

# MA3D1 Fluid Dynamics Support Class 8 - Turbulence

6th March 2015

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## 1 Turbulence

Turbulence is a flow regime characterised by rapid property changes including rapid variation of pressure and velocity in space and time. Turbulence can be described as a state of continuous instability where random features of the flow are dominant. It occurs at high Reynolds numbers and is characterised by *cascades*. Turbulence causes the formation of eddies of many different

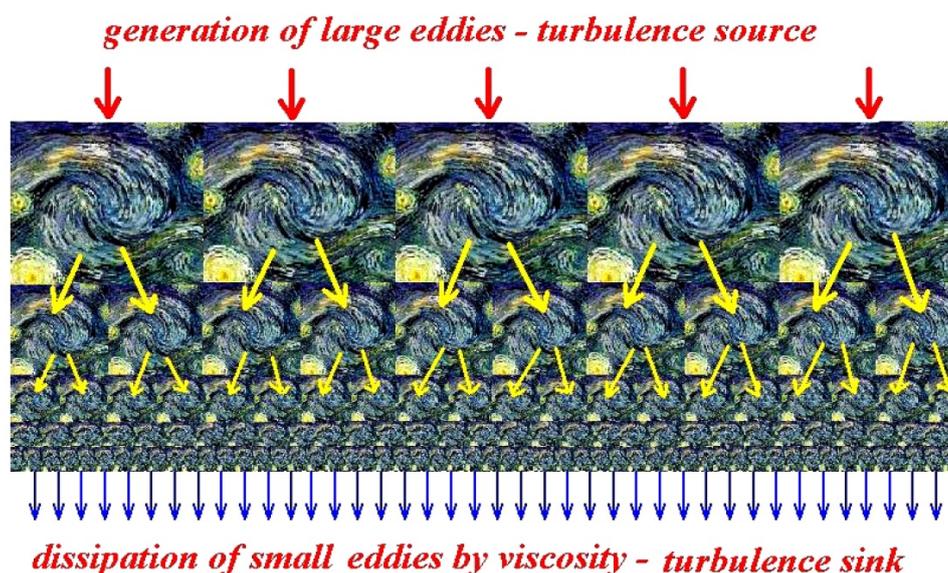


Figure 1: Turbulence characterised by transfer of energy to smaller scales.

length scales. Energy cascades from large structures down to the smaller ones until the energy is dissipated by viscous effects. The scale at which this dissipation occurs is the *Kolmogorov length scale*.

The way in which energy is distributed over the scales is given by the energy spectrum function  $E(k)$  with  $k = |\mathbf{k}|$ . Kolmogorov predicted the form of this energy spectrum with a statistical/dimensional argument. Let  $\epsilon$  be the energy dissipation rate, consider a length scale  $L > l > \eta = (\nu^3/\epsilon)^{1/4}$  where  $l = 1/k$ , let  $k_f \ll k \ll k_\eta$  with  $k_f \sim L^{-1}$  and  $k_\eta \sim \eta^{-1}$  (Kolmogorov wavenumber).

The dimensions are of the energy dissipation rate and energy are

$$[\epsilon] = \left[ \frac{U^2}{T} \right], \quad [E(k)] = \left[ \frac{U^2}{k} \right] = \left[ \frac{L^3}{T^2} \right]. \quad (1.1)$$

Assuming  $E(k) \sim \epsilon^a k^b$  then we have

$$\left[ \frac{L^3}{T^2} \right] = \left[ \frac{L^2}{T^3} \right]^a \left[ \frac{1}{b} \right]^b.$$

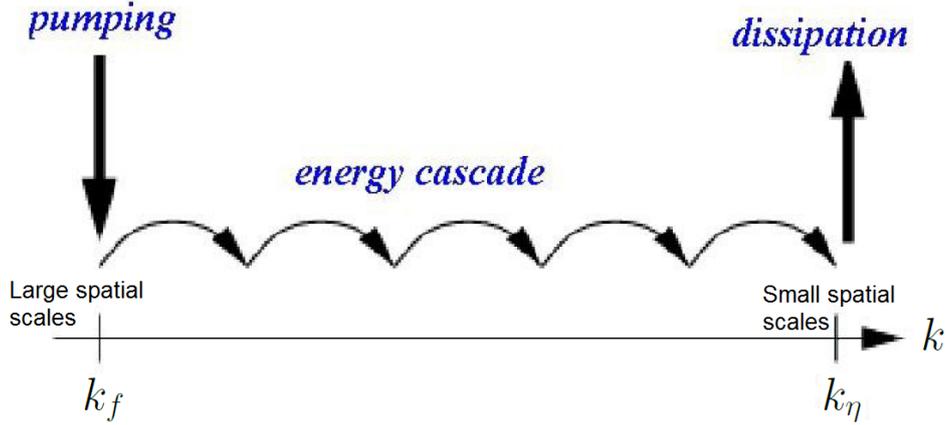


Figure 2: Visualisation of the energy cascade in Fourier space. Forcing occurs at the large spatial scales with small wavenumbers  $k_f$  and the energy cascades down to the small spatial scales with wavenumber  $k_\eta$  where dissipation occurs.

