MA4H7 Atmospheric Dynamics Support Handout 4 - Waves in Boussinesq Equations

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1 Boussinesq Equations

A standard approximation to make in geophysical fluid dynamics is the *Boussinesq Approximation*. This approximation is based on the fact that the density of a geophysical fluid does not vary greatly from a mean value. We therefore express density as

$$\rho = \rho_0 + \rho'(x, y, z, t) \tag{1}$$

where $|\rho'| \ll \rho_0$. Inserting this into the continuity equation and ignoring terms of order ρ' we get incompressibility

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{2}$$

For the x and y momentum equations in Navier-Stokes, any term multiplied by ρ is dominated by ρ_0 and terms multiplied by density variations can be ignored. With the assumption of a Newtonian fluid the stress tensor $\boldsymbol{\tau}$ is defined as

$$\begin{aligned}
\tau^{xx} &= \mu\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\right), \tau^{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \tau^{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\
\tau^{yy} &= \mu\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial y}\right), \tau^{yz} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \tau^{zz} = \mu\left(\frac{\partial w}{\partial z} + \frac{\partial w}{\partial z}\right)
\end{aligned} \tag{3}$$

where μ is the coefficient of dynamic viscosity. Dividing the x and y momentum equations by ρ_0 and setting the kinematic viscosity $\nu = \mu/\rho_0$ we have

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + f_* w - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu\Delta u \tag{4}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \nu\Delta v. \tag{5}$$

For the z momentum equation we define the hydrostatic pressure p_0 which varies only in z so that for some reference pressure P_0

$$p = p_0(z) + p'(x, y, z, t) \text{ where } p_0(z) = P_0 - \rho_0 g z. \tag{6}$$

This gives

$$\frac{dp_0}{dz} = -\rho_0 g, \tag{7}$$

then the z momentum equation becomes

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} - f_* u = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - \frac{\rho' g}{\rho_0} + \nu\Delta w. \tag{8}$$

The $\rho' g$ term is important for buoyancy force in the equations, therefore ignoring it would result in a bad geophysical model. Finally the energy equation (sometimes called density equation or buoyancy equation) with diffusion coefficient κ is

$$\frac{\partial \rho'}{\partial t} + u\frac{\partial \rho'}{\partial x} + v\frac{\partial \rho'}{\partial y} + w\frac{\partial \rho'}{\partial z} = \kappa\Delta\rho'. \tag{9}$$

Since the variables ρ and p do not appear explicitly in the equations we drop the prime notation on the variations ρ' and p' to get the *Boussinesq Equations.*

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+f_* w-fv=-\frac{1}{\rho_0}\frac{\partial p}{\partial x}+\nu\Delta u \tag{10}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+fu=-\frac{1}{\rho_0}\frac{\partial p}{\partial y}+\nu\Delta v \tag{11}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}-f_* u=-\frac{1}{\rho_0}\frac{\partial p}{\partial z}-\frac{\rho' g}{\rho_0}+\nu\Delta w \tag{12}$$

$$\nabla\cdot\boldsymbol{u}=0 \tag{13}$$

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+w\frac{\partial \rho}{\partial z}=\kappa\Delta\rho. \tag{14}$$

Example 1. *(Internal Gravity Waves) These are waves generated by buoyancy forces. We use the following linearised equations for an inviscid, stratified fluid with small 2D motion. These equations can be derived from the Boussinseq equations above by setting* $\boldsymbol{u}=\epsilon\boldsymbol{u}_1, p=p_0+\epsilon p_1, \rho=\rho_0+\epsilon\rho_1$ *and ignoring any terms of order greater than* ϵ.

$$\rho_0\frac{\partial u_1}{\partial t}=-\frac{\partial p_1}{\partial x},\qquad \rho_0\frac{\partial w_1}{\partial t}=-\frac{\partial p_1}{\partial z}-\rho_1 g,$$
$$\frac{\partial u_1}{\partial x}+\frac{\partial w_1}{\partial z}=0,\qquad \frac{\partial \rho_1}{\partial t}+w_1\frac{\partial \rho_0}{\partial z}=0.$$

Find the dispersion relation $\omega(k)$, *we assume solutions of the form*

$$u_1=\hat{u}_1 e^{i(kx+mz-\omega t)},\ \ w_1=\hat{w}_1 e^{i(kx+mz-\omega t)},$$
$$p_1=\hat{p}_1 e^{i(kx+mz-\omega t)},\ \ \rho_1=\hat{\rho}_1 e^{i(kx+mz-\omega t)}.$$

Put these into the linearised equations and eliminate $\hat{u_1}, \hat{w_1}, \hat{p_1}, \hat{\rho_1}$ *to get*

$$\omega^2=\frac{k^2N^2}{(m^2+k^2)}\ \textit{where}\ N^2=\frac{-g}{\rho_0}\frac{\partial\rho_0}{\partial z}.$$

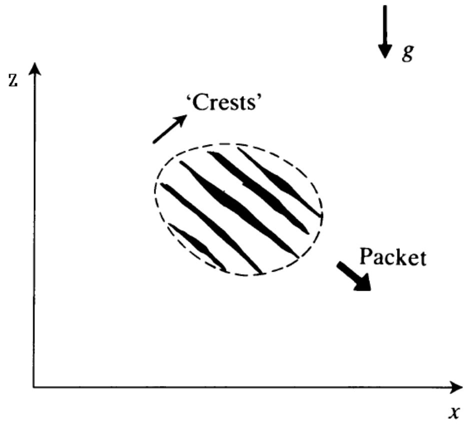

Figure 1: Propagation of a 2D packet of internal gravity waves; the crests denote lines of constant phase $kx+mz-\omega t$.

