

# MA3D1 Fluid Dynamics Support Class 5 - Shear Flows and Blunt Bodies

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## 1 2D Flows - Shear flows

### Example 1. *Flow over an inclined plane*

A layer of viscous fluid is bounded from below by a plane fixed at angle  $\theta$  to the horizontal, and from above by a free surface. The thickness of the layer is  $h$ . The air above the free surface is at atmospheric pressure  $p_0$ . Consider a stationary flow resulting from the balance between gravity and viscosity, see figure 1.

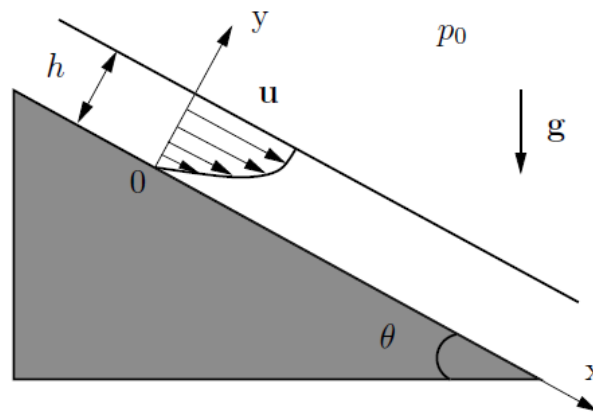


Figure 1: Flow over an inclined plane.

1. Formulate the no-slip boundary condition for the velocity field at the bottom plane and the pressure boundary condition at the free surface.

**Answer:** Clearly the flow is plane parallel, therefore

$$\mathbf{u}(x, y) = (u(y), 0), \quad (1.1)$$

where function  $u(y)$  is to be found. The no-slip condition in this case is  $u(0) = 0$  with no flow through automatically satisfied since  $v = 0$ , and the pressure boundary condition at the free surface is  $p(x, h) = p_0$  as the free surface is in contact with the atmosphere.

2. Show that absence of stress at the free surface means that  $\frac{\partial u}{\partial y} = 0$  at  $y = h$ .

**Answer:** Stress at the free surface is given by the tangential to the surface component of the viscous stress tensor,

$$\sigma_{ij} = \rho\nu(\partial_{x_i}u_j + \partial_{x_j}u_i), \quad (1.2)$$

namely by  $\sigma_{xy} = \rho\nu\partial_y u(y)$ . Absence of the stress at the free surface due to the lack of fluid above therefore means  $\partial_y u = 0$  at  $y = h$ .

3. Solve the stationary Navier-Stokes equations and find the velocity and the pressure fields (the velocity is parallel to the plane).

**Answer:** It is clear that the pressure field in this problem is independent of  $x$ . Also the perpendicular and parallel components to the plane of gravity are  $-g \cos(\theta)$  and  $g \sin(\theta)$  respectively. Thus, the stationary Navier-Stokes equations written in components are:

$$\frac{1}{\rho} \partial_y p + g \cos(\theta) = 0 \quad (1.3)$$

$$\nu \partial_{yy} u + g \sin(\theta) = 0. \quad (1.4)$$

Integrating equation (1.3) and using the pressure boundary condition  $p(x, h) = p_0$  we have

$$p = p_0 + \rho g (h - y) \cos(\theta). \quad (1.5)$$

Integrating equation (1.4) twice and using the no-slip boundary condition  $u(0) = 0$  with  $\partial_y u = 0$  at  $y = h$ , we have

$$u = \frac{g \sin(\theta)}{2\nu} y(2h - y). \quad (1.6)$$

4. Find the mass flux per unit length in the transverse to the plane of motion direction, i.e. mass of fluid  $Q$  passing through the cross-section of the layer per unit  $z$  and per unit time.