Matching powers of  $L$  and  $T$  gives  $b = -5/3$  and  $a = 2/3$ . Therefore the only dimensionally consistent energy spectrum is

$$E(k) = C\epsilon^{\frac{2}{3}}k^{-\frac{5}{3}} \quad (1.2)$$

where we hope  $C$  is a universal constant.

**Example 1. Dual cascade in evolving two-dimensional turbulence** Given information: the energy and the enstrophy densities in the two-dimensional plane,  $E$  and  $Z$  respectively, are expressed in terms of the one-dimensional energy spectrum  $\hat{E}(k)$  as follows

$$E = \int_0^\infty \hat{E}(k) dk, \text{ and } Z = \int_0^\infty k^2 \hat{E}(k) dk, \quad (1.3)$$

where  $k = |\mathbf{k}|$  is the wavevector length.

The energy and the enstrophy centroids describe the wavevectors containing most of energy and enstrophy (at a particular moment of time). They are defined respectively as

$$k_E = \frac{1}{E} \int_0^\infty k \hat{E}(k) dk \quad (1.4)$$

$$k_Z = \frac{1}{Z} \int_0^\infty k^3 \hat{E}(k) dk. \quad (1.5)$$

The Cauchy-Schwarz inequality is

$$\left| \int_0^\infty f(k)g(k) dk \right| \leq \left| \int_0^\infty f^2(k) dk \right|^{1/2} \left| \int_0^\infty g^2(k) dk \right|^{1/2} \quad (1.6)$$

for any functions  $f, g \in L^2$ .

Usually, Fjørtoft's argument finds the directions of the energy and the enstrophy cascades in stationary 2D turbulence in presence of forcing and dissipation. Below, we will consider evolving 2D turbulence (no forcing or dissipation), and will re-formulate Fjørtoft's argument in terms of the energy and enstrophy centroids.

1. Explain why the enstrophy is conserved in 2D ideal flows but not in 3D flows.

2. Assuming that the integrals defining  $E, Z, k_E, k_Z$  converge and using the Cauchy-Schwarz inequality, prove that:

$$k_E \leq \sqrt{\frac{Z}{E}} \quad (1.7)$$

$$k_E k_Z \geq \frac{Z}{E} \quad (1.8)$$

$$k_Z \geq \sqrt{\frac{Z}{E}}. \quad (1.9)$$

**Hint:** You might like to choose  $f(k)$  and  $g(k)$  from the following list:  
 $\hat{E}^{1/2}, k^{1/2} \hat{E}^{1/2}, k \hat{E}^{1/2}, k^{3/2} \hat{E}^{1/2}$ .

3. Interpret inequalities (1.7) and (1.9) in terms of the allowed directions for the energy and the enstrophy transfers in the  $k$ -space (cascades). What does inequality (1.8) say about these transfers?

**Example 2. Dispersion of particles in turbulence** *This problem considers the evolution in time of the mean distance  $\delta$  between two tracer particles (e.g. dust specks) embedded in turbulence. You may know that if such particles experienced Brownian motion then  $\delta = Dt^{1/2}$ , where  $D$  is a diffusion constant. In turbulence, the mean separation also grows as a power law,  $\delta \propto t^n$ , but the power  $n$  is different from  $1/2$ . Namely, it is given by so-called Richardson law which you will be asked to derive in this problem. To do this, you will need to use a dimensional analysis approach similar to the one we have used in question 8.3.1 to derive the famous Kolmogorov spectrum.*

1. *Consider 3D turbulence forced at a large scale at which the role of viscous dissipation is negligible. Define the inertial range of scales. Describe in words the Richardson cascade picture.*

2. *Formulate Kolmogorov's universality hypothesis.*

3. *Find the physical dimension of the energy injection rate  $\epsilon$ .*

4. Consider two tracer particles which are moving with the fluid particles in 3D turbulence. Assume that the distance between these particles corresponds to the inertial range of scales in turbulence, and use Kolmogorov's universality hypothesis and the dimensional argument to derive Richardson's law

$$\delta = C\epsilon^m t^n, \quad (1.10)$$

where  $C$  is a dimensionless constant. Find  $m$  and  $n$ .