The phase velocity is then (where $\boldsymbol{k}=(k,m)$*)*

$$c_{ph}=\frac{\omega\boldsymbol{k}}{|\boldsymbol{k}|^2}=\frac{\omega}{(k^2+m^2)}(k,m),$$

and the group velocity is then

$$c_g = \left(\frac{\partial}{\partial k}, \frac{\partial}{\partial m}\right)\omega = \frac{\omega m}{k(k^2+m^2)}(m, -k).$$

So we can see that $c_{ph} \propto (k, m)$ is perpendicular to $c_g \propto (m, -k)$. As time proceeds the crests move in direction (k, m) and the packet moves in the perpendicular direction $(m, -k)$.

Example 2. *(Question from McWilliams, Fundamentals of Geophysical Fluid Dynamics) Consider inertia-gravity waves with small amplitude fluctuations in 3D Boussinesq equations with $\rho = \rho_0(1 - \alpha T)$ for basic state of rest (this is the Boussinesq approximation for density in seawater) with uniform rotation $f = f_0$ and stratification $\bar{\rho}(z) = \rho_0(1 - N_0^2 z/g)$ in an unbounded domain. The linearised equations are*

$$\partial_t u - fv = \partial_x \phi \tag{15}$$

$$\partial_t v + fu = \partial_y \phi \ \textit{(Momentum equations)} \tag{16}$$

$$\partial_t w = \partial_z \phi + b \tag{17}$$

$$\nabla \cdot \boldsymbol{u} = 0 \ \textit{(Incompressibility)} \tag{18}$$

$$\partial_t b + N^2 w = 0 \ \textit{(Conservation of internal energy)} \tag{19}$$

with $\phi = p/\rho_0$ the geopotential and $b = g\rho/\rho_0$ the buoyancy.

(a) Derive the dispersion relation.
Answer: *Look for solutions $u = u_0 e^{i(kx+ly+mz-\omega t)}, v = v_0 e^{i(kx+ly+mz-\omega t)}, w = w_0 e^{i(kx+ly+mz-\omega t)}$, $\phi = \phi_0 e^{i(kx+ly+mz-\omega t)}, b = b_0 e^{i(kx+ly+mz-\omega t)}$. Follow a similar method as for the inertial waves in a rotating fluid and substitute solutions into the equation to get the dispersion relation,*

$$\omega = \pm\left(\frac{(k^2+l^2)N^2 + m^2 f^2}{k^2+l^2+m^2}\right)^{\frac{1}{2}} \tag{20}$$

(This calculation is messy, but don't worry about it too much.)

(b) Demonstrate that ω depends only on the direction of $\boldsymbol{u}$ and not its magnitude $K = |\boldsymbol{u}|$.
Answer: *Express $\boldsymbol{k} = (k, l, m)$ in polar form $\boldsymbol{u} = (K, \theta, \lambda)$ with $k = K\cos(\theta)\cos(\lambda), l = K\cos(\theta)\sin(\lambda), m = K\sin(\theta)$. Then*

$$\omega = \pm(N^2\cos^2(\theta) + f^2\sin^2(\theta))^{\frac{1}{2}}. \tag{21}$$

So it depends on the orientation θ but is independent of the magnitude K.

(c) Show that N and f are the largest/smallest frequencies allowed for inertial-gravity modes. (Assume $f < N$).
Answer: *If $f < N$ then*

$$\max|\omega| = N, \ \textit{for } m = 0, l = 0, k = K \ \ (\theta = 0, n\pi \ \textit{for } n \in \mathbb{N})$$
$$\min|\omega| = f, \ \textit{for } m = K, l = 0, k = 0 \ \ (\theta = n\pi/2 \ \textit{for } n \ \textit{odd}).$$

(d) Demonstrate that the phase and group velocities

$$c_g = \frac{\omega}{\boldsymbol{k}} = \frac{\omega \boldsymbol{k}}{K^2}, \ \ c_{ph} = \frac{\partial \omega}{\partial \boldsymbol{k}} = \left(\frac{\partial \omega}{\partial k}, \frac{\partial \omega}{\partial l}, \frac{\partial \omega}{\partial m}\right),$$

are orthogonal and have opposite signed vertical components for the inertial-gravity mode.

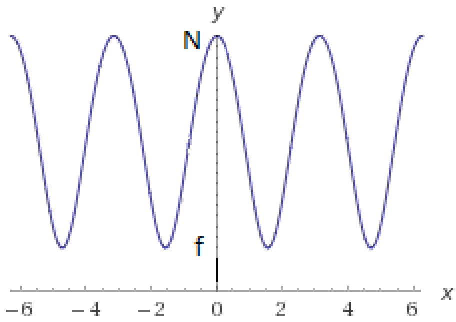

Figure 2: Plot of the dispersion relation (21).

Answer:

$$c_{ph} = \frac{\omega \boldsymbol{k}}{K^2} = \frac{\omega}{K^2}(k, l, m), \;\; c_g = \frac{\partial \omega}{\partial \boldsymbol{k}} = \frac{1}{\omega K^2}(k(N^2-\omega^2), l(N^2-\omega^2), m(f^2-\omega^2)).$$

Taking the dot product shows they are orthogonal, $c_g \cdot c_{ph} = 0$*. Also*

$$sign[c_{ph}] = sign[\omega]sign[m], \;\; sign[c_g] = sign[\omega]sign[m]sign[f^2-\omega^2],$$

and since $f < N$

$$\omega^2 = \frac{(k^2+l^2)N^2+m^2f^2}{k^2+l^2+m^2} > \frac{(k^2+l^2)f^2+m^2f^2}{k^2+l^2+m^2} = f^2$$

so $f^2 - \omega^2 < 0$.