**Answer:** The mass of fluid passing through the cross-section of the layer, per unit length in the transverse direction, per unit time is

$$Q := \int_0^h \rho u \, dy = \int_0^h \frac{\rho g}{\nu} \left[ yh - \frac{1}{2} y^2 \right] \sin(\theta) \, dy = \frac{\rho g h^3}{3\nu} \sin(\theta). \quad (1.7)$$

**Example 2. Rain flow over an inclined roof**

A violent rainstorm hits a roof inclined at an angle  $\theta$  from the horizontal as shown in figure 2. The rain pours down at a mass flow rate  $Q$  per unit horizontal area, each drop falls at a velocity  $V$ . Soon a steady-state water (density  $\rho$ ) layer is established, while raindrops splash on the top part of the thin layer. The angle of the water surface relative to the roof is small ( $dh/dx \ll 1$ ), and friction between the roof and water may be neglected.

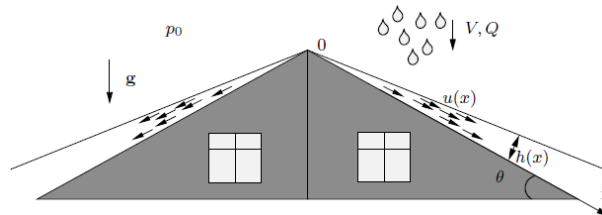


Figure 2: Flow down a roof.

1. What is the pressure distribution in the water in the direction perpendicular to the roof? Parallel to the roof? Find the pressure within the fluid layer.

2. By considering the balance of the momentum and mass in a small volume of fluid between  $x$  and  $x + dx$ , find the evolution equations for the layer thickness  $h(x)$  and the flow velocity  $u(x)$ .

3. Derive a criterion for when the rain is so violent that the shape of the water layer on the roof can be regarded as independent of gravity. Obtain a solution for  $h(x)$  in this case.

## 2 Modelling Bodies with Point Sources and Sinks

### Example 3. Flows past a zeppelin and a balloon

Flows past blunt bodies of revolution, e.g. a zeppelin-shaped body, see figure 3, can be constructed by combining a uniform flow  $\mathbf{u} = U\hat{x}$  with a set of point sources and sinks on the  $x$ -axis (here  $\hat{x}$  is a unit vector along the  $x$ -axis).

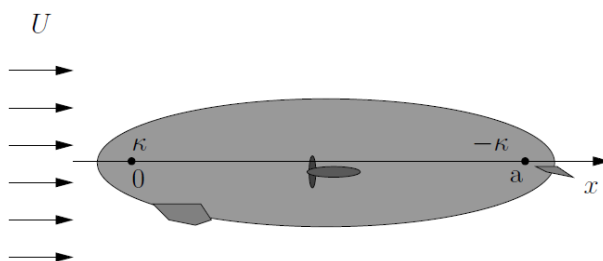


Figure 3: Flow around a zeppelin.

1. Let us first study a case with only one source and no sinks. Consider a three-dimensional flow whose velocity potential is a sum of the individual contributions corresponding to a uniform flow with velocity  $U$  along the  $x$ -axis and a source of strength  $\kappa$  at  $x = (0, 0, 0)$ . Find the velocity field of such a flow.

2. Find the stagnation point at which the velocity is zero.

3. Describe properties of the streamline passing through the stagnation point. How does this streamline behave far downstream of the stagnation point? Explain why this streamline corresponds to the shape of the body of revolution to which the considered flow is a solution with the free-slip boundary condition.

4. Now let us consider a case with only one source and one sink. Namely, let the velocity potential be a sum of the individual contributions corresponding to a uniform flow with velocity  $U$  along the  $x$ -axis, a source of strength  $\kappa$  at  $x = (0, 0, 0)$  and a sink of the same strength  $\kappa$  located at  $x = (a, 0, 0)$ , see figure 3. Find the velocity field of such a flow.

5. Consider the limit  $a \rightarrow 0$  keeping  $\mu = \kappa a$  constant. Find the velocity potential  $\phi$  and the radial component of velocity  $u_r$ . Find the surface where  $u_r = 0$ . What is the shape of the body of revolution in the limit  $a \rightarrow 0$ ? (Hint: think of the free-slip boundary condition in terms of  $u_r$ ).