5. In his 1926 paper (Atmospheric diffusion shown on a distance-neighbour graph), Richardson obtained an equation for  $P(\delta, t)$ , the probability density function defined so that  $P(\delta, t)d\delta$  is the probability of the two-particle separation to be in the range from  $\delta$  to  $\delta + d\delta$ . He wrote

$$\frac{\partial P}{\partial t} = \frac{1}{\delta^2} \frac{\partial}{\partial \delta} \left( \delta^2 D(\delta) \frac{\partial P}{\partial \delta} \right), \quad (1.11)$$

where  $D(\delta) = C'\epsilon^q \delta^s$  ( $C'$  is a dimensionless constant) and he deduced the value of the index  $s$  from experimental data, because it was not until 1941 that Kolmogorov's dimensional argument was discovered. For the same reason, the dependence on  $\epsilon$  was not found (nor even mentioned) by Richardson. Please use the dimensional argument and find  $q$  and  $s$ .

## 2 Identifying Kolmogorov Spectrum - 2013/14 Material

Plotting the energy spectrum of a flow over several different times will help to identify the regime that the energy spectrum follows.

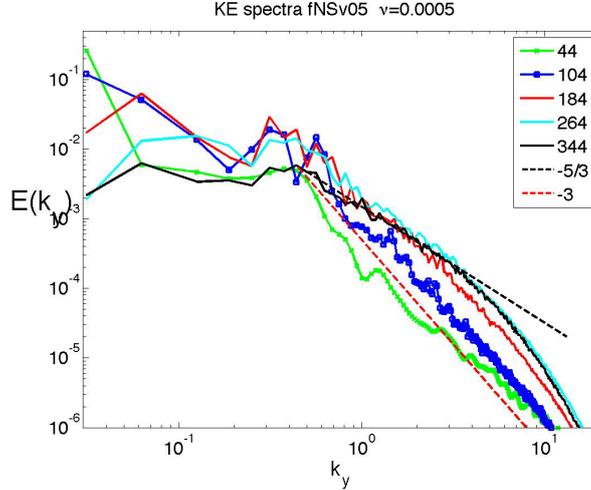


Figure 3: Energy spectrum for 5 different times in a flow. Plotting the  $k^{-3}$  and  $k^{-5/3}$  slopes suggests that early times follow a  $k^{-3}$  regime whereas late times follow a  $k^{-5/3}$  regime.

Plotting a compensated energy spectrum  $E(k)k^{5/3}\epsilon^{-2/3}$  will give a strong indication to a Kolmogorov energy spectrum if it flattens out with zero slope. Early times will not reach a turbulent structure hence will not display a Kolmogorov energy cascade. Note that a  $k^{-5/3}$  spectrum does not necessarily indicate turbulence.

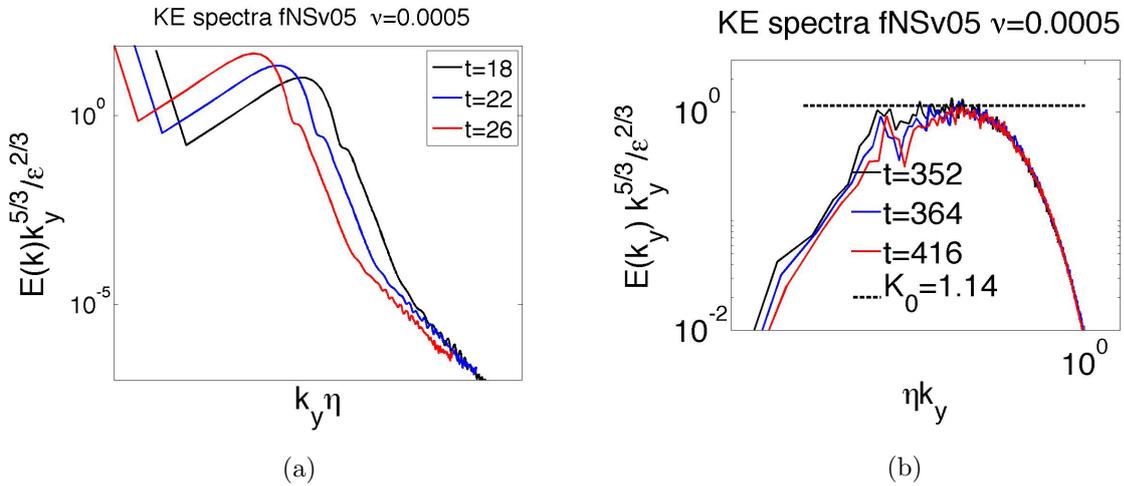


Figure 4: Compensated energy spectra of (a) early times where a turbulent structure has not formed so a Kolmogorov  $k^{-5/3}$  regime is not evident, and (b) late times where there is a flattening to zero slope indicating a Kolmogorov energy cascade.

3D turbulence displays a forward energy cascade with small scale structures forming as energy moves from large to small scales followed by strong dissipation. There is also strong enstrophy production.