uv}) + \frac{\partial}{\partial x}(\overline{u^2}) = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right)$$

$\overline{uv}$: average of (uv)

Cross-stream (y-component V):

$$U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} + \frac{\partial}{\partial x}(\overline{uv}) + \frac{\partial}{\partial y}(\overline{v^2}) = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right)$$

Then make some assumptions about magnitudes of various things and reduces to

Reduce RANS

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + \frac{\partial}{\partial y}(\overline{uv}) = 0. \qquad (1)$$

Now assume similarity solution (looks the same at all scales)

$$U = U_s(x)\, f\left(\frac{y}{\ell(x)}\right)$$

$U_s(x)$: velocity scale

$\ell(x)$: length scale

f: some function to be determined.

Use the eddy viscosity assumption

$$\tau = -\overline{uv} = \nu_T\frac{\partial U}{\partial y} = U_s^2\, g\left(\frac{y}{\ell}\right)$$

ν_T: eddy viscosity

Substituting similarity solution into (1)
then provides the Self-preservation equations.
(The equation for f, f', f'' etc).
(Different for wake/jet).

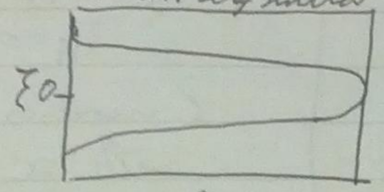
Momentum integral is required to provide constraints to be able to find the values for m and n in the power law solutions
$$U_s(x) \sim x^m \quad , \quad L(x) \sim x^n$$
These describe how mean velocity profile changes downstream.

Example: Turbulent Jet 2-13. . .

2.13 Turbulent Jet

Consider plane jet with mean flow profile

Mean self similar profile.



f

Self preservation equation for a jet is

$$\left[\frac{l}{U_s}\frac{dU_s}{\partial x}\right] f^2 - \left[\frac{\partial l}{\partial x}\right] f' \int_0^{\xi} f\, d\eta - \left[\frac{l}{U_s}\frac{dU_s}{\partial x}\right] f' \int_0^{\xi} f\, d\eta = \frac{1}{R_T} g'$$

(a) How does downstream length scale l change?

__Answer__ For self-preservation, ie. for shapes of f and g to be the same downstream, we need the coefficients of f to be independent of x, that is

$\frac{l}{U_s}\frac{dU_s}{\partial x}$ and $\frac{dl}{\partial x}$ independent of x

$\Rightarrow l \sim Bx$

(b) Velocity scale is determined by the constant momentum integral

$$M = \int_{-\infty}^{\infty} U^2(x,y)\,dy = [U_s^2 l] \int_{-\infty}^{\infty} f^2 d\xi$$

How does velocity scale U_s change with distance downstream?

__Answer__ We assume $U_s(x) = Ax^n$

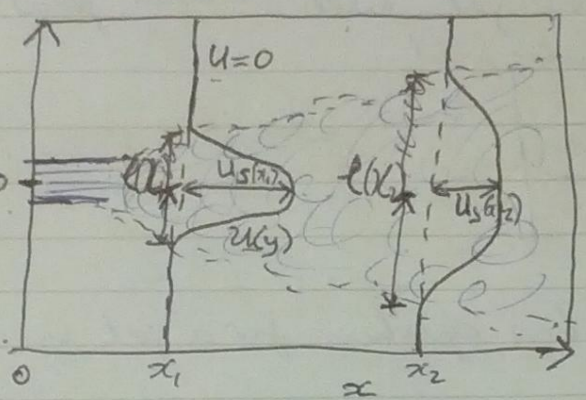
So $[U_s^2 l] \sim AB\, x^{2n} x^m$

and we need $U_s^2 l$ to be independent of x, therefore

$2n + m = 0$, we have $m = 1$

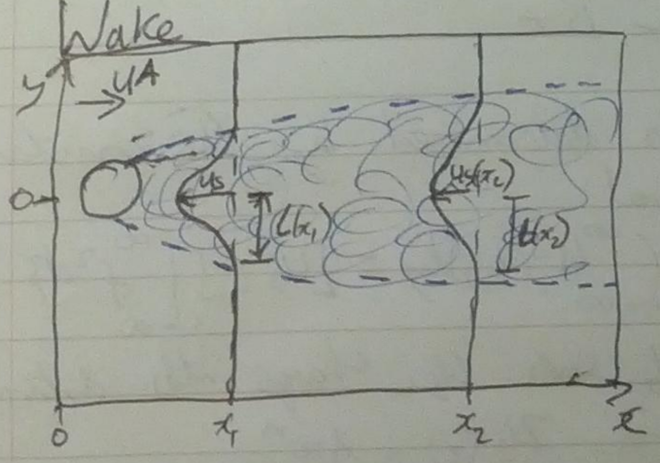
$\Rightarrow n = -\frac{1}{2}$ So $U_s \sim Ax^{-1/2}$

(c) Sketch how a turbulent jet grows downstream.

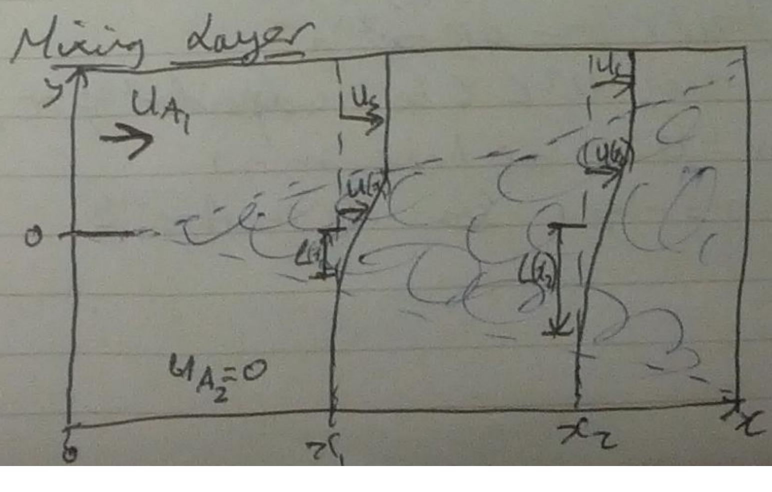


$u_s = \max |u_A - U|$
$u_A = 0$ mean flow scale
ℓ increases linearly with x, $(\ell \sim x)$
U_s decreases as $x^{-1/2}$.

(d) Where are the constant stress layers in a turbulent jet?
<u>Answer</u> Close to a wall (or stationary fluid)
there is a layer between the viscous layer and the laminar flow where the Reynolds stress is constant. These are located either side of the centre of the jet.



U_s decreases as $x^{-1/2}$
ℓ grows as $x^{1/2}$.



ℓ grows linearly but U_s does not change downstream.